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On torsion free crystallographic groups

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Abstract

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The torsion free crystallographic groups arise as fundamental groups of compact flat Riemannian manifolds. Let R be a crystallographic group with point group G and translation group T. In this paper we consider the $\mathbb{Q}G$-module $T \otimes_{\mathbb{Z}} \mathbb{Q}$, for which we prove: If R is torsion free, then G does not act irreducibly on $T \otimes_{\mathbb{Z}} \mathbb{Q}$. A proof of this theorem for solvable groups G was first given by G. Cliff. The theorem proves a conjecture made by the second author. The proof of the theorem uses the classification of the finite simple groups.

1. Results

Let G be a finite group and let X be a $\mathbb{Z}G$-lattice. An element $\alpha \in H^2(G, X)$ is called special, if $\operatorname{res}^G_C \alpha \neq 0$ for all nontrivial cyclic subgroups $C \leq G$. In this paper we prove the following theorem:

Theorem. *Let G be a nontrivial finite group. Suppose X is a $\mathbb{Z}G$-lattice such that G is faithfully represented on X and that $X \otimes_{\mathbb{Z}} \mathbb{Q}$ is irreducible. Then there is no special element in $H^2(G, X)$.*

A proof of this theorem for solvable groups G was first given by G. Cliff.

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Corollary. *Let R be a crystallographic group with point group G and translation group T. If R is torsion free, i.e., has no elements of finite nontrivial order, then G does not act irreducibly on $T \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Proof. The extension of T by G, giving the torsion free crystallographic group R, is described by a special element in $H^2(G, T)$. □

The torsion free crystallographic groups are exactly the fundamental groups of compact flat Riemannian manifolds. So the Theorem proves Conjecture 1.2 made by the second author in [27].

The proof of the theorem uses the classification of the finite simple groups.

2. Reductions

In this section we shall show how to reduce the proof of the theorem to the study of certain properties of characters of finite simple groups.

Throughout this section let G be a finite group. For any prime p, we let $\mathbb{Z}_p$ denote the ring of p-adic integers and $\mathbb{Q}_p$ the field of p-adic numbers. For any field K let $\bar{K}$ denote a fixed algebraic closure. Tensor products will always be taken over the integers $\mathbb{Z}$. A $\mathbb{Z}G$-lattice X is called irreducible if $X \otimes_{\mathbb{Z}} \mathbb{Q}$ is an irreducible $\mathbb{Q}G$-module.

Lemma 2.1. *Let X be an irreducible $\mathbb{Z}G$-lattice. Suppose that $X \otimes \mathbb{Z}_p$ contains an indecomposable direct summand in the principal $\mathbb{Z}_pG$-block. Then every irreducible constituent of $X \otimes \mathbb{C}$ is in the principal p-block of G.*

Proof. Let ζ be a primitive $|G|$-th root of unity and let $K = \mathbb{Q}(\zeta) \subset \bar{\mathbb{Q}} \subset \mathbb{C}$. Then K is a splitting field for G and the K-characters of G are the same as the $\mathbb{C}$-characters. Since $X \otimes \mathbb{Q}$ is irreducible, any two distinct irreducible constituents of $X \otimes K$ are conjugate under an element of the Galois group of $K/\mathbb{Q}$ (see [18, Theorem 9.21(c)]). This group permutes the p-blocks of $\mathrm{Irr}_K(G)$ fixing the principal block (see [6, Lemmas IV.4.9 and IV.4.12]). It is therefore enough to show that one of the constituents of $X \otimes K$ is in the principal p-block.

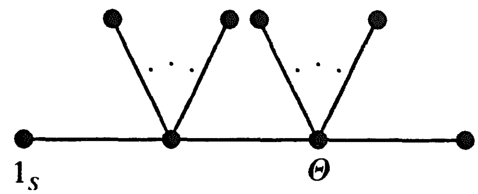
Let $\varphi : K \to \bar{\mathbb{Q}}_p$ be an embedding over $\mathbb{Q}$. Let $L = \varphi(K)\mathbb{Q}_p$. Then φ induces a bijection between $\mathrm{Irr}_K(G)$ and $\mathrm{Irr}_L(G)$ which preserves p-blocks (see [12, (7.10)]). In order to finish the proof, it suffices to show that $X \otimes L$ has a constituent in the principal p-block. This, however, follows from our assumption that $X \otimes \mathbb{Z}_p$ has a direct summand in the principal block (see the introductory discussion of [6, Section VI.1]). □

Lemma 2.2. *Let X be an irreducible $\mathbb{Z}G$-lattice such that $H^2(G, X)$ contains a special element. Let $\mathcal{G}$ denote the set of irreducible characters of G arising from* *constituents of $X\otimes\mathbb{C}$. Let S denote a simple component in the socle (the product of all minimal normal subgroups) of G. Finally, let $\mathcal{S}$ denote the set of irreducible characters of S arising from constituents of ϑ_S, for $\vartheta\in\mathcal{G}$. Then we have:*

(a) *If $\vartheta\in\mathcal{G}$, then ϑ is in the principal p-block for every prime p dividing $|G|$.*

(b) *If $\psi\in\mathcal{S}$, then ψ is in the principal p-block of S for every p dividing $|G|$.*

(c) *Let p be a prime dividing $|S|$ such that a Sylow p-subgroup of S is cyclic. Then there is $\Theta\in\mathcal{S}$ which has the following position on the oriented Brauer tree:*



(1)

Proof. (a) Let p be a prime dividing $|G|$. Since $H^2(G, X)$ contains a special element, it follows from [24, Remark II.1(ii)], that $H^2(G, X\otimes\mathbb{Z}_p)\neq 0$. Write $X\otimes\mathbb{Z}_p = U_1\oplus\cdots\oplus U_n$ with indecomposable $\mathbb{Z}_pG$-lattices U_i. Then $H^2(G, U_i)\neq 0$ for some i. By [16, Lemma 2.2.25], U_i is in the principal $\mathbb{Z}_pG$-block. Lemma 2.1 now shows that the irreducible constituents of $X\otimes\mathbb{C}$ lie in the principal p-block.

(b) The result follows from (a) and an iterated application of [6, Lemma IV.4.10], since S is a subnormal subgroup of G.

(c) We shall use the terminology of [24]. Let α be a special element in $H^2(G, X)$. Then $\beta = \mathrm{res}_S^G\,\alpha$ is a special element in $H^2(S, X)$. Since a Sylow p-subgroup of S is cyclic, this implies that the p-adic S-extension with kernel $X\otimes\mathbb{Z}_p$ corresponding to β is p-torsion free (see [24, Remark II.1(ii)]). By choosing a minimal subextension we obtain a Frattini S-extension (see [24, Proposition II.2]). The kernel of this Frattini S-extension contains an irreducible $\mathbb{Q}_p$-character which has the position of Θ on the Brauer tree (of the principal $\mathbb{Z}_pS$-block). This assertion is proved in Section III of [24]. An irreducible L-constituent of this character gives rise to an element in $\mathcal{S}$, which has the same position on the Brauer tree. This completes the proof of the lemma. □

3. The simple groups

In this section we shall use the classification of the finite simple groups to show that there is no finite nonabelian simple group which has a character satisfying conclusions (b) and (c) of Lemma 2.2.

Let us shortly describe the strategy we are going to follow. Let S be a nonabelian finite simple group and $\mathcal{S}$ the subset of those irreducible characters χ of S satisfying:

(i) χ is in the principal p-block of S for every prime p dividing $|S|$.

(ii) For every prime p dividing $|S|$ such that a Sylow p-subgroup of S is cyclic, χ has the position of Θ on the Brauer tree (1) of the principal p-block.

In particular, if there is a prime p as in (ii), the set $\mathcal{S}$ has at most 1 element. We intend to show that $\mathcal{S}$ in fact is empty. By the classification theorem, S is a sporadic group, an alternating group or a finite group of Lie type. For each such S we find a prime r such that a Sylow r-subgroup of S is cyclic. The Brauer tree of the principal r-block is not known in all cases. However, sufficient information on the ordinary irreducible characters of S is available to restrict the number of possibilities for the character Θ in (1).

We then determine a prime r' with the property that none of these possible Θ's is in the principal r'-block. In most of the cases this is done by showing that Θ is of r'-defect 0, i.e., that $|S|/\Theta(1)$ is not divisible by r'. Hence $\mathcal{S}$ is empty.

3.1. The sporadic groups

The Brauer trees for these groups are almost all completely known [15]. In any case it is possible to determine the character Θ of (1) for suitable primes r for which S has a cyclic Sylow r-subgroup. For each sporadic group the prime r, the degree of the character Θ and the prime r' are given in Table 1. Except for J_2 and He, Θ is always of r'-defect 0. In J_2 and He, Θ is in a 3-block of defect 1, as can

Table 1
Primes and characters for sporadic groups

Group	r	$\Theta(1)$	r'
M_{11}	11	45	5
M_{12}	5	176	11
J_1	5	76	19
M_{22}	5	231	7
J_2	7	288	3
M_{23}	11	1035	23
HS	11	2520	7
J_3	17	3078	19
M_{24}	11	3312	23
McL	11	1750	7
He	17	22050	3
Ru	7	81432	13
Suz	7	168960	11
O'Nan	11	175770	31
C_3	23	5544	11
C_2	23	37422	11
F_{22}	7	2555904	13
Ha	7	267520	11
Ly	7	38734375	37
Th	31	30507008	19
F_{23}	7	166559744	13
C_1	23	4100096	11
J_4	5	1183406741	23
F_{24}	11	9100908180	13
BM	11	422966584586250	13
M	17	5057254202494959840375000	19

be seen from the tables in [15]. Since these two groups have order divisible by 9, Θ is not in the principal 3-block. This completes the proof for the sporadic groups.

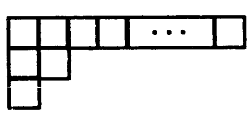
3.2. *The alternating groups*

We start with some small examples. The results for these small alternating groups are collected in Table 2. The symbols have the same meaning as in Table 1. The Brauer trees for these examples are well known and easily calculated. (See also [20, Theorem 6.1.46].)

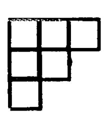
Now let $S = A_n$, $n > 11$. Let $\chi \in \mathcal{S}$ and let r be the largest prime less than n. It easily follows from Bertrand's Postulate (see [11, p. 420]) that $r \geq (n+3)/2$. Then a Sylow r-subgroup of S is cyclic, and χ has the position of Θ on the Brauer tree (1) of the principal r-block. Before we deal with the general case, we have to consider some extreme cases first.

As in the results of Table 2, we use the description of the r-blocks of A_n via the Nakayama conjecture for S_n (see [20, Theorem 6.1.46]). The Brauer trees for S_n can easily be determined using [19, Corollary 12.2]. From this description it follows that the Young diagram corresponding to the character Θ is distinct from its conjugate. Hence the restriction of χ to A_n is irreducible.

Case A: $r = n - 1$. Then the Young diagram corresponding to χ is


(2)

Since $n - 6$ is even, the 2-core of (2) is

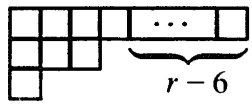


Hence χ is not in the principal 2-block of A_n.

Table 2
Primes and characters for small alternating groups

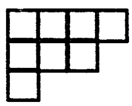
Group	r	$\Theta(1)$	r'
A_5	3	4	2
A_6	5	8	2
A_7	7	15	5
A_8	5	56	7
A_9	7	142	3
A_{10}	7	288	2
A_{11}	11	45	7

Case B: $r = n - 2$. Then the Young diagram corresponding to χ is



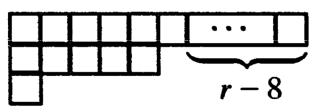
(3)

Let r' be a prime dividing $r - 6$. Then $r' \neq 3$, since $r \neq 3$. Therefore, the r'-core of (3) is



Hence χ is not in the principal r'-block of A_n.

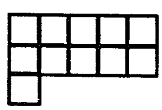
Case C: $r = n - 4$. Since $n > 11$ and r is the maximal prime $<n$, we have $r \geq 13$. The Young diagram corresponding to χ is



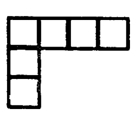
(4)

Suppose first that $r - 8 = 3^a$ is a power of 3. Then $r - 7$ is not a power of 2. For suppose $r - 7 = 2^b$. Then $3^a = 2^b - 1$. It is easy to see that this equation has only two integral solutions, namely $a = 0$, $b = 1$ and $a = 1$, $b = 2$. This contradicts $r \geq 13$.

So let r' be an odd prime dividing $r - 7$. Then $r' \geq 5$. If $r' > 5$, the r'-core of (4) is

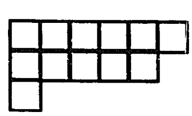


since $r' \neq 7$. If $r' = 5$, the 5-core of (4) is

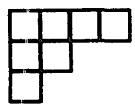


In neither case is χ in the principal r'-block.

Suppose now that $r - 8$ is not a power of 3. Let $r' \neq 3$ be a prime dividing $r - 8$. If $r' \neq 5$, the r'-core of (4) is

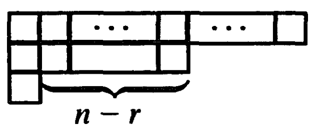


The 5-core of (4) is



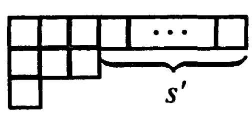
In neither case is χ in the principal r'-block.

Case D: $(n+3)/2 \leq r \leq n-3$, $r \neq n-4$. The Young diagram corresponding to χ is of shape



(5)

Suppose first that $n-r=2^a$, $a \geq 3$. Let r' be an odd prime dividing $n-r-2$, which exists since $a \geq 3$. Beginning from the right, we first remove r'-hooks from the second row of (5), and then from the first, until we obtain the diagram



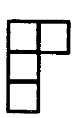
where $s' < r'$. This has the following hook lengths:

$$\begin{array}{lllllll} s'+5 & s'+3 & s'+2 & s' & s'-1 & \cdots & 1 \\ 4 & 2 & 1 & & & & \\ 1 & & & & & & \end{array}$$

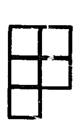
If this is not the r'-core, r' is one of the numbers $s'+5$, $s'+3$, $s'+2$. If $r'=s'+5$, the r'-core of (5) is

(6)

If $r'=s'+3$, the r'-core of (5) is



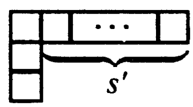
and if $r'=s'+2$, and $r'>3$, the r'-core of (5) is



Finally, if $r'=s'+2=3$, the r'-core of (5) is again given by (6).

To conclude that χ is not in the principal r'-block, we have to show that (6) is not an r'-core of (5) (so that several of the above possibilities do not occur at all). If (6) is the r'-core of (5), we have $n \equiv 2 \pmod{r'}$. On the other hand, by the definition of r', $n - r \equiv 2 \pmod{r'}$. This implies $r' = r$, which is impossible, since $r > n/2$.

We finally consider the case that $n - r$ is not a power of 2. Let r' be an odd prime dividing $n - r$. By successively removing r'-hooks from (5), we end up with the following diagram



(7)

with $s' < r'$. This has hook lengths

$$\begin{array}{lllll} s'+3 & s' & s'-1 & \cdots & 1 \\ 2 & & & & \\ 1 & & & & \end{array}$$

If (7) is not the r'-core of (5), we have $r' = s' + 3$. But this means that the r'-core of (5) is the empty partition, and so $r' \mid n$. On the other hand, $r' \mid n - r$, hence $r' = r$, contradicting $r > n/2$. Thus (7) is indeed the r'-core of (5), showing that χ is not in the principal r'-block.

3.3. *The simple groups of Lie type*

A simple group of Lie type is either the Tits simple group or else of the form $G/Z(G)$, where $G = \mathbf{G}^F$ is the set of fixed points of a Frobenius map F on a simple, simply connected linear algebraic group $\mathbf{G}$ (see [3, Section 1.19]).

The Brauer tree for the Tits simple group in characteristic 13 is given in [13]. The character Θ has degree 2048, hence is of 2-defect 0.

Now let $G/Z(G)$ be a simple group of Lie type, which is neither the Tits group nor a Suzuki or Ree group. To G is associated a positive integer $q = p^a$, where p is a prime number, the so-called defining characteristic of G (see [3, Section 1.9, p. 35]; q is an integer, since we have excluded the Suzuki and Ree groups from our consideration). The order of $G = G(q)$ may be found in [3, p. 75ff]. It can be written as a product of a power of q and cyclotomic polynomials, evaluated at q. We shall write Φ_n for the nth cyclotomic polynomial. A primitive divisor of $\Phi_s(q)$ is a prime r such that $r \mid \Phi_s(q)$ but $r \nmid \Phi_t(q)$ if $t < s$. Primitive divisors exist for all pairs (q, s) of positive integers, except for $(q, s) = (2^b - 1, 2)$, $b \geq 1$ and $(q, s) = (2, 6)$ (see [17, Theorem 8.3]).

Given G, let m_G be the largest integer such that $\Phi_{m_G}(q)$ divides $|G(q)|$ for all q. If $(q, m_G) \neq (2, 6)$ and $(q, m_G) \neq (2^b - 1, 2)$, let r be a primitive divisor of

$\Phi_{m_G}(q)$. Such a prime is called a Coxeter prime for $G(q)$. If $G = {}^2B_2(q^2)$, $q^2 = 2^{2m+1}$, $m \geq 1$, a Coxeter prime for G by definition is a prime divisor of $q^2 - \sqrt{2}q + 1$. For the Ree groups ${}^2G_2(q^2)$, $q^2 = 3^{2m+1}$, $m \geq 1$ and ${}^2F_4(q^2)$, $q^2 = 2^{2m+1}$, $m \geq 1$, Coxeter primes are defined to be the prime divisors of $q^2 - \sqrt{3}q + 1$ respectively $q^4 - \sqrt{2}q^3 + q^2 - \sqrt{2}q + 1$. If r is a Coxeter prime, a Sylow r-subgroup of G is cyclic (see [14]).

When considering unipotent characters of G, it does no harm to pass from $\mathbf{G}$ to the adjoint group $\mathbf{G}_{\mathrm{ad}}$. Unipotent characters of G have $Z(G)$ in their kernel (see [3, Section 12.1, p. 380]). Let

$$\mu : \mathbf{G} \to \mathbf{G}_{\mathrm{ad}}$$

be a surjective homomorphism of algebraic groups. Then $\ker(\mu) = Z(\mathbf{G})$ (cf. [3, p. 25]). Let G_0 be the image of G in $(\mathbf{G}_{\mathrm{ad}})^F$. Then $G_0 \cong G/Z(G)$. The unipotent characters of G_0 are exactly the restrictions to G_0 of the unipotent characters of $(\mathbf{G}_{\mathrm{ad}})^F$ (see [5, Proposition 7.10]).

In the following, let $S = G/Z(G)$ be the simple nonabelian component of a simple group of Lie type G. Let $\chi \in \mathcal{S}$ and let r be a Coxeter prime for G (if it exists). Then a Sylow r-subgroup of S is cyclic and nontrivial, and χ has the position of Θ in the Brauer tree (1) of the principal r-block. If χ is unipotent we can think of it as a character of G or $(\mathbf{G}_{\mathrm{ad}})^F$. We are now going to describe the possible characters χ for the classical groups. Again we consider some small cases first, mainly the cases where no Coxeter primes exist. These results are collected in Table 3. We start the general investigation with a preliminary result.

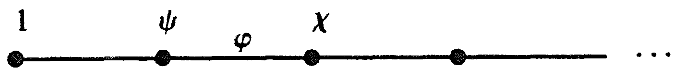
Lemma 3.1. *Let $G = G(q)$ be a classical group, defined over the field with q elements. Let r be a prime such that a Sylow r-subgroup of G is cyclic. Suppose also that for any fundamental reflection s in the Weyl group W of G, $r \nmid |P_s|$, where P_s denotes the corresponding standard parabolic subgroup. We assume furthermore, that W has rank at least 2. Let χ be an irreducible character in the position of*

Table 3
Primes and characters for some small classical groups

Group	r	$\Theta(1)$	r'
$A_5(2)$	31	6480	5
$A_6(2)$	127	5208	31
${}^2A_3(2)$	5	81	3
${}^2A_4(2)$	11	1024	2
${}^2A_5(2)$	11	8064	7
$B_3(2)$	7	120	5
$B_4(2)$	17	4200	5
$D_4(2)$	7	4096	2
$D_5(2)$	17	9300	5
${}^2D_4(2)$	17	1344	7

Θ on the Brauer tree (1) *of the principal r-block of G. Then χ is in the principal series, i.e.,* $(\chi, \mathbf{1}_B^G) \neq 0$, *if B denotes the Borel subgroup of G.*

Proof. By a result of Robinson [25, Theorem 10], the Brauer tree is an open polygon



Let φ be the irreducible Brauer character corresponding to the edge of the tree connecting ψ and χ. We extend φ by 0 to all of G. Then $\varphi = (\chi - \psi + \mathbf{1})\gamma$, if γ denotes the characteristic function on the r-regular classes of G. If P is a subgroup of G with $r \nmid |P|$, then $\mathbf{1}_P^G$ is a projective character, and we have:

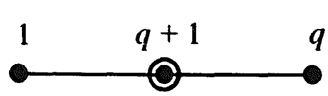
$$\begin{aligned} 0 &\leq (\varphi, \mathbf{1}_P^G) \\ &= ((\chi - \psi + \mathbf{1})\gamma, \mathbf{1}_P^G) \\ &= (\chi - \psi + \mathbf{1}, \mathbf{1}_P^G) \\ &= (\chi, \mathbf{1}_P^G) - (\psi, \mathbf{1}_P^G) + 1 . \end{aligned}$$

In particular, we have $(\chi, \mathbf{1}_B^G) \geq (\psi, \mathbf{1}_B^G) - 1$. We proceed to show that $(\psi, \mathbf{1}_B^G) > 1$. With the same arguments we have used so far, we can show that $(\psi, \mathbf{1}_B^G) > 0$. Thus ψ lies in the principal series. By [4, Theorem 68.24], we have $(\psi, \mathbf{1}_B^G) = \lambda(1)$, where λ is the character of W which corresponds to ψ.

Suppose $\lambda(1) = 1$. Choose a fundamental reflection $s \in W$ such that $\lambda(s) \neq 1$. This is possible since λ is a nontrivial homomorphism of W, and W is generated by a set of fundamental reflections. Set $P = P_s$. Then, by [4, Theorem 68.24], we have $(\mathbf{1}_P^G, \psi) = (\mathbf{1}_{\langle s \rangle}^W, \lambda) = (\mathbf{1}_{\langle s \rangle}, \lambda_{\langle s \rangle}) = 0$. This is a contradiction, since by assumption $\mathbf{1}_P^G$ is a projective character. This completes the proof of the lemma. □

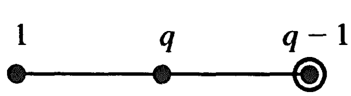
Suppose $G = A_1(q)$, $q \geq 4$. Let r be an odd prime dividing $q^2 - 1$. Then the Sylow r-subgroup of G is cyclic. We distinguish two cases:

Case I: $r \mid q - 1$. Then the Brauer tree of the principal r-block of G is



Hence $\chi(1) = q$ and χ is of p-defect 0.

Case II: $r \mid q + 1$. Then the Brauer tree of the principal r-block of G is



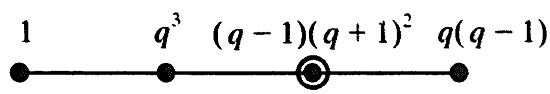
If $q-1$ is a power of 2, let $r'=2$. Otherwise, let r' be an odd prime dividing $q-1$. Then χ is of r'-defect 0 (for $G/Z(G)=S=PSL_2(q)$).

Let $G=A_l(q)$, $l\geq 2$. Since $m_G=l+1$, a Coxeter prime r exists except for $l=5$, $q=2$. So let us assume $(q,l)\neq(2,5)$ in the following. By [8], the Brauer tree of the principal r-block of G is completely known. The partition labelling χ is $(1,1,l-1)$. We have by [7, p. 115]

$$\chi(1)=q^3\,\frac{(q^l-1)(q^{l-1}-1)}{(q^2-1)(q-1)}\,.$$

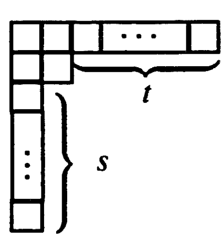
If $l=2$, let $r'=p$. If $l>2$ and $(q,l)\neq(2,6)$, let r' be a primitive divisor of $\Phi_l(q)$. Then χ is of r'-defect 0.

Now let G be of type ${}^2A_l(q)$, $l\geq 2$. We consider the case of ${}^2A_2(q)$ first. Since $PSU_3(2)$ is not simple, we may assume that $q>2$. Let r be a primitive divisor of $\Phi_6(q)$. Then a Sylow r-subgroup of G is cyclic, and the Brauer tree of the principal r-block is



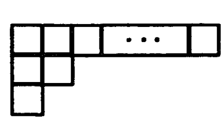
(see [10]). Thus χ is one of the exceptional characters and $\chi(1)=(q-1)(q+1)^2$. Thus χ is of 2-defect 0 for $S=G/Z(G)$.

Next let G be of type ${}^2A_l(q)$, $l\geq 3$, odd. Suppose that $(q,l)\neq(2,3)$. Let r be a primitive divisor of $\Phi_{2l}(q)$. By the description of the r-blocks of G given by Fong and Srinivasan in [7], two unipotent characters of G lie in the same r-block, if and only if their corresponding Young diagrams have the same e-core, where e is the order of q^2 modulo r. By our choice of r, we have $e=l$. The l-core of the partition (1^{l+1}), which corresponds to the trivial character, is the partition (1). Thus χ is either the Steinberg character, or else has one of the following diagrams:



(8)

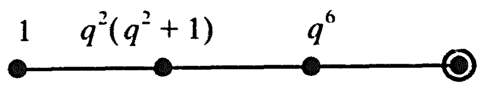
with $s\geq 0$. We may assume that χ is not the Steinberg character, since otherwise we are done with $r'=p$. We claim that the diagram (8) for χ has no $(l-2)$-hooks. For suppose it has such a hook. Then it must be the diagram



(9)

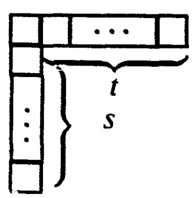
But then χ is not in the principal series, since the number of odd respectively even hook lengths in (9) differ by more than 1 (cf. [3, Section 13.8, p. 466]). This contradicts Lemma 3.1. Hence the diagram for χ has no $(l-2)$-hooks, and so $\Phi_{2(l-2)}(q) \mid \chi(1)$.

If $l=3$, the Brauer tree of the principal r-block given below is easily determined.



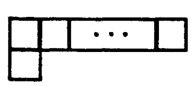
Thus χ is the Steinberg character and we are done. Suppose now that $l>3$ and that $(q,l) \neq (2,5)$. Let r' be a primitive divisor of $\Phi_{2(l-2)}(q)$. Then χ is of r'-defect 0. This follows from $\Phi_{2(l-2)}(q) \mid \chi(1)$ and $r' \nmid \gcd(\Phi_{2(l-2)}(q), \Phi_{2l}(q))$.

Now let G be of type ${}^2A_l(q)$, $l \geq 4$, even. Let r be a primitive divisor of $\Phi_{2(l+1)}(q)$. Since χ is in the principal series by Lemma 3.1, it is unipotent. This time the order of q^2 modulo r equals $l+1$. Therefore, the Young diagram of χ is an $(l+1)$-hook, i.e., of shape



(10)

We may suppose that $s>0$ and $t>0$. Then the diagram (10) has no $(l-1)$-hooks. For otherwise this diagram would be



(11)

so that χ is not in the principal series for the same reason as above. Hence $\Phi_{2(l-1)}(q) \mid \chi(1)$. Let r' be a primitive divisor of $\Phi_{2(l-1)}(q)$ if $(q,l) \neq (2,4)$. Then χ is of r'-defect 0 which follows from the above and $r' \nmid \gcd(\Phi_{2(l+1)}(q), \phi_{2(l-1)}(q))$.

We now consider the remaining classical, i.e., the symplectic and orthogonal groups. Let G be such a group and r a Coxeter prime for G. We may assume that the Weyl group of G has rank at least 2. The hypotheses of Lemma 3.1 are clearly satisfied, so that a character $\chi \in \mathscr{S}$ is in the principal series, hence unipotent. The unipotent characters of G are parametrized by symbols. For the definition of symbols, their basic properties and all other facts we are going to use here, we refer the reader to [3, Section 13.8].

For our purposes, a symbol is an array of integers

$$\Lambda = \begin{pmatrix} \lambda_1 & \cdots & \lambda_a \\ \mu_1 & \cdots & \mu_b \end{pmatrix}, \tag{12}$$

satisfying certain additional conditions, e.g.:

$$0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_a , \tag{13}$$

$$0 \leq \mu_1 < \mu_2 < \cdots < \mu_b . \tag{14}$$

$$\lambda_1, \mu_1 \text{ not both equal to } 0 . \tag{15}$$

If χ_Λ is the unipotent character corresponding to Λ, then the degree of χ_Λ is given by a rather complicated formula, which we do not need to reproduce here. A simplified version of this formula can be found in [23, Proposition 5]. We shall, however, need the notion of a cohook of a symbol Λ. Let e denote a positive integer. A cohook of Λ of length e is a pair of integers (κ, ν) such that $0 \leq \kappa < \nu$, $\kappa - \nu = e$ and either $\kappa \in \{\lambda_1, \ldots, \lambda_a\}$ and $\nu \not\in \{\mu_1, \ldots, \mu_b\}$ or $\kappa \in \{\mu_1, \ldots, \mu_b\}$ and $\nu \not\in \{\lambda_1, \ldots, \lambda_a\}$.

Let us first consider the cases of $B_l(q)$ and $C_l(q)$. Here, the symbols parametrizing the unipotent characters in the principal series satisfy:

$$b = a - 1 \geq 0 , \tag{16}$$

$$\sum_{i=1}^{a} \lambda_i + \sum_{j=1}^{b} \mu_j = l + (a-1)^2 . \tag{17}$$

Lemma 3.2. *Let Λ be a symbol satisfying* (13)–(17). *Suppose furthermore that Λ has a cohook of length l. Then Λ is one of the following symbols*:

$$\binom{l}{\ }, \begin{pmatrix} 0 & 1 & \cdots & m-1 & l \\ 1 & 2 & \cdots & m & \end{pmatrix}, \quad 1 \leq m \leq l . \tag{18}$$

Proof. (a) We first show that $\lambda_a \leq l$, $\mu_b \leq l$. Suppose that $\lambda_1 \neq 0$. Then

$$\sum_{i=1}^{a-1} \lambda_i + \sum_{j=1}^{b} \mu_j \geq \tfrac{1}{2}((a-1)a + (b-1)b) = (a-1)^2 , \tag{19}$$

$$\sum_{i=1}^{a} \lambda_i + \sum_{j=1}^{b-1} \mu_j \geq \tfrac{1}{2}((a+1)a + (b-2)(b-1)) = (a-1)^2 + 2 . \tag{20}$$

Suppose now that $\lambda_1 = 0$. Then, by (15), $\mu_1 \neq 0$ and we have

$$\sum_{i=1}^{a-1} \lambda_i + \sum_{j=1}^{b} \mu_j \geq \tfrac{1}{2}((a-2)(a-1) + (b+1)b) = (a-1)^2 ,$$

$$\sum_{i=1}^{a} \lambda_i + \sum_{j=1}^{b-1} \mu_j \geq \tfrac{1}{2}((a-1)a + (b-1)b) = (a-1)^2 .$$

In any case we have $\lambda_a \leq l$ and $\mu_b \leq l$ by (17).

(b) Since Λ has a cohook of length l, we must have $\lambda_a = l$ or $\mu_b = l$. We start by investigating the case $\lambda_a = l$. Suppose first that $\lambda_1 \neq 0$. If $a = 1$, then $\Lambda = ({}^l)$. Suppose now that $a > 1$. Then, from $\lambda_a = l$ and (19) we obtain

$$(a-1)^2 = \sum_{i=1}^{a-1} \lambda_i + \sum_{j=1}^{a-1} \mu_j \geq (a-1)^2 .$$

This implies $\lambda_i = i$, $1 \leq i \leq a-1$, $\mu_j = j-1$, $1 \leq j \leq a-1$. But this symbol has no cohook of length l. So we must have $\lambda_1 = 0$, $\mu_1 \neq 0$. This leads to the symbols (18), which do have cohooks of length l.

(c) Now suppose $\lambda_a < l$. Then, in order to get a cohook of length l, we must have $\mu_b = l$, $\lambda_1 \neq 0$. In this case we get a contradiction by using (20). □

Remember that r is a Coxeter prime for G and that $\chi \in \mathcal{S}$. Then χ is in the principal series by Lemma 3.1. Let e be the order of q^2 modulo r. Then $e = l$, since r is a Coxeter prime. By [23, Corollary 7], a unipotent character χ_Λ is of r-defect 0, unless Λ has a cohook of length l. Thus χ is of the form χ_Λ, where Λ is one of the symbols (18).

If $l = 2$, then, as is easy to see, the position of Θ on the Brauer tree is occupied by the Steinberg character. Hence we may assume $l > 2$ in the following. Now $\chi \in \mathcal{S}$ is neither the trivial nor the Steinberg character. The symbol $({}^l)$ corresponds to the trivial, the symbol

$$\begin{pmatrix} 0 & 1 \cdots l-1 & l \\ 1 & 2 \cdots \quad l & \end{pmatrix}$$

to the Steinberg character. The remaining symbols have no cohooks of length $l-1$. Hence, if r' is a primitive divisor of $\Phi_{2(l-1)}(q)$, then χ_Λ is of r'-defect 0, and we are done. The method fails for the pairs $(q, l) \in \{(2,3), (2,4)\}$. These are considered in Table 3.

Now let $G = D_l(q)$, $l \geq 4$. Here, the symbols parametrizing the unipotent characters in the principal series are exactly those which satisfy:

$$b = a \geq 1 , \tag{21}$$

$$\sum_{i=1}^{a} \lambda_i + \sum_{j=1}^{b} \mu_j = l + a(a-1) . \tag{22}$$

Furthermore, symbols with the arrays of λ's and μ's interchanged, represent the same unipotent character. We note the following:

Lemma 3.3 *Let Λ be a symbol satisfying* (13)–(15), (21) *and* (22). *Suppose furthermore that Λ has a cohook of length $l-1$. Then Λ is one of the following symbols*:

$$\binom{l}{0},\ \binom{l-1}{1},\ \begin{pmatrix}0 & 1 & 2 \cdots m-2 & l-1\\ 1 & 2 & 3 \cdots m-1 & m\end{pmatrix},\quad 2 \le m \le l\,. \tag{23}$$

Proof. The proof is omitted, since it is similar to the proof of Lemma 3.2. □

Let Λ be one of the symbols (23), such that χ_Λ is neither the trivial nor the Steinberg character. Then Λ has no cohooks of length $l-2$. Thus χ_Λ is of r'-defect 0, if r' is a primitive divisor of $\Phi_{2(l-2)}(q)$. The method fails for the pairs $(q, l) \in \{(2,4),(2,5)\}$. These are considered in Table 3.

We finally have to consider the case of $G = {}^2D_l(q)$, $l \ge 4$. The symbols we are interested in satisfy:

$$b = a - 2 \ge 0\,, \tag{24}$$

$$\sum_{i=1}^{a} \lambda_i + \sum_{j=1}^{b} \mu_j = l + (a-2)(a-1)\,. \tag{25}$$

We note the following:

Lemma 3.4. *Let Λ be a symbol satisfying* (13)–(15), (24) *and* (25). *Suppose furthermore that Λ has a cohook of length l. Then Λ is one of the following symbols*:

$$\binom{0\ \ l}{\ },\ \begin{pmatrix}0 & 1 & 2 \cdots m-1 & m & l\\ 1 & 2 & 3 \cdots & m & \end{pmatrix},\quad 1 \le m \le l-1\,. \tag{26}$$

Proof. Omitted. □

Let r be a primitive divisor of $\Phi_{2l}(q)$. Now a symbol of type (26) has no cohook of length $l-1$, unless it corresponds to the trivial or the Steinberg character. So we are done except in the case $(q, l) = (2, 4)$, which is considered in Table 3. This completes the proof for the classical groups. It should be remarked that in a classical group $G(q)$, where q is odd, all Brauer trees are known by [9].

It remains to investigate the exceptional groups of Lie type. We start with the following lemma:

Lemma 3.5. *Let G be an exceptional group of Lie type arising from a simple, simply connected algebraic group* $\mathbf{G}$ *such that $S = G/Z(G)$ is simple. Let r be a Coxeter prime for G. Then the nonexceptional characters in the principal r-block of S are unipotent. The exceptional characters in this block are of r'-defect* 0 *for some prime $r' \mid |S|$.*

Proof. Let $\hat{S} = (\mathbf{G}_{\mathrm{ad}})^F$. Then S is the commutator subgroup of $\hat{S}$. We shall show that the assertions are true for $\hat{S}$. Since the unipotent characters of $\hat{S}$ restrict irreducibly to the unipotent characters of S as we have already remarked above, the same will be true for the exceptional characters in the principal r-block, and hence the assertions are also true for S.

Consider the set of characters $\mathscr{E}_r(\hat{S}, 1)$ (see [1] for a definition). This is a union of r-blocks, which follows from [1] if $\hat{S}$ is neither a Suzuki nor a Ree group. If $\hat{S}$ is one of the latter, it easily follows from the Jordan decomposition of characters by considering character degrees. Hence $\mathscr{E}_r(\hat{S}, 1)' := \mathscr{E}_r(\hat{S}, 1)\backslash\{\text{defect 0 characters}\}$ contains the principal block. We remark that **G** is the dual group of $\mathbf{G}_{\mathrm{ad}}$. Since the r-elements in G, the dual group of $\hat{S}$, are regular, $\mathscr{E}_r(\hat{S}, 1)'$ contains only unipotent characters and exactly one further family of characters, which all have the same restriction to the r-regular classes. Thus the first assertion follows.

Let χ be an exceptional character. Then $\chi \in \mathscr{E}_r(\hat{S}, 1)'\backslash\mathscr{E}(\hat{S}, 1)$. This means that $\chi \in \mathscr{E}(\hat{S}, t)$ for some r-element t in G. We then have

$$\chi(1) = |G : C_G(t)|_{p'} = |G : T|_{p'},$$

where T is the maximal torus containing t, and p is the defining characteristic of G. Now take any prime $r' \neq p$ with $r' \mid |G|$, $r' \nmid |T|$. It is easy to see, using the theory of primitive divisors, that such a prime always exists. In a Suzuki or Ree group we even have $\gcd(|G : T|, |T|) = 1$. This completes the proof of the lemma. □

If G is a simple Suzuki group and r a Coxeter prime for G, then the position of Θ in the Brauer tree (1) of the principal r-block is occupied by the exceptional characters (see [2]). We are done in this case by Lemma 3.5. If G is of type G_2 or ${}^3D_4$, then this position is occupied by the Steinberg character. For G_2 this is proved in [26], for ${}^3D_4$ it is very easy to see. In the remaining cases we proceed as follows. The degrees of the unipotent characters are given in Lusztig's book [21]. We exclude those which are of r-defect 0. Now let χ be one of the unipotent characters we are left with. Then χ is of r'-defect 0 for suitable primes r'. We may choose r' as a primitive divisor of a certain $\Phi_s(q)$, $\Phi_s'(q)$, respectively $\Phi_s''(q)$ according to Table 4. We leave the details to the reader.

Here, $\Phi_{12}''(q) = q^2 + \sqrt{3}q + 1$, $\Phi_{24}'(q) = q^4 + \sqrt{2}q^3 + q^2 + \sqrt{2}q + 1$. Thus

Table 4

Group	r' divides one of
$F_4(q)$	$\Phi_8(q)$, $\Phi_6(q)$ ($r' = 7$ for $q = 2$)
$E_6(q)$	$\Phi_9(q)$, $\Phi_8(q)$
${}^2E_6(q)$	$\Phi_{12}(q)$, $\Phi_{10}(q)$
$E_7(q)$	$\Phi_{14}(q)$, $\Phi_{12}(q)$
$E_8(q)$	$\Phi_{24}(q)$, $\Phi_{20}(q)$
${}^2G_2(q^2)$, $q^2 = 3^{2m+1}$, $m \geq 1$	$\Phi_{12}''(q)$, $\Phi_4(q)$
${}^2F_4(q^2)$, $q^2 = 2^{2m+1}$, $m \geq 1$	$\Phi_8(q)$, $\Phi_{24}'(q)$, $\Phi_{12}(q)$

every nontrivial character in the principal r-block is of r'-defect 0 for some prime r' dividing the group order. This completes the proof for the simple groups of Lie type.

4. The proof of the theorem

In this section we complete the proof of the theorem. Let X be an irreducible $\mathbb{Z}G$-lattice such that $H^2(G, X)$ contains a special element. Suppose furthermore that G is faithfully represented on X. Choose a minimal normal subgroup N of G and a simple direct factor S of N. It follows from Lemma 2.2 and the results of the preceding section, that S is abelian. Hence N is an elementary abelian p group for some prime p.

Now let $\chi = \sum_{i=1}^{n} \chi_i$ be the character of $X \otimes \mathbb{C}$. Suppose there is a prime $q \mid |G|$ and $q \neq p$. By Lemma 2.2(a), each χ_i is in the principal q-block, and so has $N \leq O_{q'}(G)$ in its kernel (see [6, Lemma IV.4.12]). But then N is in the kernel of χ, which contradicts the fact that G is faithfully represented on X.

Finally suppose that G is a p-group. Let $C = \langle g \rangle$ be a cyclic subgroup of order p in the center of G. Let $X^C = \{x \in X \mid xc = x \text{ for all } c \in C\}$ the set of fixed points of C in X and $X' = X(1 + g + \cdots + g^{p-1})$.

Then $H^2(C, X) \cong X^C/X'$ (see [22, Theorem IV.7.1]). Since C is central, X^C is a $\mathbb{Z}G$-submodule of X. But $X \otimes \mathbb{Q}$ is irreducible and represents G faithfully, which implies $X^C = 0$, and so $H^2(C, X) = 0$. Then $H^2(G, X)$ can have no special element, a final contradiction.

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