Spin structures on flat manifolds with cyclic holonomy

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1 Introduction

Let M denote a compact, connected, flat Riemannian manifold (flat manifold for short) of dimension n with fundamental group Γ. Then Γ is a Bieberbach group of rank n, i.e., Γ is torsion free and there is a short exact sequence of groups

$$0 \to \Lambda \to \Gamma \xrightarrow{\pi} G \to 1, \tag{1}$$

where G is finite, the holonomy group of Γ, and Λ is a free abelian group of rank n. Moreover, Λ is a maximal abelian subgroup of Γ, called the translation subgroup or translation lattice of Γ.

The conjugation action of Γ on Λ yields a linear, faithful action of G on Λ, and the Riemannian structure on M induces a G-invariant scalar product on Λ. We thus obtain a homomorphism $\rho : \Gamma \to \mathrm{O}(n)$, the holonomy representation of Γ. The manifold M is oriented if and only if ρ maps Γ into $\mathrm{SO}(n)$, in which case we also say that Γ is oriented.

In this paper we are interested in spin structures on flat oriented manifolds M. It is well known (see [3]), that the spin structures on M are classified by the lifts of ρ to $\mathrm{Spin}(n)$, i.e., by the homomorphisms $\varepsilon : \Gamma \to \mathrm{Spin}(n)$ satisfying $\rho = \lambda \circ \varepsilon$, where λ denotes the vector representation of $\mathrm{Spin}(n)$. If such a lift exists, we also say that the pair (ρ, Γ) has a spin structure.

Different G-invariant scalar products on Λ may lead to different holonomy representations ρ_i of Γ, $i = 1, 2$, which are of course conjugate by an element of $\mathrm{GL}(n, \mathbb{R})$, but not necessarily by one of $\mathrm{O}(n)$. In Proposition 2.1 we show that in this situation there is a natural correspondence between the spin structures of (ρ_1, Γ) and those of (ρ_2, Γ) . As a consequence we generalize a result of Dekimpe, Sadowski and Szczepański [2]. If the holonomy representation of Γ is a direct sum of two $\mathbb{R}$-equivalent oriented representations, then M has a spin structure (Corollary 2.2).

Let $\pi : \Gamma \to G$ denote the natural epimorphism and let G_2 be a Sylow 2-subgroup of G. Then $\Gamma_2 := \pi^{-1}(G_2)$ is a Bieberbach group and it is known that (ρ, Γ) has a spin structure if and only if $(\rho|_{\Gamma_2}, \Gamma_2)$ has one (see, e.g., [2, Proposition 1]). So in order to investigate the existence of spin structures, we may restrict to the case that G is a 2-group. Also, if H is any subgroup of G (or of G_2) and $(\rho|_{\pi^{-1}(H)}, \pi^{-1}(H))$ does not have a spin structure, then (ρ, Γ) does not have one either.

In Section 3 of our paper we consider the case that G is a cyclic 2-group. Under this hypothesis we show that (ρ, Γ) has a spin structure unless G has order 4 (Theorem 3.1). Using the Heller-Reiner classification of indecomposable integral representations of $\mathbb{Z}_4$, and the subsequent classification of the flat $\mathbb{Z}_4$-manifolds due to Hiller, we obtain a necessary and sufficient condition for the exixtence of a spin structure on a $\mathbb{Z}_4$-manifold, based on the embedding of its holonomy group G into $\mathrm{SO}(n)$ and the cocycle describing the fundamental group Γ as and extension of G by Λ (Theorem 3.2).

In the final section we consider flat oriented manifolds with Klein four holonomy groups. Our main result here asserts that such a manifold always has a spin structure provided its first Betti number is 0 (Theorem 4.1). This theorem is based on the classification of these manifolds due to Tirao. Examples show that one cannot drop the assumption on the first Betti number, and that for elementary abelian holonmy groups of order greater than four an analogous result does not hold.

2 Spin structures on flat oriented manifolds

Here we introduce our notation and some basic, well known facts about spin structures. The cyclic group of order m is denoted by $\mathbb{Z}_m$. As usual, we write $\mathrm{O}(n) = \mathrm{O}(n, \mathbb{R})$ for the orthogonal group of $\mathbb{R}^n$ with respect to the standard scalar product; $\mathrm{SO}(n) = \mathrm{SO}(n, \mathbb{R})$ is the special orthogonal group and $\mathrm{Spin}(n) = \mathrm{Spin}(n, \mathbb{R})$ its universal covering group. We also write $\lambda : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ for the natural homorphism, also called the vector representation of $\mathrm{Spin}(n)$.

Let $A(n)$ and $E(n)$ denote the affine group and the group of rigid motions, respectively, acting from the left on n-dimensional euclidean space. Thus $A(n) = \mathrm{GL}(n, \mathbb{R}) \ltimes \mathbb{R}^n$, and $E(n) = \mathrm{O}(n) \ltimes \mathbb{R}^n$, viewed as a subgroup of $A(n)$. We write (A, v) with $A \in \mathrm{GL}(n, \mathbb{R})$ and $v \in \mathbb{R}^n$ for an element of $A(n)$. For two elements (A, v) and (B, w) of $A(n)$ we have the multiplication rule $(A, v)(B, w) = (AB, Aw + v)$. Every Bieberbach group Γ as in (1) can be embedded into $E(n)$ in such a way that $\Lambda = \Gamma \cap \mathbb{R}^n$.

Let F be a group and

$$\rho : F \to \mathrm{SO}(n) \tag{2}$$

a homomorphism. Following [2, Definition 1], we say that the pair (ρ, F) has a *spin structure*, if ρ lifts to $\mathrm{Spin}(n)$, i.e., if there exisits a homomorphism $\varepsilon : F \to \mathrm{Spin}(n)$ such that $\rho = \lambda \circ \varepsilon$. If $F = \Gamma$ is the fundamental group of a flat oriented n-manifold M, then M has a spin structure if and only if (ρ, Γ) has a spin structure, where $\rho : \Gamma \to \mathrm{SO}(n)$ is the holonomy representation of Γ (see [3]). The existence of a spin structure on M is independent of the particular Riemannian structure on M (see [10, Remark 1.9]). This can be made more precise as follows.

Proposition 2.1 *Suppose that $n \geq 2$. Let M_i be flat oriented n-manifolds with fundamental groups Γ_i, $i = 1, 2$. Suppose that Γ_1 is isomorphic to Γ_2 (as abstract groups). Then M_1 has a spin structure if and only if M_2 has one.*

Proof. We may assume that Γ_1 and Γ_2 are subgroups of $E(n)$, and that $M_i = \mathbb{R}^n/\Gamma_i$, $i = 1, 2$. By Bieberbach's second theorem (see [1, Theorem I.4.1]), there is an element $\alpha \in A(n)$ such that $\Gamma_2 = \alpha\Gamma_1\alpha^{-1}$.

Let $\rho_i : \Gamma_i \to \mathrm{SO}(n)$ denote the holonomy representations of Γ_i, $i = 1, 2$. Suppose that $\alpha = (X, v)$ with $X \in \mathrm{GL}(n, \mathbb{R})$ and $v \in \mathbb{R}^n$. Then

$$\rho_2(\alpha\gamma\alpha^{-1}) = X\rho_1(\gamma)X^{-1}, \qquad \text{for all } \gamma \in \Gamma_1.$$

Let $\mathrm{ML}(n)$ denote the metalinear group, a twofold covering of the general linear group $\mathrm{GL}(n, \mathbb{R})$, and let $\lambda : \mathrm{ML}(n) \to \mathrm{GL}(n, \mathbb{R})$ denote the covering homomorphism. Then, $\lambda^{-1}(\mathrm{SO}(n)) = \mathrm{Spin}(n)$. Choose $\tilde{X} \in \mathrm{ML}(n)$ with $\lambda(\tilde{X}) = X$.

Suppose that $\pi_1 : \Gamma_1 \to \mathrm{Spin}(n)$ is a spin structure of (ρ_1, Γ_1). Define $\pi_2 : \Gamma_2 \to \mathrm{Spin}(n)$ by $\pi_2(\alpha\gamma\alpha^{-1}) := \tilde{X}\pi_1(\gamma)\tilde{X}^{-1}$ for $\gamma \in \Gamma_1$. Then π_2 is a spin structure of (ρ_2, Γ_2). By symmetry, this completes the proof. $\square$

The above proof shows in fact that there is a natural one-to-one correspondence between the spin structures on M_1 and those on M_2. Moreover, the proof can easily be translated to the more general situation of Pin or Pin^- structures (using a twofold covering of the general linear group containing the Pin^- group) on flat manifolds which are not oriented.

Corollary 2.2 *Let M be a flat oriented n-manifold, $n \geq 4$, with fundamental group Γ and holonomy group G. Suppose that the translation lattice Λ of Γ is of the form $\Lambda = \Lambda_1 \oplus \Lambda_2$ with G-invariant sublattices Λ_1 and Λ_2, such that the $\mathbb{R}$-representations ρ_i' of G afforded by Λ_i map into $\mathrm{SO}(m)$, $i = 1, 2$, and that ρ_1' and ρ_2' are equivalent (over $\mathbb{R}$). Then M has a spin structure.*

Proof. We may assume that $\Gamma \in E(n)$ and that $M = \mathbb{R}^n/\Gamma$. Our hypothesis implies that Γ is conjugate in $A(n)$ to a Bieberbach group $\tilde{\Gamma} \in E(n)$ with translation lattice $\tilde{\Lambda} = \tilde{\Lambda}_1 \oplus \tilde{\Lambda}_2$, such that the representations $\tilde{\rho}_i' : G \to \mathrm{SO}(m)$ afforded by $\tilde{\Lambda}_i$ are in fact equal.

The "double construction" in the proof of [2, Theorem 1] implies that $\tilde{M} := \mathbb{R}^n/\tilde{\Gamma}$ has a spin structure. By Proposition 2.1, the manifold M also has a spin structure. $\square$

We recall a well known criterion, due to Griess [6] and Gagola and Garrison [4], for the non-existence of a spin structure of the pair (ρ, F) from (2) in case F is finite. The criterion was used by these authors to construct non-trivial double covers for certain groups. We are indebted to Klaus Lux for pointing out reference [4]. The criterion is based on the following lemma. By $|x|$ we denote the order of the element x of some group.

Lemma 2.3 (Griess [6], Gagola-Garrison [4]) *Let $A \in \mathrm{SO}(n)$ be of order 2 and let $a \in \lambda^{-1}(A)$. Then $|a| = 4$ if and only if*

$$\frac{1}{2}\,(n - \mathrm{Trace}(A)) \equiv 2 \,(\mathrm{mod}\ 4).$$

Proof. Let d denote the dimension of the (-1)-eigenspace of A. Then d is even. By [4, Corollary 4.3], there is an inverse image $a \in \mathrm{Spin}(n)$ of A with $a^2 = (-1)^{d(d-1)/2}$. Now $d = (n - \mathrm{Trace}(A))/2$ and the result follows. $\square$

Proposition 2.4 (Griess [6], Gagola-Garrison [4]) *Let F be a finite group and $\rho : F \to \mathrm{SO}(n)$ a homomorphism with character χ.*

Let $g \in F$ have order 2. If

$$\frac{1}{2}\left(\chi(1) - \chi(g)\right) \equiv 2 \pmod 4,$$

then there is no $\varepsilon : F \to \mathrm{Spin}(n)$ such that $\rho = \lambda \circ \varepsilon$.

Proof. This is a direct consequence of Lemma 2.3. $\square$

Example 2.5 Consider the flat oriented manifold $\tilde{M}_1$ of [11, Table 1, p. 327]. Its holonomy representation is a "double" one (each irreducible component has multiplicity two), but the sum of the three distinct irreducible components yields a representation of $\mathbb{Z}_2 \times \mathbb{Z}_2$ into $\mathrm{O}(3)$ but not into $\mathrm{SO}(3)$.

The fundamental group Γ of $\tilde{M}_1$ is generated by its translation subgroup $\mathbb{Z}^6$ and the two generators (copied from [11, Table 1, p. 327])

$$\gamma_1 = ([1, 1, 1, -1, -1, 1], (0, 0, 1/2, 0, 0, 0)^t)$$

and

$$\gamma_2 = ([1, 1, -1, 1, 1, -1], (1/2, 1/2, 0, 0, 0, 0)^t).$$

Here the first bracket denotes a diagonal (6×6)-matrix with the given entries.

Let A_2 and A_3 denote the linear parts of γ_2 and $\gamma_1\gamma_2$ respectively. Then $(6 - \mathrm{Trace}(A_2))/2 = 2$ and $(6 - \mathrm{Trace}(A_3))/2 = 4$. Thus, by Lemma 2.3, inverse images of A_2 and A_3 in $\mathrm{Spin}(6)$ have orders 4 and 2 respectively. Hence a spin structure $\varepsilon : \Gamma \to \mathrm{Spin}(6)$ would map γ_2 and $\gamma_1\gamma_2$ to elements of order 4 and 2, respectively. But $\gamma_1\gamma_2 = ([1, 1, -1, -1, -1, -1], (1/2, 1/2, 1/2, 0, 0, 0))$ and $(\gamma_2)^2 = (\gamma_1\gamma_2)^2$. Thus there is no spin structure on $\tilde{M}_1$.

This example shows that the hypothesis of Corollary 2.2, namely that the ρ_i' map into $\mathrm{SO}(m)$ cannot be dropped.

3 Cyclic holonomy

In this section we study spin structures on manifolds with cyclic holonomy groups of 2-power order.

Let $A \in \mathrm{SL}(n, \mathbb{Z})$ have order 2^m, $m \geq 1$, and let $G := \langle A \rangle$. Choose an embedding $\rho' : G \to \mathrm{SO}(n)$. We want to investigate Bieberbach groups with holonomy group G and translation lattice $\Lambda = \{(\mathrm{id}, v) \mid v \in \mathbb{Z}^n\} \leq E(n)$. We identify the (multiplicatively written) group Λ with the natural $\mathbb{Z}G$-module $\mathbb{Z}^n$.

Any element $\delta \in \mathbb{Q}^n$ gives rise to a derivation $\hat{\delta} : G \to \mathbb{Q}^n/\mathbb{Z}^n$ by sending A to $\delta + \mathbb{Z}^n$. Extending the above convention, we identify $\mathbb{Q}^n/\mathbb{Z}^n$ with the $\mathbb{Z}G$-module $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda/\Lambda$. If the cohomology class of $\hat{\delta}$ in $H^1(G, \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda/\Lambda)$ is special, then δ defines a Bieberbach group

$$\Gamma = \langle (A, \delta), \Lambda \rangle \leq A(n) \tag{3}$$

with holonomy group G and translation lattice Λ. Composing ρ' with the canonical map $\pi : \Gamma \to G$, we obtain the holonomy representation $\rho := \rho' \circ \pi : \Gamma \to \mathrm{SO}(n)$. The natural action of G on $\mathbb{Z}^n$ gives $\mathbb{Z}^n/2\mathbb{Z}^n$ the structure of an $\mathbb{F}_2G$-module, isomorphic to the $\mathbb{F}_2G$-module $\bar{\Lambda} := \Lambda/\Lambda^2$ arising from the conjugation action of Γ on Λ. We write $\bar{v}$ for the image of $v \in \mathbb{Z}^n$ in $\mathbb{Z}^n/2\mathbb{Z}^n$, which we identify with $\bar{\Lambda}$. Recall that the radical of a module is the intersection of its maximal submodules.

Theorem 3.1 *Let Γ and δ be as in* (3). *Then the following holds.*

(1) *If $\lambda^{-1}(\rho(A))$ contains an element of order $2^m = |A|$, then (ρ, Γ) has a spin structure.*

(2) *$\lambda^{-1}(\rho(A))$ contains an element of order $2^m = |A|$ if and only if*

$$\frac{1}{2}\left(n - \mathrm{Trace}(A^{2^{m-1}})\right) \equiv 0 \pmod 4.$$

(3) *If $m = 1$ or $m \geq 3$, then (ρ, Γ) has a spin structure.*

(4) *Suppose that $m = 2$ and that $\lambda^{-1}(\rho(A))$ does not contain elements of order 4. Put $\delta' := A^3\delta + A^2\delta + A\delta + \delta$. Then $\delta' \in \Lambda$ and (ρ, Γ) has a spin structure if and only if $\bar{\delta}'$ is not contained in the radical of $\bar{\Lambda}$.*

Proof. Let $a \in \lambda^{-1}(A)$.

(1) Since $|a| = |A|$, the isomorphism $\varepsilon : \Gamma \to \mathrm{Spin}(n)$ defined by sending (A, δ) to a and Λ to 1 defines the required spin structure.

(2) Clearly, $\lambda^{-1}(\rho(A))$ contains an element of order $2^m = |A|$, if and only if $\lambda^{-1}(\rho(A^{2^{m-1}}))$ contains an involution. This is equivalent to the given condition by Lemma 2.3.

(3) First consider the case $m = 1$. Using the well known integral representation theory of the group of order 2, we may assume that $\Lambda = \Lambda_1 \oplus \Lambda_2$ with G-invariant sublattices Λ_1 and Λ_2 such that A acts diagonally on Λ_1 and $H^1(G, \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_2/\Lambda_2) = 0$. Write $\delta = \delta_1 + \delta_2$ with $\delta_i \in \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_i$, $i = 1, 2$. Then $\delta_2 \in \Lambda_2$ and we may assume that $\delta_2 = 0$, i.e., $\delta \in \Lambda_1$.

Put $\delta' := A\delta + \delta$. The fact that $\delta \mapsto \delta + \Lambda$ defines a nonzero element of $H^1(G, \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda/\Lambda)$ implies that $\bar{\delta}' \in \bar{\Lambda}$ is non-zero, i.e., $\delta' \notin \Lambda_1^2$. Let Λ_0 be a sublattice of index 2 in Λ with $\Lambda_1^2 + \Lambda_2 \leq \Lambda_0$ and $\delta' \notin \Lambda_0$. Then Λ_0 is normal in Γ since G acts trivially on Λ_1/Λ_1^2. Hence Γ/Λ_0 is a cyclic group of order 4, generated by the image of (A, δ). It follows that (ρ, Γ) has a spin structure.

Now suppose that $m \geq 3$. Let χ denote the character of the embedding $G \to \mathrm{GL}(n, \mathbb{C})$. Let s denote the multiplicity of an irreducible complex faithful character ζ of G in χ. Then the multiplicity of ζ^j in χ equals s for all odd integers $1 \leq j \leq 2^m - 1$ since χ is rational valued.

The matrix A is equivalent to a block diagonal matrix with (1×1)-blocks containing the entries ± 1, and (2×2)-blocks of the form

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix},$$

where $0 < \alpha < \pi$ with $2^m\alpha \in 2\pi i\mathbb{Z}$. Such a matrix contributes the value -2 to the trace of $A^{2^{m-1}}$, if and only if $2^{m-1}\alpha \notin 2\pi i\mathbb{Z}$. By the above, the matrix contains exactly $s2^{m-2}$ such blocks. Hence $n - \mathrm{Trace}(A^{2^{m-1}}) = 2^m s$. The result follows from (1) and (2).

(4) In this case, $|a| = 2|A| = 8$ and thus $a^4 = -1$. Consider the $\mathbb{F}_2G$-module $\bar{\Lambda} = \Lambda/\Lambda^2$. Suppose first, that (ρ, Γ) has a spin structure ε. Then $\varepsilon(\mathrm{id}, \delta') = -1$ since $(A, \delta)^4 = (\mathrm{id}, \delta')$. Hence $(\mathrm{id}, \delta') \notin K := \ker(\varepsilon) \leq \Lambda$. On the other hand, $\Lambda^2 \leq K$ and $\bar{K}$ is a maximal $\mathbb{F}_2G$-submodule of $\bar{\Lambda}$. Thus $\bar{\delta}'$ is not contained in the radical of $\bar{\Lambda}$.

Suppose now that $\bar{\delta}'$ is not contained in the radical of $\bar{\Lambda}$. Then there is a normal subgroup K of Γ with $\Lambda^2 \leq K \leq \Lambda$ and $|\Lambda\colon K| = 2$ such that $(\mathrm{id}, \delta') \notin K$. Clearly, Γ/K is cyclic of order 8, and we obtain a spin structure by sending (A, δ) to a and K to 1. $\square$

Using the description of flat $\mathbb{Z}_4$-manifolds of Hiller [8], we further investigate this case. The classification of the (finitely many) isomorphism classes of indecomposable integral representations of a cyclic group G of order 4 is due to Heller and Reiner, and is reproduced in [8, Theorem 1.3]. The corresponding low degree cohomology is computed in [8, Propositions 2.2, 2.3]. For the convenience of the reader, these results are presented in Table 1. The notation for the $\mathbb{Z}G$-lattices is taken from Hiller's paper. On M_1, M_2, and M_4 the subgroup of order 2 of G acts trivially, the other modules afford faithful representations of G. If X is an indecomposable $\mathbb{Z}G$-lattice, then either $H^2(G, X)$ is trivial or has two elements. In the latter case, the last column of Table 1 describes the non-trivial element in $H^2(G, X)$. These elements are special only for $X = M_1$, M_4 or $M_9(0)$ (see [8, Proposition 2.7]).

Theorem 3.2 *Let $A \in \mathrm{SL}(n, \mathbb{R})$ be of order 4, satisfying $(n - \mathrm{Trace}(A^2))/2 \equiv 2 \pmod 4$. Put $G = \langle A \rangle$, and let δ and Γ be as in (3). Choose an embedding $\rho : \Gamma \to E(n)$.*

Write $\Lambda = \oplus_{i=1}^m \Lambda_i$ with indecomposable $\mathbb{Z}G$-lattices Λ_i, $i = 1, \ldots, m$. Decompose δ accordingly as $\delta = \sum_{i=1}^m \delta_i$ with $\delta_i \in H^1(G, \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_i/\Lambda_i)$.

Then (ρ, Γ) has a spin structure if and only if, for some $1 \leq i \leq m$, (Λ_i, δ_i) is equivalent to $(M_9(0), (1/4, 0, 0, 0)^t)$.

Proof. By Theorem 3.1(4) we have to investigate when $\bar{\delta}'$ lies in the radical of $\bar{\Lambda}$ (viewed as an $\mathbb{F}_2G$-module). The radical of a direct sum of modules is the direct sum of their radicals. Hence $\bar{\delta}'$ lies in the radical of $\bar{\Lambda}$, if and only if $\bar{\delta}'_i$ lies in the radical of $\bar{\Lambda}_i := \Lambda_i/\Lambda_i^2$ for all $1 \leq i \leq m$.

Note that $\delta'_i \in \Lambda_i$ is a fixed vector of A. Hence if there are no nonzero fixed vectors in Λ_i, we have $\bar{\delta}'_i = 0$ and hence it lies in the radical of $\bar{\Lambda}_i$. This is the case for the lattices M_2, M_3 and $M_8(0)$.

Table 1: Indecomposable integral representations of $\mathbb{Z}_4$

Name	Matrix	Cocycle
M_1	(1)	$(\frac{1}{4})$
M_2	(-1)	
M_3	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	
M_4	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} \\ 0 \end{pmatrix}$
M_5	$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{4} \\ 0 \\ 0 \end{pmatrix}$
$M_6(0)$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	
$M_6(1)$	$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{2} \\ 0 \\ 0 \\ 0 \end{pmatrix}$
$M_8(0)$	$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$	
$M_9(0)$	$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} \frac{1}{4} \\ 0 \\ 0 \\ 0 \end{pmatrix}$

Next suppose that $\bar{\Lambda}_i$ has a unique nonzero fixed vector v_i. Then v_i lies in the radical of $\bar{\Lambda}_i$. Moreover, either $\bar{\delta}_i' = 0$ or $\bar{\delta}_i' = v_i$. This is the case for the lattices M_1, M_4, M_5, and $M_6(0)$.

Suppose next that Λ_i is equivalent to $M_6(1)$. In this case, $\delta' = (1,1,0,0)^t$. Observe that $\bar{\Lambda}_i = U_1 \oplus U_2$ as $\mathbb{F}_2G$-modules with

$$U_1 = \langle (0,0,1,0)^t, (0,0,0,1)^t, (1,1,1,0^t \rangle \quad \text{and} \quad U_2 = \langle (1,0,1,1)^t \rangle.$$

Hence the radical of $\bar{\Lambda}_i$ equals the radical of U_1 which is spanned by the fixed vector $(1,1,0,0)^t$. Thus, again, either $\bar{\delta}_i' = 0$ or lies in the radical of $\bar{\Lambda}_i$.

It remains to consider the case that Λ_i is equivalent to $M_9(0)$. Here, as in the $M_6(1)$ case, $\bar{\delta}_i' \in \langle (1,1,0,0)^t \rangle$. But contrary to that case, $\bar{\Lambda}_i = \langle (1,1,0,0)^t \rangle \oplus \langle (0,0,1,0)^t, (0,0,0,1)^t, (1,1,1,0^t \rangle$ as $\mathbb{F}_2G$-modules. Hence if $\bar{\delta}_i' \neq 0$, it does not lie in the radical of $\bar{\Lambda}_i$. This completes the proof. $\square$

Example 3.3 (a) Let

$$A := \left(\begin{array}{cc|cccc} 0 & 1 & & & & \\ 1 & 0 & & & & \\ \hline & & 1 & 0 & 0 & 1 \\ & & 0 & 1 & 0 & 1 \\ & & 0 & 0 & 0 & -1 \\ & & 0 & 0 & 1 & 0 \end{array}\right) \in \mathrm{SL}(6, \mathbb{Z}).$$

Then A has order 4 and $\left(6 - \mathrm{Trace}(A^2)\right)/2 = 2$. (Observe that $\mathbb{Z}^6$ is a direct sum of the $\mathbb{Z}\langle A \rangle$-modules M_4 and $M_9(0)$ of Table 3.2.

(1) Let $\delta := (1/2, 0, 0, 0, 0, 0)^t \in \mathbb{Q}^6$. Then δ gives rise to a special cocycle, and we let Γ be defined as in (3). By Theorem 3.2, there is no spin structure on Γ.

(2) Now let $\delta := (0, 0, 1/4, 0, 0, 0)^t \in \mathbb{Q}^6$. Then, again, δ gives rise to a special cocycle, and we let Γ be defined as in (3). By Theorem 3.2, Γ has a spin structure.

This yields another example of two Bieberbach groups with the same holonomy representation, one with and one without spin structure. The first example appears in [11, Table 1, p. 327].

(b) A 5-dimensional $\mathbb{Z}_4$-manifold without a spin structure is provided by the module $M_2 \oplus M_3 \oplus M_4$ and the special cocyle arising from M_4. In this case we may take

$$A := \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \in \mathrm{SL}(5, \mathbb{Z}),$$

and $\delta = (0, 0, 0, 1/2, 0)^t$.

Let G be a finite group. As in [2, Definition 2], we let $s(G)$ denote the minimal dimension of a flat oriented spin manifold with holonomy group G.

Corollary 3.4 *Let m be a positive integer. Then $s(\mathbb{Z}_{2^m}) = 2^{m-1} + 1$ if $m > 1$, and $s(\mathbb{Z}_2) = 3$.*

Proof. The smallest degree of a flat oriented $\mathbb{Z}_2$-manifold equals 3, so that the result for $m = 1$ follows from Theorem 3.1.

Assume that $m > 1$. Let r be minimal such that there is an $(r \times r)$-matrix B over $\mathbb{Z}$ of order 2^m. Then $r = 2^{m-1}$ and there is no Bieberbach group of dimension r with holonomy group isomorphic to $\langle B \rangle$ (see [7]). The minimal polynomial of B is the 2^mth cyclotomic polynomial $X^{2^{m-1}} + 1$, and thus B has determiant 1.

Now put $n := r + 1$ and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix},$$

and let $\delta \in \mathbb{Q}^n$ have entry $1/2^{m-1}$ in the first coordinate, and zeroes elsewhere. Then δ gives rise to a special cocycle and we define Γ as in (3). In this case δ' is the first standard basis vector. Clearly, $\bar{\delta}'$ is not contained in the radical of the $\mathbb{F}_2G$-module $\mathbb{Z}_2^n$, so that Γ has a spin structure by Theorem 3.1. $\square$

4 Flat oriented manifolds with holonomy $\mathbb{Z}_2^2$

In [12] Tirao has given a classification of all oriented Bieberbach groups with holonomy groups isomorphic to the Klein four group and first Betti number equal to 0. We use this classification to show that all the corresponding flat oriented manifolds have spin structures. We begin with a more general result.

Theorem 4.1 *Let Γ be an oriented Bieberbach group with translation subgroup Λ and holonomy group $G = \mathbb{Z}_2^2$. Suppose that there is a decomposition of Λ into a direct sum of Γ-invariant sublattices*

$$\Lambda = \Lambda_1 \oplus \Lambda_2$$

with $\mathrm{rk}(\Lambda_1) = 3$ and $\mathrm{rk}(\Lambda_2) = n - 3$, such that Γ/Λ_2 is the Hantzsche-Wendt Bieberbach group (the fundamental group of the flat oriented 3-dimensional manifold with non-cyclic holonomy). Then M has a spin structure.

Proof. Let $G = \langle a_1, a_2 \rangle$ with $a_1^2 = a_2^2 = a_3^2 = 1$, where $a_3 = a_1 a_2$. Let us write ρ' and ρ'_1 for the homomorphism of G induced by the conjugation actions of Γ on Λ and Λ_1, respectively. Since Γ/Λ_2 is the Hantzsche-Wendt Bieberbach group, we may and will assume that $\Lambda_1 = \mathbb{Z}^3$, and that

$$\rho'_1(a_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \text{and} \qquad \rho'_1(a_2) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let $\hat{\delta} \in H^1(G, \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda/\Lambda)$ be the cocycle defining the extension

$$0 \to \Lambda \to \Gamma \to G \to 1.$$

Then $\hat{\delta} = \hat{\delta}_1 + \hat{\delta}_2$ with $\hat{\delta}_i \in H^1(G, \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_i/\Lambda_i)$, $i = 1, 2$. More specifically, we may assume (see [12, Definition on p. 229]) that

$$\hat{\delta}_1(a_1) = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix} + \Lambda_1, \quad \hat{\delta}_1(a_2) = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix} + \Lambda_1, \quad \hat{\delta}_1(a_3) = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} + \Lambda_1.$$

Now let $B_i := \rho'(a_i)$, and let $u_i \in \mathrm{Spin}(n)$ be such that $\lambda(u_i) = B_i$, $i = 1, 2, 3$. Then $u_3 = \pm u_1 u_2$. Also, $u_i^2 = \pm 1$, $i = 1, 2, 3$. Let $\gamma_i := (B_i, \delta(a_i))$, $i = 1, 2, 3$. Then $\gamma_i^2 = (\mathrm{id}, e_i)$, $i = 1, 2, 3$, where $e_i := e_i' + 0$ with the standard basis vector e_i' of $\Lambda_1 = \mathbb{Z}^3$, $i = 1, 2, 3$.

Thus there exist a G-invariant sublattice $\Lambda_0 \leq \Lambda$ with $\Lambda_2 \leq \Lambda_0$, $|\Lambda/\Lambda_0| = 2$ such that

$$|\bar{\gamma}_i^2| = |u_i^2|, \qquad i = 1, 2, 3.$$

Here, $^- : \Gamma \to \Gamma/\Lambda_0$ denotes the natural homomorphism. This implies that there is an isomorphism $\bar{\varepsilon}' : \Gamma/\Lambda_0 \to \lambda^{-1}(\rho(G))$ mapping $\bar{\gamma}_i$ to u_i, $i = 1, 2, 3$. Hence there is an epimorphism $\varepsilon' : \Gamma \to \lambda^{-1}(\rho(G))$ mapping γ_i to u_i, $i = 1, 2, 3$. Thus if ε denotes the composition of ε' with the embedding of $\lambda^{-1}(\rho(G))$ into $\mathrm{Spin}(n)$, we have $\lambda \circ \varepsilon = \rho$. $\square$

Corollary 4.2 *Let M be a flat oriented manifold with first Betti number equal to 0 and holonomy group $G = \mathbb{Z}_2^2$. Then M has a spin structure.*

Proof. Let Γ denote the fundamental group of M. By [12, Theorems 2.7, 4.5], there is a decomposition of translation subgroup Λ of Γ as in Theorem 4.1 and the result follows. $\square$

Remark 4.3 By varying the lattice Λ in the above corollary, we get all possible 2-fold coverings of $G = \mathbb{Z}_2^2$ as inverse images $\lambda^{-1}(\rho(G))$. By [12, Theorem 2.7], there is a decomposition of Λ into a direct sum of Γ-invariant sublattices

$$\Lambda = m_1 H_1 \oplus m_2 H_2 \oplus m_3 H_3 \oplus k_1 R_1 \oplus k_2 R_2 \oplus k_3 R_3 \oplus sM \oplus tN, \qquad (4)$$

with non-negative integers $m_i \geq 1$, k_i, s, and t. Here, H_i and R_i are of rank 1 and 2, respectively, $i = 1, 2, 3$, and M and N of rank 3. With the notation of the proof of Theorem 4.1 we have

$$\Lambda_1 = H_1 \oplus H_2 \oplus H_3$$

and

$$\Lambda_2 = (m_1 - 1)H_1 \oplus (m_2 - 1)H_2 \oplus (m_3 - 1)H_3 \oplus k_1 R_1 \oplus k_2 R_2 \oplus k_3 R_3 \oplus sM \oplus tN.$$

Using the fact that $|u_i| = 4$ if and only if $(n - \mathrm{Trace}(B_i))/2 \equiv 2 \pmod 4$, we get the following table.

Case	m_1	m_2	m_3	k_1	k_2	k_3	s	t	$\rho'(a_i)$	$\lvert u_i\rvert$
									B_1	4
I	1	1	1	0	0	0	0	0	B_2	4
									B_3	4
									B_1	2
II	2	2	1	0	0	1	0	0	B_2	2
									B_3	4
									B_1	4
III	2	2	1	0	0	1	1	0	B_2	4
									B_3	2
									B_1	2
IV	1	2	1	0	1	0	0	0	B_2	2
									B_3	2

In Case I, which just gives the usual Hantzsche-Wendt manifold, we have $\lambda^{-1}(\rho(G)) \cong Q_8$, the quaternion group of order 8. In Case II we obtain $\lambda^{-1}(\rho(G)) \cong D_8$, the dihedral group of order 8. Case III yields $\lambda^{-1}(\rho(G)) \cong \mathbb{Z}_4 \times \mathbb{Z}_2$. Finally, in Case IV we have $\lambda^{-1}(\rho(G)) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Example 4.4 (a) Example 2.5 already shows that we cannot drop the assumption of the first Betti number being equal to 0. Here we shall give an example of a Bieberbach group with holonomy $\mathbb{Z}_2^2$ of dimension 5 without spin structure and with the first Betti number equal to 1.

Let Γ be the Bieberbach group generated by $\mathbb{Z}^5$ and the following two elements of $E(5)$:

$$\gamma_1 := ([1, -1, -1, -1, -1], (1/2, 0, 0, 0, 0)^t)$$

and

$$\gamma_2 := ([1, 1, 1, -1, -1], (0, 1/2, 0, 0, 0)^t).$$

Here, $\gamma_1^2 = (\gamma_1\gamma_2)^2$. Using Lemma 2.3 we can show as in Example 2.5, that the manifold corresponding to Γ does not have a spin structure.

(b) Im and Kim have constructed (see [9, Theorem on p. 270]), for every integer $k \geq 3$, a flat oriented manifold with holonomy group $\mathbb{Z}_2^k$, first Betti number equal to 0, without a spin structure. It follows that the case of the Klein four group is different.

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Although it is not explicitly visible in the final version of our paper, the possibility of quickly performing numerous experiments with GAP [5] was of great help for us.

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