Flat manifolds with holonomy group $\mathbb{Z}_2^k$ of diagonal type

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1 Introduction

Let M^n be a flat manifold of dimension n. By definition, this is a compact connected, Riemannian manifold without boundary with sectional curvature equal to zero. From the theorems of Bieberbach ([2]) the fundamental group $\pi_1(M^n) = \Gamma$ determines a short exact sequence:

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} G \to 0, \tag{1}$$

where $\mathbb{Z}^n$ is a torsion free abelian group of rank n and G is a finite group which is isomorphic to the holonomy group of M^n. The universal covering of M^n is the Euclidean space $\mathbb{R}^n$ and hence Γ is isomorphic to a discrete cocompact subgroup of the isometry group $Isom(\mathbb{R}^n) = O(n) \times \mathbb{R}^n = E(n)$. Conversely, given a short exact sequence of the form (1), it is known that the group Γ is (isomorphic to) the fundamental group of a flat manifold if and only if Γ is torsion free. In this case Γ is called a Bieberbach group. We can define a holonomy representation $\phi : G \to GL(n, \mathbb{Z})$ by the formula:

$$\forall g \in G, \phi(g)(e_i) = \tilde{g} e_i (\tilde{g})^{-1}, \tag{2}$$

where $e_i \in \Gamma$ are generators of $\mathbb{Z}^n$ for $i = 1, 2, ..., n$, and $\tilde{g} \in \Gamma$ such that $p(\tilde{g}) = g$. In this article we shall consider only the case

$$G = \mathbb{Z}_2^k, 1 \leq k \leq n-1, \text{ with } \phi(\mathbb{Z}_2^k) \subset D \subset GL(n, \mathbb{Z}), \tag{3}$$

where D is the group of all diagonal matrices. We want to consider relations between two families of flat manifolds with the above property (3): the family $\mathcal{RBM}$ of real Bott manifolds and the family $\mathcal{GHW}$ of generalized Hantzsche-Wendt manifolds. In particular, we shall prove (Proposition 1) that the intersection $\mathcal{GHW} \cap \mathcal{RBM}$ is not empty.

In the next section we consider some class of real Bott manifolds without Spin and $\mathrm{Spin}^{\mathbb{C}}$ structure. There are given conditions (Theorem 1) for the existence of such structures. As an application a list of all 5-dimensional oriented real Bott manifolds without Spin structure is given , see Example 2. In this case we generalize the results of L. Auslaneder and R. H. Szczarba, [1] from 1962, cf. Remark 1. At the end we formulate a question about cohomological rigidity of $\mathcal{GHW}$ manifolds.

2 Families

2.1 Generalized Hantzsche-Wendt manifolds

We start with the definition of generalized Hantzsche-Wendt manifold.

Definition 1 ([16, Definition]) *A generalized Hantzsche-Wendt manifold (for short $\mathcal{GHW}$-manifold) is a flat manifold of dimension n with holonomy group $(\mathbb{Z}_2)^{n-1}$.*

Let $M^n \in \mathcal{GHW}$. In [16, Theorem 3.1] it is proved that the holonomy representation (2) of $\pi_1(M^n)$ satisfies (3).

The simple and unique example of an oriented 3-dimensional generalized Hantzsche-Wendt manifold is a flat manifold which was considered for the first time by W. Hantzsche and H. Wendt in 1934, [8].

Let $M^n \in \mathcal{GHW}$ be an oriented, n-dimensional manifold (a HW-manifold for short). In 1982, see [16], the second author proved that for odd $n \geq 3$ and for all i, $H^i(M^n, \mathbb{Q}) \simeq H^i(\mathbb{S}^n, \mathbb{Q})$, where $\mathbb{Q}$ are the rational numbers and $\mathbb{S}^n$ denotes the n-dimensional sphere. Moreover, for $n \geq 5$ the commutator subgroup of the fundamental group $\pi_1(M^n) = \Gamma$ is equal to the translation subgroup ($[\Gamma, \Gamma] = \Gamma \cap \mathbb{R}^n$), [15]. The number $\Phi(n)$ of affine non equivalent HW-manifolds of dimension n growths exponetially, see [13, Theorem 2.8], and for $m \geq 7$ there exist many isospectral manifolds non pairwise homeomorphic, [13, Corollary 3.6]. The manifolds have an interesting connection with Fibonacci groups [17] and the theory of quadratic forms over the field $\mathbb{F}_2$, [18]. HW-manifolds have no Spin-structure, [12, Example 4.6 on page 4593].

The (co)homology groups and cohomology rings with coefficients in $\mathbb{Z}$ or $\mathbb{Z}_2$, of generalized Hantzsche-Wendt manifolds are still not known, see [4] and [5]. We finish this overview with an example of generalized Hantzsche-Wendt manifolds which have been known already in 1974.

Example 1 *Let* $M^n = \mathbb{R}^n/\Gamma_n, n \geq 2$ *be manifolds defined in* [11] (*see also* [16, page 1059]), *where* $\Gamma_n \subset E(n)$ *is generated by* $\gamma_0 = (I = id, (1, 0, ..., 0))$ *and*

$$\gamma_i = \left(\begin{bmatrix} 1 & 0 & 0 & . & . & ... & 0 \\ 0 & 1 & 0 & . & . & ... & 0 \\ . & . & . & . & . & ... & \\ 0 & ... & 1 & 0 & 0 & ... & 0 \\ 0 & ... & 0 & -1 & 0 & ... & 0 \\ 0 & ... & 0 & 0 & 1 & ... & 0 \\ . & . & . & . & . & ... & \\ 0 & . & ... & 0 & 0 & 0 & 1 \end{bmatrix}, \begin{pmatrix} 0 \\ 0 \\ ... \\ 0 \\ 0 \\ \frac{1}{2} \\ ... \\ 0 \end{pmatrix} \right) \in E(n), \tag{4}$$

where the -1 *is placed in the* (i, i) *entry and the* $\frac{1}{2}$ *as an* $(i + 1)$ *entry,* $i = 1, 2, ..., n - 1$. Γ_2 *is the fundamental group of the Klein bottle.*

2.2 Real Bott manifolds

We follow [3], [10] and [14]. To define the second family let us introduce a sequence of $\mathbb{R}P^1$-bundles

$$M_n \overset{\mathbb{R}P^1}{\to} M_{n-1} \overset{\mathbb{R}P^1}{\to} ... \overset{\mathbb{R}P^1}{\to} M_1 \overset{\mathbb{R}P^1}{\to} M_0 = \{\text{a point}\} \tag{5}$$

such that $M_i \to M_{i-1}$ for $i = 1, 2, ..., n$ is the projective bundle of a Whitney sum of a real line bundle L_{i-1} and the trivial line bundle over M_{i-1}. We call the sequence (5) a *real Bott tower* of height n, [3].

Definition 2 ([10]) *The top manifold* M_n *of a real Bott tower* (5) *is called a real Bott manifold.*

Let γ_i be the canonical line bundle over M_i and set $x_i = w_1(\gamma_i)$. Since $H^1(M_{i-1}, \mathbb{Z}_2)$ is additively generated by $x_1, x_2, .., x_{i-1}$ and L_{i-1} is a line bundle over M_{i-1}, one can uniquely write

$$w_1(L_{i-1}) = \Sigma_{k=1}^{i-1} a_{k,i} x_k \tag{6}$$

with $a_{k,i} \in \mathbb{Z}_2 = \{0, 1\}$ and $i = 2, 3, ..., n$.

From above $A = [a_{ki}]$ is an upper triangular matrix [1] of size n whose diagonal entries are 0 and other entries are either 0 or 1. Summing up, we can say that the tower (5) is completly determined by the matrix A.

[1] $a_{k,i} = 0$ unless $k < i$.

From [10, Lemma 3.1] we can consider any real Bott manifold $M(A)$ in the following way. Let $M(A) = \mathbb{R}^n/\Gamma(A)$, where $\Gamma(A) \subset E(n)$ is generated by elements

$$s_i = \left(\begin{bmatrix} 1 & 0 & 0 & . & . & ... & 0 \\ 0 & 1 & 0 & . & . & ... & 0 \\ . & . & . & . & . & ... & \\ 0 & ... & 0 & 1 & 0 & ... & 0 \\ 0 & ... & 0 & 0 & (-1)^{a_{i,i+1}} & ... & 0 \\ . & . & . & . & . & ... & \\ 0 & ... & 0 & 0 & 0 & ... & (-1)^{a_{i,n}} \end{bmatrix}, \begin{pmatrix} 0 \\ . \\ 0 \\ \frac{1}{2} \\ 0 \\ . \\ 0 \\ 0 \end{pmatrix} \right) \in E(n), \tag{7}$$

where $(-1)^{a_{i,i+1}}$ is placed in $(i+1, i+1)$ entry and $\frac{1}{2}$ as an (i) entry, $i = 1, 2, ..., n-1$. $s_n = (I, (0, 0, ..., 0, 1)) \in E(n)$. From [10, Lemma 3.2,3.3] $s_1^2, s_2^2, ..., s_n^2$ commute with each other and generate a free abelian subgroup $\mathbb{Z}^n$. It is easy to see that it is not always a maximal abelian subgroup of $\Gamma(A)$. Moreover, we have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r] & \Gamma(A) \arrow[r] & \mathbb{Z}_2^k \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{Z}^n \arrow[r] \arrow[u, "i"] & \Gamma(A) \arrow[r] \arrow[u, equal] & \mathbb{Z}_2^n \arrow[r] \arrow[u, "p"] & 0 \end{tikzcd}$$

where $k = rk_{\mathbb{Z}_2}(A), N$ is the maximal abelian subgroup of $\Gamma(A)$, and $p : \Gamma(A)/\mathbb{Z}^n \to \Gamma(A)/N$ is a surjection induced by the inclusion $i : \mathbb{Z}^n \to N$. From the first Bieberbach theorem, see [2], N is a subgroup of all translations of $\Gamma(A)$ i.e. $N = \Gamma(A) \cap \mathbb{R}^n = \Gamma(A) \cap \{(I, a) \in E(n) \mid a \in \mathbb{R}^n\}$.

Definition 3 ([3]) *A binary square matrix A is a Bott matrix if $A = PBP^{-1}$ for a permutation matrix P and a strictly upper triangular binary matrix B.*

Let $\mathcal{B}(n)$ be the set of Bott matrices of size n. [2] Since two different upper triangular matrices A and B may produce (affinely) diffeomorphic ($\sim$) real Bott manifolds $M(A), M(B)$, see [3] and [10], there are three operations on $\mathcal{B}(n)$, denoted by (Op1), (Op2) and (Op3), such that $M(A) \sim M(B)$ if and only if the matrix A can be transformed into B through a sequence of the above operations, see [3, part 3]. The operation (Op1) is a conjugation by a permutation matrix,

[2]Sometimes $\mathcal{B}(n)$ is defined to be the set of strictly upper triangular binary matrices of size n.

(Op2) is a bijection $\Phi_k : \mathcal{B}(n) \to \mathcal{B}(n)$

$$\Phi_k(A)_{*,j} := A_{*,j} + a_{kj} A_{*,k}, \tag{8}$$

for $k, j \in \{1, 2, ..., n\}$ such that $\Phi_k \circ \Phi_k = 1_{\mathcal{B}(n)}$.
Finally (Op3) is, for distinct $l, m \in \{1, 2, .., n\}$ and the matrix A with $A_{*,l} = A_{*,m}$

$$\Phi^{l,m}(A)_{i,*} := \begin{cases} A_{l,*} + A_{m,*} & \text{if } i = m \\ A_{i,*} & \text{otherwise} \end{cases} \tag{9}$$

Here $A_{*,j}$ denotes j-th column and $A_{i,*}$ denotes i-th row of the matrix A.

Let us start to consider the relations between these two classes of flat manifolds. We start with an easy observation

$$\mathcal{RBM}(n) \cap \mathcal{GHW}(n) = \{M(A) \mid rank_{\mathbb{Z}_2} A = n - 1\} =$$

$$= \{M(A) \mid a_{1,2} a_{2,3} ... a_{n-1,n} = 1\}.$$

These manifolds are classified in [3, Example 3.2] and for $n \geq 2$

$$\#(\mathcal{RBM}(n) \cap \mathcal{GHW}(n)) = 2^{(n-2)(n-3)/2}. \tag{10}$$

There exists the classification, see [16] and [3], of diffeomorphism classes of $\mathcal{GHW}$ and $\mathcal{RBM}$ manifolds in low dimensions. For dim ≤ 6 we have the following table.

dim	number of GHW manifolds		number of RBM manifolds		number of $GHW \cap RBM$ manifolds
	total	oriented	total	oriented	total
1	0	0	1	1	0
2	1	0	2	1	0
3	3	1	4	2	1
4	12	0	12	3	2
5	123	2	54	8	8
6	2536	0	472	29	64

Proposition 1 $\Gamma_n \in \mathcal{GHW} \cap \mathcal{RBM}$.

Proof: It is enough to see that the group $(G, 0)\Gamma_n(G, 0)^{-1} = \Gamma(A)$, where $G = [g_{ij}], 1 \leq i, j \leq n$,

$$g_{ij} := \begin{cases} 1 & \text{if } j = n - i + 1 \\ 0 & \text{otherwise} \end{cases}$$

and $A = [a_{ij}], 1 \leq i, j \leq n$, with

$$a_{ij} := \begin{cases} 1 & \text{if } j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

□

3 Existence of Spin and $\mathrm{Spin}^{\mathbb{C}}$ structures on real Bott manifolds

In this section we shall give some condition for the existence of Spin and $\mathrm{Spin}^{\mathbb{C}}$ structures on real Bott manifolds. We use notations from the previous sections. There are a few ways to decide whether there exists a Spin structure on an oriented flat manifold M^n, see [6]. We start with the following. A closed oriented differential manifold N has such a structure if and only if the second Stiefel-Whitney class $w_2(N) = 0$. In the case of an oriented real Bott manifold $M(A)$ we have the formula for w_2.
Recall, see [10], that for the Bott matrix A

$$H^*(M(A); \mathbb{Z}_2) = \mathbb{Z}_2[x_1, x_2, ..., x_n]/(x_j^2 = x_j \Sigma_{i=1}^n a_{i,j} x_i \mid j = 1, 2, ..., n) \quad (11)$$

as graded rings. Moreover, from [11, (3.1) on page 3] the k-th Stiefel-Whitney class

$$w_k(M(A)) = (B(p))^* \sigma_k(y_1, y_2, ..., y_n) \in H^k(M(A); \mathbb{Z}_2), \quad (12)$$

where σ_k is the k-th elementary symmetric function,

$$p : \pi_1(M(A)) \to G \subset O(n)$$

a holonomy representation, $B(p)$ is a map induced by p on the classification spaces and $y_i \overset{(6)}{=} w_1(L_{i-1})$. Hence,

$$w_2(M(A)) = \Sigma_{1 \leq i < j \leq n} y_i y_j \in H^2(M(A); \mathbb{Z}_2). \quad (13)$$

There exists a general condition, see [4, Theorem 3.3], for the calculation of the second Stiefel-Whitney for flat manifolds with $(\mathbb{Z}_2)^k$ holonomy of diagonal type but we prefer the above explicit formula (13). [3] Its advantage follows from the knowledge of the cohomology ring (11) of real Bott manifolds.

[3] We use it in Example 2.

An equivalent condition for the existence of a Spin structure is as follows. An oriented flat manifold M^n (a Bieberbach group $\pi_1(M^n) = \Gamma$) has a Spin structure if and only if there exists a homomorfism $\epsilon : \Gamma \to \mathrm{Spin}(n)$ such that $\lambda_n \epsilon = p$. Here $\lambda_n : \mathrm{Spin}(n) \to SO(n)$ is the covering map, see [6]. We have a similar condition, under assumption $H^2(M^n, \mathbb{R}) = 0$, for the existence of $\mathrm{Spin}^{\mathbb{C}}$ structure, [6, Theorem 1]. In this case M^n (a Bieberbach group Γ) has a $\mathrm{Spin}^{\mathbb{C}}$ structure if an only if there exists a homomorphism

$$\bar{\epsilon} : \Gamma \to \mathrm{Spin}^{\mathbb{C}}(n) \tag{14}$$

such that $\bar{\lambda_n}\bar{\epsilon} = p$. $\bar{\lambda_n} : \mathrm{Spin}^{\mathbb{C}}(n) \to SO(n)$ is the homomorphism induced by λ_n, see [6]. We have the following easy observation. If there existe $H \subset \Gamma$, a subgroup of finite index, such that the finite covering $\tilde{M^n}$ with $\pi_1(\tilde{M^n}) = H$ has no Spin ($\mathrm{Spin}^{\mathbb{C}}$) structure, then M^n has also no such structure.

We shall prove.

Theorem 1 *Let A be a matrix of an orientable real Bott manifold $M(A)$ of dimension n.*
I. *Let $l \in \mathbb{N}$ be an odd number. If there exist $1 \leq i < j \leq n$ and rows $A_{i,*}, A_{j,*}$ such that*

$$\#\{m \mid a_{i,m} = a_{j,m} = 1\} = l \tag{15}$$

and

$$a_{i,j} = 0, \tag{16}$$

then $M(A)$ has no Spin *structure.*
Moreover, if

$$\#\{J \subset \{1, 2, ..., n\} \mid \#J = 2, \Sigma_{j \in J} A_{*,j} = 0\} = 0, \tag{17}$$

then $M(A)$ has no $\mathrm{Spin}^{\mathbb{C}}$ *structure.*

II. *If there exist $1 \leq i < j \leq n$ and rows*

$$A_{i,*} = (0, ..., 0, a_{i,i_1}, ..., a_{i,i_{2k}}, 0, ..., 0),$$

$$A_{j,*} = (0, ..., 0, a_{j,i_{2k}+1}, ..., a_{j,i_{2k}+2l}, 0, ..., 0)$$

such that $a_{i,i_1} = a_{i,i_2} = ... = a_{i,i_{2k}} = 1, a_{i,m} = 0$ for $m \notin \{i_1, i_2, ..., i_{2k}\}$ $a_{j,i_{2k}+1} = a_{j,i_{2k}+2} = ... = a_{j,i_{2k}+2l} = 1, a_{j,r} = 0$ for $r \notin \{i_{2k}+1, i_{2k}+2, ..., i_{2k}+2l\}$ and l, k odd then $M(A)$ has no Spin *structure.*

Proof: From [10, Lemma 2.1] the manifold $M(A)$ is orientable if and only if for any $i = 1, 2, .., n$,

$$\Sigma_{k=i+1}^{n} a_{i,k} = 0 \mod 2.$$

Assume that $\epsilon : \pi_1(M(A)) \to \mathrm{Spin}(n)$ defines a Spin structure on $M(A)$. Let $a_{i,i_1}, a_{i,i_2}, ..., a_{i,i_{2m}}, a_{j,j_1}, a_{j,j_2}, ..., a_{j,j_{2p}} = 1$ and let s_i, s_j be elements of $\pi_1(M(A))$ which define rows i, j of A, see (7). Then

$$\epsilon(s_i) = \pm e_{i_1} e_{i_2} ... e_{i_{2m}},$$

$$\epsilon(s_j) = \pm e_{j_1} e_{j_2} ... e_{j_{2p}}$$

and

$$\epsilon(s_i s_j) = \pm e_{k_1} e_{k_2} ... e_{k_{2r}}.$$

From (15) $2r = 2m + 2p - 2l$. Moreover $\epsilon(s_i^2) = (-1)^m, \epsilon(s_j^2) = (-1)^p$ and $\epsilon((s_i s_j)^2) = (-1)^{m+p-l} = (-1)^{m+p+l}$. Since from (16) (see also [10, Lemma 3.2]) $s_i s_j = s_j s_i$ we have $\epsilon((s_i)^2)\epsilon((s_j)^2) = \epsilon((s_i s_j)^2)$. Hence

$$(-1)^{m+p} = (-1)^{m+p+l}.$$

This is impossible since l is an odd number and we have a contradiction.

For the existence of the $\mathrm{Spin}^{\mathbb{C}}$ structure it is enough to observe that the condition (17) is equivalent to equation $H^2(M(A), \mathbb{R}) = 0$, see [3, formula (8.1)]. Hence, we can apply the formula (14). Let us assume that $\bar{\epsilon} : \pi_1(M(A) \to \mathrm{Spin}^{\mathbb{C}}(n)$ defines a $\mathrm{Spin}^{\mathbb{C}}$ structure. Using the same arguments as above, see [6, Proposition 1], we obtain a contradiction. This finished the proof of **I.**

For the proof **II** let us observe that $s_i^2 = (s_i s_j)^2$. Hence $(-1)^k = \epsilon((s_i)^2 = \epsilon((s_i s_j)^2)) = (-1)^{k+l} = 1$. This is impossible.

$\square$

In the above theorem rows of number i and j correspond to generators s_i, s_j which define a finite index subgroup $H \subset \pi_1(M(A))$. It is a Bieberbach group with holonomy group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. We proved that H (if it exists) has no Spin ($\mathrm{Spin}^{\mathbb{C}}$) structure, (see the discussion before Theorem 1). In the next example we give the list of all 5-dimensional real Bott manifolds (with) without Spin($\mathrm{Spin}^{\mathbb{C}}$) structure.

Example 2 *From* [14] *we have the list of all 5-dimensional oriented real Bott manifolds. There are 7 such manifolds without the torus. Here are* *their matrices:*

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A_{23} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{29} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A_{37} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{40} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, A_{48} = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_{49} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

From the first part of Theorem 1 above, for $i = 1, j = 2$ *the manifold* $M(A_4)$ *has no* Spin$^{\mathbb{C}}$ *structure. For the same reasons (for* $i = 1, j = 2$*) manifolds* $M(A_{40})$ *and* $M(A_{48})$ *have no* Spin *structures. The manifold* $M(A_{23})$ *has no a* Spin *structure, because it satisfies for* $i = 1, j = 3$ *the second part of the* Theorem 1. *Since any flat oriented manifold with* $\mathbb{Z}_2$ *holonomy has* Spin *strucure,* [9, Theorem 3.1] *manifolds* $M(A_{29}), M(A_{49})$ *have it. Our last example, the manifold* $M(A_{37})$ *has* Spin *structure and we leave it as an exercise.*

In all these cases it is possible to calculate the w_2 *with the help of (6), (13) and (11). In fact,* $w_2(M(A_4)) = (x_2)^2 + x_1x_3, w_2(M(A_{23})) = x_1x_3, w_2(M(A_{40})) = w_2(M(A_{48})) = x_1x_2$. *In all other cases* $w_2 = 0$.

Example 3 *Let*

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

be a family of Bott matrices, with $* \in \mathbb{Z}_2$. It is easy to check that the first two rows satisfy the condition of Theorem 1. Hence the oriented real Bott manifolds $M(A)$ have no the Spin *structure.*

Remark 1 *In* [1] *on page 6 an example of the flat (real Bott) manifold M without* Spin *structure is considered. By an immediate calculation the Bott matrix of M is equal to*

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

4 Concluding Remarks

The tower (5) is an analogy of a Bott tower

$$W_n \to W_{n-1} \to \dots \to W_1 = \mathbb{C}P^1 \to W_0 = \{\text{a point}\}$$

where W_i is a $\mathbb{C}P^1$ bundle on W_{i-1} i.e.; $W_i = P(1 \oplus L_{i-1})$ and L_{i-1} is a holomorphic line bundle over W_{i-1}. As in (5) $P(1 \oplus L_{i-1})$ is projectivisation of the trivial linear bundle and L_{i-1}. It was introduced by Grossberg and Karshon [7]. As is well known, see [3] for the complete bibliography, W_n is a toric manifold.

There is an open problem: Is it true that two toric manifolds are diffeomorphic (or homeomorphic) if their cohomology rings with integer coefficients are isomorphic as graded rings ? In some cases it has partial affirmative solutions (see [10]). For real Bott manifolds the following is true.

Theorem ([10, Theorem 1.1]) *Two real Bott manifolds are diffeomorphic if and only if their cohomology rings with $\mathbb{Z}_2$ coefficients are isomorphic as graded rings. Equivalently, they are cohomological rigid.*

All of this suggests the following:

Question Are $\mathcal{GHW}$-manifolds cohomological rigid ?

The answer to the above question is positive for manifolds from $\mathcal{GHW} \cap \mathcal{RBM}$. It looks the most interesting for oriented GHW-manifolds. However, for $n = 5$ there are two oriented Hantzsche-Wendt manifolds. From direct calculations with the help of a computer we know that they have different cohomology rings with $\mathbb{Z}_2$ coefficients.

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