Tangent bundles of Hantzsche-Wendt manifolds

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Abstract: We formulate a condition for an existence of a $\mathrm{Spin}^{\mathbb{C}}$-structure on an oriented flat manifold M^n with $H^2(M^n, \mathbb{R}) = 0$. We prove that M^n has a $\mathrm{Spin}^{\mathbb{C}}$-structure if and only if there exist a homomorphism $\epsilon : \pi_1(M^n) \to \mathrm{Spin}^{\mathbb{C}}(n)$ such $\bar{\lambda}_n \circ \epsilon = h$, where $h : \pi_1(M^n) \to \mathrm{SO}(n)$ is a holonomy homomorphism and $\bar{\lambda}_n : \mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{SO}(n)$ is a standard homomorphism defined on page 2. As an application we shall prove that all cyclic Hantzsche - Wendt manifolds have not the $\mathrm{Spin}^{\mathbb{C}}$-structure.

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1. Introduction

Let M^n be a flat manifold of dimension n. By definition, this is a compact connected, Riemannian manifold without boundary with sectional curvature equal to zero. From the theorems of Bieberbach ([2]) the fundamental group $\pi_1(M^n) = \Gamma$ determines a short exact sequence:

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{h} F \to 0, \tag{1}$$

where $\mathbb{Z}^n$ is a torsion free abelian group of rank n and F is a finite group which is isomorphic to the holonomy group of M^n. The universal covering of M^n is the Euclidean space $\mathbb{R}^n$ and hence Γ is isomorphic to a discrete cocompact subgroup of the isometry group $\mathrm{Isom}(\mathbb{R}^n) = \mathrm{O}(n) \ltimes \mathbb{R}^n = \mathrm{E}(n)$. In the above short exact sequence $\mathbb{Z}^n \cong (\Gamma \cap \mathbb{R}^n)$ and h can be considered as the projection $h : \Gamma \to F \subset \mathrm{O}(n) \subset \mathrm{E}(n)$ on the first component. Conversely, given a short sequence of the form (1), it is known that the group Γ is (isomorphic to) a Bieberbach group if and only if Γ is torsion free.

By Hantzsche-Wendt manifold (for short HW-manifold) M^n we shall understand any oriented flat manifold of dimension n with a holonomy group $(\mathbb{Z}_2)^{n-1}$. It is easy to see that n is always an odd number. Moreover, (see [12] and [17]) HW-manifolds are rational homology spheres and its holonomy representation [1] is diagonal, [16]. Hence $\pi_1(M^n)$ is

[1] That is a representation $\phi_\Gamma : F \to \mathrm{GL}(n, \mathbb{Z})$, given by a formula $\phi_\Gamma(f)(z) = \bar{f} z \bar{f}^{-1}$, where $\bar{f} \in \Gamma, f \in F, z \in \mathbb{Z}^n$ and $p(\bar{f}) = f$.

generated by $\beta_i = (B_i, b_i) \in \mathrm{SO}(n) \ltimes \mathbb{R}^n, 1 \le i \le n$, where

$$B_i = \mathrm{diag}(-1, -1, ..., -1, \underbrace{1}_{i}, -1, -1, ..., -1) \text{ and } b_i \in \{0, 1/2\}^n. \tag{2}$$

Let us recall some other properties of M^n. For $n \ge 5$ the commutator subgroup of the fundamental group is equal to the translation subgroup ($[\Gamma, \Gamma] = \Gamma \cap \mathbb{R}^n$), ([14]). The number $\Phi(n)$ of affinian not equivalent HW-manifolds of dimension n growths exponetially, see [12, Theorem 2.8] and for $m \ge 7$ there exist many pairs of isospectral manifolds all not homeomorphic to each other, [12, Corollary 3.6]. These manifolds have interesting connection with Fibonacci groups [18] and the theory of quadratic forms over a field $\mathbb{F}_2$, [19]. HW-manifolds have not a Spin-structure, [11, Example 4.6 on page 4593]. Hence tangent bundles of HW-manifolds are not trivial. There are still not known their (co)homology groups with coefficients in $\mathbb{Z}$. Here we send reader to [4] where are presented results for low dimensions and an algorithm. Finally, let us mention about properties related to the theory of fixed points. HW-manifolds satisfy so called Anosov relation. This means for any continious map $f : M^n \to M^n, \mid L(f) \mid = N(f)$, where $L(f)$ is the Lefschetz number of f and $N(f)$ is the Nielsen number of f, see [3].

In this note we are interested in properties of the tangent bundle of HW-manifolds. We shall prove that they are line element parallelizable (Proposition 1) and we shall define an infinite family of HW-manifolds without $\mathrm{Spin}^{\mathbb{C}}$-structure (Theorem 2). However, the main result of this article is related to an existence $\mathrm{Spin}^{\mathbb{C}}$-structures on oriented flat manifolds. The group $\mathrm{Spin}^{\mathbb{C}}(n)$ is given by $\mathrm{Spin}^{\mathbb{C}}(n) = (\mathrm{Spin}(n) \times S^1)/\{1, -1\}$ where $\mathrm{Spin}(n) \cap S^1 = \{1, -1\}$. Moreover, there is a homomorphism of groups $\bar{\lambda}_n : \mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{SO}(n)$ given by $\bar{\lambda}_n[g, z] = \lambda_n(g)$, where $g \in \mathrm{Spin}(n), z \in S^1$ and $\lambda_n : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the universal covering. We shall prove:

Theorem 1 *Let M be an oriented flat manifold with $H^2(M, \mathbb{R}) = 0$. M has a* $\mathrm{Spin}^{\mathbb{C}}$*-structure if and only if there exists a homomorphism $\epsilon : \Gamma \to \mathrm{Spin}^{\mathbb{C}}(n)$ such that*

$$\bar{\lambda}_n \circ \epsilon = h. \tag{3}$$

As an application we prove Theorem 2.

Theorem 2 *All cyclic* HW*-manifolds have not the* $\mathrm{Spin}^{\mathbb{C}}$*-structure.*

For a description of cyclic HW-manifolds see Definition 2. We conjecture that all HW-manifolds have not the $\mathrm{Spin}^{\mathbb{C}}$-structure.

2. Hantzsche-Wendt manifolds are line element parallelizable

We keep notations from the introduction. For any discrete group G, we have a universal principal G-bundle with the total space $\mathrm{E}\,G$ and the base space $\mathrm{B}\,G$. $\mathrm{B}\,G$ is called the classifying space of a group G and is unique up to homotopy. In our case $\mathbb{R}^n$ is the total space of a principal Γ-bundle with a base space M^n. Here $\mathrm{E}\,\Gamma = \mathbb{R}^n$ and $\mathrm{B}\,\Gamma = M^n$, see [20, page 369]. Now $G \to \mathrm{B}\,G$ behaves more or less like a functor, and in particular, from the surjection $h : \Gamma \to h(\Gamma) = F$ we can construct a corresponding map $B(h) : \mathrm{B}\,\Gamma \to \mathrm{B}\,F$. Finally, the inclusion $i_n : F \to \mathrm{O}(n)$ yields a map $B(i_n) : \mathrm{B}\,F \to \mathrm{B}(\mathrm{O}(n))$. The universal

n-dimensional vector bundle over $\mathrm{B}(\mathrm{O}(n))$ yields, via this map a vector bundle η_n over $\mathrm{B}\,F$.

Lemma 1. ([20, Proposition 1.1]) *$B(h)^*(\eta_n)$ is equivalent to the tangent bundle of M^n.*

Proof: (See [20, page 369]) We have a commutative diagram as follows

$$\begin{array}{ccc} \mathbb{R}^n = \mathrm{E}\,\Gamma & \xrightarrow{E(h)} & \mathrm{E}\,F \\ \downarrow & & \downarrow \\ M^n = \mathrm{B}\,\Gamma & \xrightarrow{B(h)} & \mathrm{B}\,F \end{array}$$

where $E(h)(g \cdot e) = h(g) \cdot E(h)(e)$ for all $g \in \Gamma$ and $e \in \mathrm{E}\,\Gamma = \mathbb{R}^n$. Let the total space of η_n be $\mathrm{E}\,F \times \mathbb{R}^n/F$ where $f \in F$ acts via $f(e, v) = (f \cdot e, f \cdot v)$. Now clearly the total space τ of the tangent bundle of $M^n = \mathrm{B}\,\Gamma$ can be taken to be $\mathbb{R}^n \times \mathbb{R}^n/\Gamma$ where Γ acts via $g(v_1, v_2) = (gv_1, h(g)v_2)$. Thus we have a commutative diagram as follows:

$$\begin{array}{ccc} \tau = \mathbb{R}^n \times \mathbb{R}^n/\Gamma & \longrightarrow & \mathrm{E}\,F \times \mathbb{R}^n/F \\ \downarrow & & \downarrow \\ M^n = \mathbb{R}^n/\Gamma & \xrightarrow{B(h)} & \mathrm{B}\,F \end{array}$$

where F acts on $\mathrm{E}\,F \times \mathbb{R}^n$ as followings $\{v_1, v_2\} \to \{E(h)(v_1), v_2\}$. This finishes the proof. □

Remark 1. From the above Lemma we can observe that the tangent bundle is flat in sense of [1, page 272].

Let us present the main result of this section.

Proposition 1. *Let M^n be a* HW*-manifold of dimension n. Then its tangent bundle is line element parallelizable,* (*is a sum of line bundles*).

Proof: By definition the fundamental group $\Gamma = \pi_1(M^n)$ is a subgroup of $\mathrm{SO}(n) \ltimes \mathbb{R}^n$ and $h(\Gamma) = (\mathbb{Z}_2)^{n-1} \subset \mathrm{SO}(n)$ is a group of all diagonal orthogonal matrices. It is also an image of the holonomy representation $\phi_\Gamma : (\mathbb{Z}_2)^{n-1} \to \mathrm{SO}(n)$. Let us recall a basic facts about line bundles. It is well known that the classification space for line bundles is $\mathbb{R}P^\infty$, the infinite projective space. Hence any line bundle $\xi : L \to M^n$ is isomorphic to $f^*(\eta_1)$, where

$$f \in [M^n, \mathbb{R}P^\infty] \simeq H^1(M^n, \mathbb{Z}_2) \simeq Hom(\Gamma, \mathbb{Z}_2) \overset{(*)}{\simeq} (\mathbb{Z}_2)^{n-1}$$

is a classification map and $\eta_1 \in H^1(\mathbb{R}P^\infty, \mathbb{Z}_2) = \mathbb{Z}_2$ is not a trivial element. Here η_1 represents the universal line vector bundle and the isomorphism $(*)$ follows from [14, Cor. 3.2.,]. Since $(\mathbb{Z}_2)^{n-1}$ is an abelian group,

$$\phi_\Gamma = \bigoplus_{i=1}^{n} (\phi_\Gamma)_i,$$

where $(\phi_\Gamma)_i : (\mathbb{Z}_2)^{n-1} \to \{\pm 1\}$ are irreducible representations of $(\mathbb{Z}_2)^{n-1}$, for $i = 1, 2, ..., n$. From Lemma 1 and [7, Theorem 8.2.2] the tangent bundle

$$\tau(M^n) = B(h)^*(\eta_n) = \bigoplus_{i=1}^{n} B(h_i)^*(\eta_1),$$

where $h_i = (\phi_\Gamma)_i \circ h$. This finishes the proof.

□

3. Spin$^{\mathbb{C}}$-structure

It is well known (see [11, Example 4.6 on page 4593]) that HW-manifolds have not the Spin-structure. In this section we shall consider the question: Do HW-manifolds have the Spin$^{\mathbb{C}}$-structure ?

On the beginning let us recall some facts about the group Spin$^{\mathbb{C}}$, which was defined in the Introduction. We start with homomorphisms ([5, page 25]):

- $i : \mathrm{Spin}(n) \to \mathrm{Spin}^{\mathbb{C}}(n)$ is the natural inclusion $i(g) = [g, 1]$.
- $j : S^1 \to \mathrm{Spin}^{\mathbb{C}}(n)$ is the natural inclusion, $j(z) = [1, z]$.
- $l : \mathrm{Spin}^{\mathbb{C}}(n) \to S^1$ is given by $l[g, z] = z^2$.
- $p : \mathrm{Spin}^{\mathbb{C}}(n) \to \mathrm{SO}(n) \times S^1$ is given by $p([g, z]) = (\lambda_n(g), z^2)$. Hence $p = \lambda_n \times l$.

Since $S^1 = \mathrm{SO}(2)$, there is a natural map $k : \mathrm{SO}(n) \times \mathrm{SO}(2) \to \mathrm{SO}(n + 2)$. Then we can describe $\mathrm{Spin}^{\mathbb{C}}(n)$ as the pullback by this map of the covering map

$$\begin{array}{ccc} \mathrm{Spin}^{\mathbb{C}}(n) & \longrightarrow & \mathrm{Spin}(n+2) \\ \downarrow & & \downarrow \lambda_n \\ \mathrm{SO}(n) \times \mathrm{SO}(2) & \xrightarrow{k} & \mathrm{SO}(n+2) \end{array}$$

Let W^n be an n-dimensional, compact oriented manifold and let $\delta : W^n \to \mathrm{BSO}(n)$ be the classification map of its tangent bundle TW^n. We now recall the definition of a Spin$^{\mathbb{C}}$-structure ([9, page 34], [5, page 47]).

Definition 1. *A* Spin$^{\mathbb{C}}$*-structure on the manifold* W^n *is a lift of* δ *to* $\mathrm{BSpin}^{\mathbb{C}}(n)$, *giving a commutative diagram*:

$$\begin{array}{ccc} & & \mathrm{BSpin}^{\mathbb{C}}(n) \\ & \nearrow & \downarrow B(\bar{\lambda}_n) \\ W^n & \xrightarrow{\delta} & \mathrm{BSO}(n). \end{array}$$

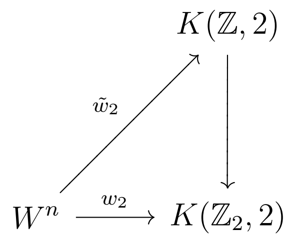
Remark 2.

(1) (See [5, Remark, d on page 49].) W^n has the $\mathrm{Spin}^{\mathbb{C}}$-structure if and only if the Stiefel-Whitney class $w_2 \in H^2(W^n, \mathbb{Z}_2)$ is $\mathbb{Z}_2$-reduction of an integral Stiefel-Whitney class $\tilde{w_2} \in H^2(W^n, \mathbb{Z})$.

(2) Let $K(\mathbb{Z}, 2)$ and $K(\mathbb{Z}_2, 2)$ be the Eilenberg-Maclane spaces. From the homotopy theory

$$H^2(W^n, \mathbb{Z}) = [W^n, BS^1] = [W^n, K(\mathbb{Z}, 2)]$$

and $H^2(W^n, \mathbb{Z}_2) = [W^n, K(\mathbb{Z}_2, 2)]$. Hence the above condition defines a commutative diagram



where the vertical arrow is induced by an epimorfizm $\mathbb{Z} \to \mathbb{Z}_2$.

From previous sections (Lemma 1) an oriented flat manifold $M = \mathrm{B}\,\Gamma$, and $\delta = \mathrm{B}(h)$ where $\Gamma = \pi_1(M)$ and $h : \Gamma \to \mathrm{SO}(n)$ is a holonomy homomorphism. Let us recall (see [15, page 323] and Remark 3) that an oriented manifold M has a Spin-structure if and only if there exists a homomorphism $e : \Gamma \to \mathrm{Spin}(n)$ such that

$$\lambda_n \circ e = h. \tag{4}$$

Hence, a condition of existence of the $\mathrm{Spin}^{\mathbb{C}}$-structure on M is very similar to the above condition (4).

Theorem 1. *Let M be an oriented flat manifold with $H^2(M, \mathbb{R}) = 0$. M has a $\mathrm{Spin}^{\mathbb{C}}$-structure if and only if there exists a homomorphism $\epsilon : \Gamma \to \mathrm{Spin}^{\mathbb{C}}(n)$ such that*

$$\bar{\lambda}_n \circ \epsilon = h. \tag{5}$$

Proof: Let us assume that there exists a homomorphism $\epsilon : \Gamma \to \mathrm{Spin}^{\mathbb{C}}(n)$ such that $\bar{\lambda}_n \epsilon = h$. We claim that conditions of Definition 1 are satisfied. In fact, $B(\bar{\lambda}_n)B(\epsilon) = B(h)$ up to homotopy. To go the other way, let us assume that $M = B\Gamma$ admits a $\mathrm{Spin}^{\mathbb{C}}$-structure. We have a commutative diagram.

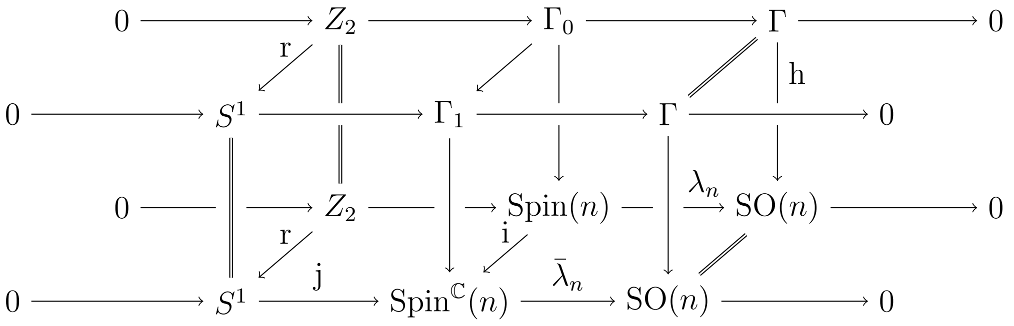


Diagram 1

where Γ_0 is defined by the second Stiefel-Whitney class $w_2 \in H^2(\Gamma, \mathbb{Z}_2)$ and Γ_1 is defined by the element $r_*(w_2) \in H^2(\Gamma, S^1)$. Here $r : \mathbb{Z}_2 \to S^1$ is a group monomorphism. Let $h^2 : H^2(\mathrm{SO}(n), K) \to H^2(\Gamma, K)$ be a homomorphism induced by the holonomy homomorphism h, for $K = \mathbb{Z}_2, S^1$. From definition, (see [10, Chapter 23.6]) there exists an element [2] $x_2 \in H^2(\mathrm{SO}(n), \mathbb{Z}_2) = \mathbb{Z}_2$ such that $h^2(x_2) = w_2$ and $h^2(r_*(x_2)) = r_*(h^2(x_2)) = r_*(w_2)$. Moreover we have two infinite sequences of cohomology which are induced by the following commutative diagram of groups

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z} & \xrightarrow{2} & \mathbb{Z} & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow r & & \\
1 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & S^1 & \longrightarrow & 1
\end{array}$$

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H^2(\Gamma, \mathbb{Z}) & \longrightarrow & H^2(\Gamma, \mathbb{Z}) & \xrightarrow{red} & H^2(\Gamma, \mathbb{Z}_2) & \longrightarrow & H^3(\Gamma, \mathbb{Z}) & \longrightarrow & \cdots \\
 & & \| & & \downarrow & & \downarrow r_* & & \| & & \\
\cdots & \longrightarrow & H^2(\Gamma, \mathbb{Z}) & \longrightarrow & H^2(\Gamma, \mathbb{R}) & \longrightarrow & H^2(\Gamma, S^1) & \longrightarrow & H^3(\Gamma, \mathbb{Z}) & \longrightarrow & \cdots
\end{array}$$

From Remark 2 $red(\tilde{w_2}) = w_2$ and since $H^2(\Gamma, \mathbb{R}) = 0, r_*(w_2) = 0$. It follows that the row

$$0 \to S^1 \to \Gamma_1 \to \Gamma \to 0$$

of the Diagram 1 splits. Hence there exists a homomorphism $\epsilon : \Gamma \to \mathrm{Spin}^{\mathbb{C}}(n)$ which satisfies (5). This proves the theorem.

□

As an immediate corollary we have.

Corollary 1. *Let M be an oriented flat manifold with the fundamental group Γ. If there exists a homomorphism $\epsilon : \Gamma \to \mathrm{Spin}^{\mathbb{C}}(n)$ such that*

$$\bar{\lambda}_n \circ \epsilon = h. \tag{6}$$

then M has a $\mathrm{Spin}^{\mathbb{C}}$-structure.

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[2] $H^*(\mathrm{SO}(n), \mathbb{Z}_2) = \mathbb{Z}_2[x_2, x_3, ..., x_n]$.

Remark 3. *A condition* (4) *of an existence of the* Spin*-structure for oriented flat manifolds also follows from the proof of the above* Theorem 1*.*

Question: Is the assumption $H^2(M,\mathbb{R}) = 0$ about the second cohomology group necessary?

Example 1.

(1) *Because of the inclusion* $i : \mathrm{Spin}(n) \to \mathrm{Spin}^{\mathbb{C}}(n)$ *each* Spin*-structure on* M *induces a* $\mathrm{Spin}^{\mathbb{C}}$*-structure.*
(2) *If* M *is any smooth compact manifold with an almost complex structure, then* M *has a canonical* $\mathrm{Spin}^{\mathbb{C}}$*-structure, see* [5, page 27]*.*

Example 2. *Any oriented compact manifold of dimension up to four has a* $\mathrm{Spin}^{\mathbb{C}}$*-structure, see* [6, page 49]*.*

From Example 2 and [15, Theorem on page 324] we have immediately.

Corollary 2. *There exist three four dimensional flat manifolds without* Spin*-structure but with* $\mathrm{Spin}^{\mathbb{C}}$*-structure.*

In [5, Example on page 50] is given a compact 5-dimensional manifold Q, without $\mathrm{Spin}^{\mathbb{C}}$-structure. However the fundamental group $\pi_1(Q) = 1$. There are also two other non-simply connected 5-dimensional examples, see [8, Eaxmples page 438]. The first one is hypersurface in $\mathbb{R}P^2 \times \mathbb{R}P^4$ defined by the equation $x_0y_0 + x_1y_1 + x_2y_2 = 0$ where $[x_0 : x_1 : x_2]$ and $[y_0 : y_1y_2 : y_3 : y_4]$ are homogeneneous coordinates in $\mathbb{R}P^2$ and $\mathbb{R}P^4$ respectively. The second one is the Dold's manifold

$$P(1,2) = \mathbb{C}P^2 \times S^1/\sim,$$

where $\sim$ is an involution, which acts on $\mathbb{C}P^2$ by complex conjugation and antipodally on S^1. Our next result gives examples of 5-dimensional flat manifolds without $\mathrm{Spin}^{\mathbb{C}}$-structure.

Proposition 2. *Two* HW*-manifolds* M_1 *and* M_2 *of dimension five have not the* $\mathrm{Spin}^{\mathbb{C}}$*-structure.*

Proof: Since $H^2(M_i,\mathbb{R}) = 0, i = 1, 2$, ([4], [16]) we can apply a condition from Theorem 1. Let $\Gamma_1 = \pi_1(M_1)$. It has the CARAT number 1-th 219.1.1, see [13]. [3] It is generated by

$$\alpha_1 = ([1,1,1,-1,-1],(0,0,1/2,1/2,0)), \alpha_2 = ([1,1,-1,-1,1],(0,1/2,0,0,0)),$$
$$\alpha_3 = ([-1,1,1,-1,1],(0,0,0,0,1/2)), \alpha_4 = ([1,-1,-1,1,1],(1/2,0,0,0,0))$$

and translations. We assume that there exists a homomorphism $\epsilon : \Gamma_1 \to \mathrm{Spin}^{\mathbb{C}}(5)$ such that $\bar{\lambda}_n \circ \epsilon = h$. From definition

$$\alpha_2\alpha_3 = \alpha_3\alpha_2$$

and $(\alpha_2\alpha_3)^2 = (\alpha_2)^2(\alpha_3)^2$. Put $\epsilon(\alpha_i) = [a_i, z_i] \in \mathrm{Spin}^{\mathbb{C}}(5), a_i \in \mathrm{Spin}(5), z_i \in S^1, i = 1, 2, 3$. Then

$$\epsilon\left((\alpha_2\alpha_3)^2\right) = \left[-1, z_2^2z_5^2\right] = \epsilon((\alpha_2))^2\epsilon\left((\alpha_3)^2\right) = \left[-1, z_2^2\right]\left[-1, z_5^2\right] = \left[1, z_2^2z_5^2\right]$$

[3]Here we use the name CARAT for tables of Bieberbach groups of dimension ≤ 6, see [13].

and $-z_2^2z_5^2 = z_2^2z_5^2$. We obtain contradiction.

Now, let us consider the second five dimensional HW-group $\Gamma_2 = \pi_1(M_2)$ which has a number 2-th. 219.1.1., (see [13]). It is generated by

$$\beta_1 = (B_1, (1/2, 1/2, 0, 0, 0)), \beta_2 = (B_2, (0, 1/2, 1/2, 0, 0)),$$
$$\beta_3 = (B_3, (0, 0, 1/2, 1/2, 0)) \text{ and } \beta_4 = (B_4, (0, 0, 0, 1/2, 1/2)).$$

Put $\beta_5 = (\beta_1\beta_2\beta_3\beta_4)^{-1} = (B_5, (1/2, 0, 0, 0, -1/2))$. Assume that there exists a homomorphism $\epsilon : \Gamma_2 \to \mathrm{Spin}^{\mathbb{C}}(5)$ which defines the $\mathrm{Spin}^{\mathbb{C}}$-structure on M_2. Let $\epsilon(\beta_i) = [a_i, z_i] \in \mathrm{Spin}^{\mathbb{C}}(5) = (\mathrm{Spin}(5) \times S^1)/\{1, -1\}$. Let $t_i = (I, (0, ..., 0, \underbrace{1}_{i}, 0, ..., 0)), i = 1, 2, 3, 4, 5$.

Since ϵ is a homomorphsim

$$\forall_{1\leq i\leq 5} \;\; \epsilon\left((\beta_i\beta_{i+2})^2\right) = \left[a_ia_{i+2}a_ia_{i+2}, z_i^2z_{i+2}^2\right] = \left[-1, z_i^2z_{i+2}^2\right]. \tag{7}$$

Moreover, by easy computation

$$\forall_{1\leq i\leq 5} \;\; (\beta_i)^2 = t_i, (\beta_i\beta_{i+2})^2 = t_{i+1}t_{i+3}^{-1} \text{ and } \epsilon(t_i) = \left[\pm 1, z_i^2\right]. \tag{8}$$

From (7), (8) and (11)

$$\left[-1, z_1^2z_3^2\right] = \left[1, z_2^2z_4^2\right] = \left[-1, z_3^2z_5^2\right] = \left[1, z_4^2z_1^2\right] = \left[-1, z_5^2z_2^2\right] = \left[1, z_1^2z_3^2\right], \tag{9}$$

which is impossible. Here indexes we read modulo 5. This finishes the proof.

□

Definition 2. *The* HW-*manifold M^n of dimension n, is cyclic if and only if $\pi_1(M^n)$ is generated by the following elements* (see [18, Lemma 1]):

$$\beta_i = (B_i, (0, 0, 0, ..., 0, \underbrace{1/2}_{i}, 1/2, 0, ..., 0)), 1 \leq i \leq n-1,$$
$$\beta_n = (\beta_1\beta_2 \ldots \beta_{n-1})^{-1} = (B_n, (1/2, 0, ..., 0, -1/2).$$

We have.

Theorem 2. *Cyclic* HW-*manifolds have not the* $\mathrm{Spin}^{\mathbb{C}}$*-structure.*

Proof: Since the above group Γ_2 satisfies our assumption the proof is generalization of arguments from the Proposition 2. Let Γ be a fundamental group of the cyclic HW-manifold of dimension ≥ 5, with set of generators $\beta_i = (B_i, b_i), i = 1, 2, .., n$. Since $H^2(\Gamma, \mathbb{R}) = 0$, ([4]) we can apply a condition from Theorem 1. Let us assume that there exist a homomorphism $\epsilon : \Gamma \to Spin^{\mathbb{C}}(n)$, which defines the $\mathrm{Spin}^{\mathbb{C}}$-structure and

$$\epsilon(\beta_i) = [a_i, z_i], a_i \in \mathrm{Spin}(n), z_i \in S^1. \tag{10}$$

From [14] the maximal abelian subgroup $\mathbb{Z}^n$ of Γ is exactly the commutator subgroup Γ. Hence $\epsilon([\Gamma, \Gamma]) \subset i\,(\mathrm{Spin}(n)) \subset \mathrm{Spin}^{\mathbb{C}}(n)$. Since $\forall_i \; \epsilon((\beta_i)^2) = [a_i^2, z_i^2]$ and $(\beta_i)^2 \in [\Gamma, \Gamma]$, $z_i^2 = \pm 1$, for $i = 1, 2, ..., n$. It follows that

$$\forall_i \;\; z_i \in \{\pm 1, \pm i\}. \tag{11}$$

Let $t_i = (I, (0, ..., 0, \underbrace{1}_{i}, 0, ..., 0))$. From (10)

$$\epsilon(t_i) = \epsilon\left((\beta_i)^2\right) = \left[\pm 1, z_i^2\right], i = 1, 2, ..., n \tag{12}$$

and also

(13) $$\forall_{1\leq i\leq n}\ \epsilon\left((\beta_i\beta_{i+2})^2\right) = \left[-1, z_i^2 z_{i+2}^2\right].$$

Moreover

(14) $$\forall_{1\leq i\leq n}\ (\beta_i\beta_{i+2})^2 = t_{i+1}t_{i+3}^{-1}.$$

From equations (12), (13) and (14) we have

$$\left[-1, z_i^2 z_{i+2}^2\right] = \left[1, z_{i+1}^2 z_{i+3}^2\right]$$

and

$$\forall_{1\leq i\leq n}\ z_i^2 z_{i+2}^2 = -z_{i+1}^2 z_{i+3}^2 = z_{i+2}^2 z_{i+4}^2.$$

Since n is odd $z_i^2 z_{i+2}^2 = -z_{i+n}^2 z_{i+2+n}^2 = -z_i^2 z_{i+2}^2$, contradiction, (cf. (9)) [4]. This finishes the proof.

□

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