Spin structures on flat manifolds

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Abstract

The aim of this paper is to present some results about spin structures on flat manifolds. We prove that any finite group can be the holonomy group of a flat spin manifold. Moreover we shall give some methods of constructing spin structures related to the holonomy representation.

1 Introduction

Let M^n be a flat manifold of dimension n. By definition, this is a compact connected, Riemannian manifold without boundary and with sectional curvature equal to zero. From the theorems of Bieberbach ([3]) the fundamental group $\pi_1(M^n) = \Gamma$ detemines a short exact sequence:

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} F \to 0, \qquad (1)$$

where $\mathbb{Z}^n$ is a torsion free abelian group of rank n and F is a finite group which is isomorphic to the holonomy group of M^n. The universal covering of M^n is the Euclidean space $\mathbb{R}^n$ and hence Γ is isomorphic to a discrete cocompact subgroup of the isometry group $Isom(\mathbb{R}^n) = \mathbb{R}^n \rtimes O(n) = E(n)$. In the above short exact sequence $\mathbb{Z}^n \cong (\Gamma \cap \mathbb{R}^n)$ and p can be considered as the projection $p : \Gamma \to F \subset O(n) \subset E(n)$ on the second component.

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Conversely, given a short sequence of the form (1), it is known that the group Γ is (isomorphic to) a Bieberbach group if and only if Γ is torsion free.

In this paper, we study the open and difficult problem of finding all spin structures (if any) on oriented flat manifolds. That is important, for example, when we want to consider the Dirac operator or η-invariant. There is a complete characterization of the flat manifolds which supports an Anosov diffeomorphism (cf. [16]) or with the first Betti number equal to zero (cf. [10]). We think, that our article is a first step towards a decsription of the oriented flat manifolds with or without spin structure. The existence of a Spin structure on M^n is equivalent to the existence of a homomorphism $\varepsilon : \Gamma \to \mathrm{Spin}(n)$ such that $\lambda_n \varepsilon = p$. Here $\lambda_n : \mathrm{Spin}(n) \to SO(n)$ is the covering map ([15], see also below). Moreover it is well known (cf. [7, page 40]) that for any oriented Riemannian manifold M^n, the existence of a spin structure is equivalent to the condition that the second Stiefel-Whitney class $w_2(M^n) \neq 0$. Hence and from (cf. [20, Corollary 1.3]) any flat manifold with holonomy of odd order has a spin stucture. In that same paper, Vasquez shows that not all flat manifolds admit a spin structure by providing an example of a flat manifold M^n with $w_2(M^n) = 0$.

All of this suggests the global question about the relations between the properties of the holonomy group and the existence of spin structures. We shall consider it in the first sections of this paper. We can also say that the class of flat manifolds with holonomy groups of order $2^k, k \geq 1$ is crucial from the point of view of the existence of spin structures. We would like to mention that a complete answer for this question in the case of an elementary abelian group $\mathbb{Z}_2^k, k \geq 1$ was given in [14], where sufficient and necessary conditions for the existence of a spin structure is given. It turns out that there are in fact many such flat manifolds admitting a spin structure, but also a lot of them not admitting a spin structure. But, for most finite groups, not of odd order, the answer is still not known. For example nothing is known for cyclic 2-groups of order greater than two. In the last section we consider the case of generalized quaternion 2-groups and define a spin structure on such flat oriented manifolds.

After having established a sufficient condition for admitting a spin structure (see Proposition 2) we are able to show (Theorem 1) that for any finite group G there exists a flat manifold M, having G as its holonomy group and also admitting a spin structure. That result suggests the question on what can be said about the minimal dimension $s(G)$ of a flat spin manifold with holonomy group G. We present some results in case the holonomy group is an elementary abelian 2-group, but we also devote a whole part of this paper to the case of generalized quaternion groups. In fact, we are able to determine (Theorem 2) the minimal dimension of a flat manifold, having such a quaternion group as its holonomy group and admitting a spin structure.

The second problem which we consider in section four is the dependence of the spin structure on the properties of the homomorphism ε. We shall give, in the case where

$$\varepsilon(\mathbb{Z}^n) = \{1\},$$

a characterization of the holonomy groups and representations of flat oriented manifolds which do not admit a spin structure (Proposition 4). We would like to mention that the case $\varepsilon(\mathbb{Z}^n) = \{1, -1\}$ is different and needs other methods. In the last section we consider non trivial examples of spin structures ε on oriented flat manifolds with generalized quaternion two-groups as holonomy groups and with the property that $\varepsilon(\mathbb{Z}^n) = \mathbb{Z}_2$.

2 A general result

¿From above any oriented flat manifold with holonomy of odd order has a spin structure. Moreover we have the following.

Proposition 1 *Let M^n and Γ be as above and let F_2 be a 2-Sylov subgroup of F. Then M^n has a spin structure if and only if $M^n(2) = \mathbb{R}^n/p^{-1}(F_2)$ has a spin structure.*

Proof: We have an injection $i : p^{-1}(F_2) \to \Gamma$, which induces a homomorphism $i^* : H^2(M^n, \mathbb{Z}_2) \to H^2(M(2), \mathbb{Z}_2)$. By [2, Proposition 10.4] i^* is a monomorphism too, so that $w_2(TM^n) = 0$ if and only if $w_2(TM^n(2)) = 0$. $\square$

Corollary 1 *Let H be a finite group with Sylow 2-subgroup of order 2. Then any oriented flat manifold with holonomy group H has a spin structure.*

Proof: From [20, Corollary 2.8] and [19] (see also [14, Proposition 4.2]) we get that any oriented flat manifold with $\mathbb{Z}_2$ holonomy has a spin structure. Hence by the above proposition the corollary follows. $\square$

Problem Classify all finite groups H, for which any oriented flat manifold with holonomy group H admits a spin structure.

3 Holonomy and Spin structures

Let us start by recalling some basic facts about the group $\mathrm{Spin}(n)$. We refer to [12, Part II, Ch. 11, Th.9.2] and [7] for more details. The n-dimensional real Clifford algebra $Cl(n)$ is a unitary real associative algebra generated by $e_1, e_2, \ldots, e_n$ staisfying the relations

$$\forall i,\ 1 \leq i \leq n:\ e_i e_i = -1,\ \ \forall i,j,\ 1 \leq i < j \leq n:\ e_i e_j = -e_j e_i.$$

So, any element of $Cl(n)$ can be written as a *polynomial* of the form

$$p(e_1, e_2, \ldots, e_n) = a^0 + \sum_{1\leq i\leq n} a_i^1 e_i + \sum_{1\leq i<j\leq n} a_{i,j}^2 e_i e_j + \cdots + a^n e_1 e_2 \ldots e_n, \quad (2)$$

where all of the coefficients $a^k_{i_1 i_2 \ldots i_k}$ belong to $\mathbb{R}$.
The norm of an element of the above form is equal to

$$\sqrt{(a^0)^2 + \sum_{1\leq i\leq n} (a_i^1)^2 + \sum_{1\leq i<j\leq n} (a_{i,j}^2)^2 + \cdots + (a^n)^2}$$

Let $\mathbb{R}^n$ be the subspace of $Cl(n)$ generated by $e_1, e_2, \ldots, e_n$. We shall denote by $\mathrm{Pin}(n)$ the group which is multiplicatitively generated by all elements of $\mathbb{R}^n$ of norm 1. The group $\mathrm{Spin}(n)$ is a subgroup of $\mathrm{Pin}(n)$ which is invariant under the automorphism $' : Cl(n) \to Cl(n)$, which maps e_i to $-e_i$ for $i = 1, 2, \ldots, n$. It consists of elements of the form (2) of norm 1, for which the coefficients appearing in front of an odd number of e_i's vanish. In other words:

$$a_i^1 = a_{i,j,k}^3 = a_{i,j,k,l,m}^5 = \ldots = 0. \quad (3)$$

For $n \geq 2$ there is a standard covering map

$$\lambda_n : \mathrm{Spin}(n) \to SO(n) : y \mapsto \lambda_n(y), \text{ with } \lambda_n(y) : \mathbb{R}^n \to \mathbb{R}^n : x \mapsto yxy^*$$

where, * is the anti-automorphism of $Cl(n)$ determined by $(e_{i_1} e_{i_2} \ldots e_{i_r})^* = e_{i_r} \ldots e_{i_2} e_{i_1}$. The kernel of λ_n equals $\{1, -1\}$. We start with the following.

Lemma 1 *Let $k_1, k_2, ..., k_l \geq 2$ be natural numbers and $\sum_{i=1}^{l} k_i = n$. Then there exists a homomorphism*

$$\nu : \mathrm{Spin}(k_1) \times \mathrm{Spin}(k_2) \times ... \times \mathrm{Spin}(k_l) \to \mathrm{Spin}(n)$$

given by the formula

$$\nu((p_1(e_1,\ldots,e_{k_1}), p_2(e_1,\ldots,e_{k_2}),\ldots,p_l(e_1,\ldots,e_{k_l}))=$$

$$p_1(e_1,\ldots,e_{k_1})p_2(e_{k_1+1},\ldots,e_{k_1+k_2})\cdots p_l(e_{k_1+\cdots+k_{l-1}+1},\ldots,e_n).$$

Here $p_i(e_1,\ldots,e_{k_i})\in Spin(k_i)$.

Proof: Without loss of generality we can assume that $l=2$.
Consider any $p_1(e_1,\ldots,e_{k_1}), q_1(e_1,\ldots,e_{k_1})\in \mathrm{Spin}(k_1)$ and $p_2(e_1,\ldots,e_{k_2})$, $q_2(e_1,\ldots,e_{k_2})\in \mathrm{Spin}(k_2)$. We have that

$$\begin{aligned}
&\nu\Big(\left(p_1(e_1,\ldots,e_{k_1}),p_2(e_1,\ldots,e_{k_2})\right)\left(q_1(e_1,\ldots,e_{k_1}),q_2(e_1,\ldots,e_{k_2})\right)\Big)\\
&\quad= \nu\left(p_1(e_1,\ldots,e_{k_1})q_1(e_1,\ldots,e_{k_1}),p_2(e_1,\ldots,e_{k_2})q_2(e_1,\ldots,e_{k_2})\right)\\
&\quad= p_1(e_1,\ldots,e_{k_1})q_1(e_1,\ldots,e_{k_1})p_2(e_{k_1+1},\ldots,e_{k_1+k_2})q_2(e_{k_1+1},\ldots,e_{k_1+k_2}).
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\nu\left(p_1(e_1,\ldots,e_{k_1}),p_2(e_1,\ldots,e_{k_2})\right)\nu\left(q_1(e_1,\ldots,e_{k_1}),q_2(e_1,\ldots,e_{k_2})\right)\\
&\quad= p_1(e_1,\ldots,e_{k_1})p_2(e_{k_1+1},\ldots,e_{k_1+k_2})q_1(e_1,\ldots,e_{k_1})q_2(e_{k_1+1},\ldots,e_{k_1+k_2})
\end{aligned}$$

But as $p_2(e_{k_1+1},\ldots,e_{k_1+k_2})$ depends only on $e_{k_1+1},e_{k_1+2},...,e_n$ and q_1 is defined on $e_1,e_2,...,e_{k_1}$ and using the fact that both p_2 and q_1 satisfy condition (3), one easily sees that

$$p_2(e_{k_1+1},\ldots,e_{k_1+k_2})q_1(e_1,\ldots,e_{k_1})=q_1(e_1,\ldots,e_{k_1})p_2(e_{k_1+1},\ldots,e_{k_1+k_2})$$

from which it follows that ν is a homomorphism. □

It is easy to see that $Ker(\nu)\cong(\mathbb{Z}_2)^{l-1}$.

Let $\Gamma\subset E(n)$ be the fundamental group of an oriented flat manifold of dimension n. Let F be the holonomy group of Γ and $\phi_\Gamma : F\to SL(n,\mathbb{Z})$ be the corresponding holonomy representation, i.e. the action of F on $\mathbb{Z}^n$ induced by the short exact sequence (1) (see also [3]). Assume that $\phi_\Gamma=\bigoplus_{i=1}^k\phi_i$ as $\mathbb{Z}$-representations and $\phi_i(F)\subset SL(n_i,\mathbb{Z})$. Via the short exact sequence (1), we can view Γ as an extension of F by $\mathbb{Z}^n$ and thus Γ is determined by an element $\beta\in H^2_{\phi_\Gamma}(F,\mathbb{Z}^n)$. The condition that Γ is torsion free has been translated into the language of cohomology. A 2-cohomology class β is called *special* if and only if for each cyclic subgroup C of F we have that the restriction map res : $H^2(F,\mathbb{Z}^n) \to H^2(C,\mathbb{Z}^n)$, maps β to a non-zero element. It has been proved that torsion free extensions correspond exactly to these special elements. So any Bieberbach group is determined by a special 2-cohomology class.

However, as an F-module, $\mathbb{Z}^n = \bigoplus_{i=1}^k \mathbb{Z}^{n_i}$, and therefore

$$H^2_{\phi_\Gamma}(F,\mathbb{Z}^n) = \bigoplus_{i=1}^k H^2_{\phi_i}(F,\mathbb{Z}^{n_i}).$$

Hence β can be seen as a sum of elements $\beta_i \in H^2_{\phi_i}(F,\mathbb{Z}^{n_i})$. So, the group (to be precise the extension) Γ determines groups $\Gamma_1,\Gamma_2,...,\Gamma_k$ which correspond to cocycles $\beta_1,\beta_2,...,\beta_k$. Note that after conjugating by an element of $GL(n_i,\mathbb{R})$, we can view ϕ_i as being a representation into $SO(n_i)$. We will refer to this conjugate representation by the symbol ϕ_i'. Analogously, we will also use ϕ_Γ' to denote the corresponding conjugate representation into $SO(n)$. Note that there exists an embedding ψ of Γ into $E(n)$ as a Bieberbach group, such that $\forall \gamma \in \Gamma: \ \psi(\gamma) = (t_\gamma, \phi_\Gamma'(\gamma))$ for some $t_\gamma \in \mathbb{R}^n$.

Definition 1 *Let $\phi : G \to SO(n)$ be an orthogonal representation of a group G. The pair (ϕ, G) has a Spin structure if there exist a homomorphism $\varepsilon : G \to Spin(n)$ such that $\lambda_n \varepsilon = \phi$.*

The main applications of the above lemma are the following.

Theorem 1 *Every finite group is the holonomy of a spin flat manifold.*

Proposition 2 *Let Γ and ϕ_Γ be as above. Assume that the dimension of any representation ϕ_i is greater than or equal to two. If for each $i = 1,2,...,k$; (ϕ_i',Γ_i) has a spin structure then (ϕ_Γ',Γ) has a spin structure.*

Proof of Proposition 2: As an extension Γ can be viewed as the set of elements $((z_1,z_2,\dots,z_k),f)$ with $f \in F$ and $z_i \in \mathbb{Z}^{n_i}$. The product is determined by a cocycle $\beta = \beta_1 \oplus \cdots \oplus \beta_k \in \bigoplus H^2_{\phi_i}(F,\mathbb{Z}^{n_i})$. Thus the product of two elements is given by

$$((z_1,z_2,\dots,z_k),f)((y_1,y_2,\dots,y_k),g) =$$

$$((z_1+\phi_1(f)y_1+\beta_1(f,g),\dots,z_k+\phi_k(f)y_k+\beta_k(f,g)),fg)$$

The group Γ_i consists of all tuples $(z,f) \in \mathbb{Z}^{n_i} \times F$ and the product is given by $(z,f)(y,g) = (z + \phi_i(f)y + \beta_i(f,g), fg)$. The existence of a Spin structure for the pair (ϕ'_i, Γ_i) implies the existence of a morphism $\varepsilon_i : \Gamma_i \to \mathrm{Spin}(n_i)$ satisfying $\lambda_{n_i}\varepsilon_i = \phi'_i$. If we now put

$$\varepsilon((z_1, z_2, \ldots, z_k), f) = \nu(\varepsilon_1(z_1, f), \varepsilon_2(z_2, f), \ldots, \varepsilon_k(z_k, f))$$

we find the desired spin structure for the pair (ϕ'_Γ, Γ). $\square$

Remark The converse statement of Proposition 2 is not true. In the proof of the Theorem 1 we will show that if one takes any oriented Bieberbach group Γ, say determined by a cocycle β and a module structure ϕ_Γ, then the "double" of Γ, that is, the group determined by a module structure $\phi_\Gamma \oplus \phi_\Gamma$ and cocycle $\beta \oplus \beta$, always has a Spin structure, while its components $\Gamma_1 = \Gamma_2 = \Gamma$ need not have a Spin structure.

It is however easy to see that the converse is true under the extra assumption that $\varepsilon(\mathbb{Z}^n) = \{1\}$. ($\mathbb{Z}^n$ being the maximal normal abelian subgroup of Γ).

Proof of Theorem 1: Let F be a finite group. Then, there exists an oriented Bieberbach group Γ with holonomy group F. We have a short exact sequence

$$0 \to \mathbb{Z}^n \to \Gamma \to F \to 0,$$

which is determined by some $\beta \in H^2(F, \mathbb{Z}^n)$. Now, we shall recall some standard construction (cf. [13, Proposition 3.3] and [16, Theorem 2.2]). Let, Γ' be the group defined by the element $\beta \oplus \beta \in H^2(F, \mathbb{Z}^n \oplus \mathbb{Z}^n) \cong H^2(F, \mathbb{Z}^n) \oplus H^2(F, \mathbb{Z}^n)$. Then Γ' is a Bieberbach group, and if the holonomy representation of Γ is given by $\phi_\Gamma : F \to SO(n)$, then the holonomy representation of Γ' is the direct sum of representations $\phi_\Gamma \oplus \phi_\Gamma$. Let us denote the flat manifold with fundamental group Γ by M^n and with fundamental group Γ' by $d(M)$. For any $f \in F$, we choose a $\widetilde{\phi_\Gamma(f)} \in Spin(n)$ such that $\lambda_n(\widetilde{\phi_\Gamma(f)}) = \phi_\Gamma(f)$. Using Lemma 1 we can define

$$\varepsilon : F \to \mathrm{Spin}(n) \times \mathrm{Spin}(n) \to \mathrm{Spin}(2n),$$

by the formula $\varepsilon(f) = \nu(\widetilde{\phi_\Gamma(f)}, \widetilde{\phi_\Gamma(f)})$.

We claim that ε is a homomorphism. In fact, let $f_1, f_2 \in F$, then

$$\varepsilon(f_1 f_2) \quad = \quad \nu(\widetilde{\phi_\Gamma(f_1 f_2)}, \widetilde{\phi_\Gamma(f_1 f_2)})$$

$$
\begin{aligned}
&= \nu(\pm\widetilde{\phi_\Gamma(f_1)}\widetilde{\phi_\Gamma(f_2)}, \pm\widetilde{\phi_\Gamma(f_1)}\widetilde{\phi_\Gamma(f_2)}) \\
&= \nu(\widetilde{\phi_\Gamma(f_1)}\widetilde{\phi_\Gamma(f_2)}, \widetilde{\phi_\Gamma(f_1)}\widetilde{\phi_\Gamma(f_2)}) \\
&= \varepsilon(f_1)\varepsilon(f_2).
\end{aligned}
$$

Moreover we have that $\lambda_{2n}\varepsilon = \phi_\Gamma \oplus \phi_\Gamma$. Indeed

$$\lambda_{2n}\varepsilon(f) = \lambda_{2n}(\nu(\widetilde{\phi_\Gamma(f)}, \widetilde{\phi_\Gamma(f)})) = \underbrace{(\lambda_n\widetilde{\phi_\Gamma(f)}, \lambda_n\widetilde{\phi_\Gamma(f)})}_{\in SO(n)\times SO(n)\subseteq SO(2n)} = (\phi_\Gamma \oplus \phi_\Gamma)(f).$$

It follows that Γ' is the fundamental group of a spin flat manifold with holonomy F. We can see that for a spin structure defined in such a way we have that $\varepsilon(\mathbb{Z}^{2n}) \cap \mathbb{Z}_2 = \{1\}$. $\square$

Remark There is another proof of Theorem 1 which uses characteristic classes. In fact let $d(M)$ denote the flat manifold of dimension $2n$ as defined in the proof of Theorem 1. Let $p_i : d(M^n) \to M^n (i = 1, 2)$ denote the maps induced by projections on the first or second coordinate at the covering space $\mathbb{R}^n \times \mathbb{R}^n$. We have

$$Td(M^n) \simeq p_1^*T(M^n) \oplus p_2^*T(M^n).$$

It follows from the definition of $d(M^n)$, that the vector bundles $p_1^*T(M^n)$ and $p_2^*T(M^n)$ are isomorphic. Moreover, the Stiefel-Whitney formula (cf. [12, Part III, Ch. 16, section 3.1]) gives us:

$$(w(Td(M)) = w(p_1^*T(M^n))w(p_2^*T(M^n)) =$$

$$= (1 + w_2(TM) + ... + w_n(TM))(1 + w_2(TM) + ... + w_n(TM).$$

Hence we can observe that $w_2(Td(M)) = 0$. In the above, we used the fact that the induced homomorphisms $p_i^* : H^*(M, \mathbb{Z}_2) \to H^*(d(M), \mathbb{Z}_2)$ are monomorphisms.

Now we can formulate a definition.

Definition 2 *For any finite group G, we denote by $s(G)$ the minimal dimension of an oriented spin flat manifold with holonomy group G.*

The number $s(G)$ is obviously related to the number $d(G)$ which is the minimal dimension of a flat manifold with holonomy group G. We have always

$d(G) \leq s(G) \leq 4d(G)$ and if the manifold of minimal dimension is oriented we have from the proof of Theorem 1 that $s(G) \leq 2d(G)$. About the number $d(G)$ we only have knowledge for a few classes of groups as for example: cyclic, elementary abelian, dihedral, semidihedral, quaternion (cf. [3], [10]). For most classes, the exact determination of $d(G)$ is still an open and difficult problem.

We have already observed in the Remark after the Proposition 2 that in some cases the spin structure on oriented flat manifold has some special conditions. In the next section we want to consider such situations.

4 Spin structure ε with property $\varepsilon(\mathbb{Z}^n) = \{1\}$

We start with a simple observation.

Proposition 3 *Let G be a finite group with $H^2(G, \mathbb{Z}_2) = 0$. Then any oriented flat manifold M with holonomy G has a spin structure.*

Proof: Let $\Gamma \subset E(n)$ be the fundamental group of M. We have to define a homomorphism $\varepsilon : \Gamma \to \mathrm{Spin}(n)$ such that $p = \lambda_n \varepsilon$. By the assumption there exists a homomorphism $p' : p(G) \to \lambda_n^{-1}(p(G))$ such that $\lambda_n p' = id_{p(G)}$. Put $\varepsilon = p'p$. $\square$

If a finite group G has a trivial Schur multiplier (i.e. $H_2(G, \mathbb{Z}) = 0$) and is perfect, then any central extension of it with a group of order 2 splits and so they satisfy the condition of the proposition. The binary icosahedral group is an interesting example of it (cf. [21, § 6.2]). Moreover the groups $L_2(8)$, $L_2(16)$, $L_3(3)$, $U_3(3)$, M_{11}, $L_2(32)$, $U_3(4)$, J_1 are examples of such groups. These examples and many more can be found in the Atlas ([4]).

Moreover we can use Proposition 3 and [17, Proposition 6.1] to give an estimate for the number $s((\mathbb{Z}_2)^n)$. We have

Corollary 2 *Suppose that $n > 3$. Let $1 \leq i \leq 8, n = 8k + i$ and $h_i = 4k + (i - 1)$ for $i = 1, 2, 3$ $h_4 = h_3$ and $h_i = 4k + 3$ for $i = 5, 6, 7, 8$, then $s((\mathbb{Z}_2)^{n-h_i-1}) \leq n$.*

Proof: It follows from [17] that for the above numbers $n - h_i$, $(\mathbb{Z}_2)^{n-h_i}$ can be seen as a subgroup of an extra special 2-group $\mathcal{G}_n \subset \mathrm{Spin}(n)$, where $\mathcal{G}_n$ denotes the preimage under λ_n of the maximal diagonal subgroup of $SO(n)$.

As $\{-1, 1\}$ is certainly a subgroup of this elementary abelian 2-group, it follows that $\lambda_n((\mathbb{Z}_2)^{n-h_i}) \cong (\mathbb{Z}_2)^{n-h_i-1}$.

Next, we need to construct an oriented flat manifold M^n of dimension n with holonomy group $(\mathbb{Z}_2)^{n-h_i-1}$, and moreover we have to require that the holonomy representation

$$\phi : p(\pi_1(M^n)) \to SO(n)$$

satisfies $\phi((\mathbb{Z}_2)^{n-h_i-1}) = \lambda_n((\mathbb{Z}_2)^{n-h_i})$.

Such a construction is possible due to Theorem 5.1 of [18]. The existence of a spin structure on M^n now follows from the definition. $\square$

At the end of this part we want to give some criterion for the non-existence of spin structures with property from the title of the section. This criterion is well known and was used by Griess [9] and Gagola and Garrison [8] to construct non-trivial double covers for certain groups. We are indebted to Klaus Lux for pointing out the latter references.

Proposition 4 *Let G be the holonomy group of a flat oriented manifold with holonomy representation $\Phi : G \to SO(n)$ with character χ. Let $g \in G$ have order 2. If*

$$\frac{1}{2}(\chi(1) - \chi(g)) \equiv 2(\bmod 4).$$

Then there is no $\varepsilon : G \to Spin(n)$ such that $\Phi = \lambda_n \varepsilon$.

Proof: Let d denote the dimension of the (-1)-eigenspace of $\Phi(g)$. Then d is even. By [8, Corollary 4.3], there is an inverse image $u \in Spin(n)$ of $\Phi(g)$ with $u^2 = (-1)^{\frac{d(d-1)}{2}}$. Now $d = \frac{\chi(1)-\chi(g)}{2}$, so u has order 4. In particular, the inverse image $\lambda_n^{-1}(\Phi(G))$ is non-split, from which th eresult follows $\square$

5 The case of quaternion holonomy

In this section we will illustrate the use of Proposition 2 and determine $s(Q_{2^\alpha})$ for the quaternion group of order 2^α ($\alpha \geq 3$), which is given by the presentation

$$Q_{2^\alpha} = \langle x, y \mid x^{2^\alpha} = 1,\ x^{2^{\alpha-1}} = y^2,\ y^{-1}xy = x^{-1} \rangle.$$

For any $\alpha \geq 3$, the center of Q_{2^α} is the group $\{1, y^2\}$ of order 2.

It will turn out that this family of groups is tractable because of the following fact.

Proposition 5 *([6]) Let $\alpha \geq 3$. Then Q_{2^α} admits exactly one (up to $\mathbb{Q}$-equivalence) irreducible and faithful representation $\psi_\alpha : Q_{2^\alpha} \to GL(n, \mathbb{Q})$ (for some $n \in \mathbb{N}$).*

Proof: Denote the nontrivial representation of $\mathbb{Z}_2 = \{1, y^2\} \to GL(1, \mathbb{Q})$ by ϕ. Then this unique faithful and irreducible representation is the induced representation

$$\psi_\alpha = \mathrm{Ind}_{\{1,y^2\}}^{Q_{2^\alpha}} \phi \qquad \text{(and so } n = 2^{\alpha-1}\text{)}$$

□

Of course as Q_{2^α} is a finite group, the representation ψ_α can also be viewed as a representation into $O(n)$, and in fact even into $SO(n)$. We will follow this point of view in Lemma 3.

Lemma 2 *Let H be a group containing G as a subgroup of index 2. Let l be a positive integer and suppose that $\phi : G \to SO(4l)$ is a representation, such that the pair (ϕ, G) has a Spin structure. Then also the pair $(\mathrm{Ind}_G^H \phi, H)$ has a spin structure.*

Proof: Fix an $h \in H$ such that $H = G \cup Gh$. Then any element $x \in H$ is either of the form $x = g$ for some unique $g \in G$ or of the form $x = gh$ for some unique $g \in G$.
Throughout the proof of this lemma, we will use $k = 2l$ and $n = 4l = 2k$. By [5], we know that $\forall g \in G$,

$$\mathrm{Ind}_G^H \phi(g) = \begin{pmatrix} \phi(g) & 0 \\ 0 & \phi(h^{-1}gh) \end{pmatrix} \text{ and } \mathrm{Ind}_G^H \phi(gh) = \begin{pmatrix} 0 & \phi(gh^2) \\ \phi(h^{-1}gh) & 0 \end{pmatrix}$$

If I_n denote the $n \times n$-identity matrix, then we also have that

$$\mathrm{Ind}_G^H \phi(g) = \begin{pmatrix} \phi(g) & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \phi(h^{-1}gh) & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \quad (4)$$

and

$$\mathrm{Ind}_G^H \phi(gh) = \begin{pmatrix} \phi(gh^2) & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \begin{pmatrix} \phi(h^{-1}gh) & 0 \\ 0 & I_n \end{pmatrix}. \quad (5)$$

We know there exists a morphism $\varepsilon : G \to \mathrm{Spin}(n)$ such that $\lambda_n \varepsilon = \phi$. Let us denote by $\tilde{\varepsilon} : G \to \mathrm{Spin}(2n)$, the map which sends each element of the form (2) to the same *polynomial* but now seen as an element of $\mathrm{Spin}(2n)$. So, in fact, using the notations of Lemma 1 (with $k_1 = k_2 = n$), we have that $\forall g \in G : \tilde{\varepsilon}(g) = \nu(\varepsilon(g), 1)$. So, $\tilde{\varepsilon}$ is a morphism and

$$\forall g \in G : \lambda_{2n}(\tilde{\varepsilon}(g)) = \begin{pmatrix} \phi(g) & 0 \\ 0 & I_n \end{pmatrix}.$$

Let us fix the following element of $\mathrm{Spin}(2n)$:

$$s = \frac{1}{2^k}(e_1 - e_{n+1})(e_2 - e_{n+2}) \dots (e_n - e_{2n})$$

It is easy to see that s indeed belongs to $\mathrm{Spin}(2n)$ and that

$$\lambda_{2n}(s) = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \in SO(2n).$$

Using the fact that $n = 4l$, one easily computes that $s^2 = 1$ (if one caries out the same computation for $n = 4l + 2$, one will find that $s^2 = -1$). Using s define a new map $\varepsilon' : H \to \mathrm{Spin}(2n)$, by the formulas

$$\forall g \in G : \varepsilon'(g) = \tilde{\varepsilon}(g) s \tilde{\varepsilon}(h^{-1}gh) s \text{ and } \varepsilon'(gh) = \tilde{\varepsilon}(gh^2) s \tilde{\varepsilon}(h^{-1}gh).$$

From its definition and the expressions (4) and (5), one immediately gets that

$$\lambda_{2n}\varepsilon' = \mathrm{Ind}_G^H \phi.$$

The only thing left to show is that ε' is a morphism of groups. This can be done by checking the following 4 conditions for all $g_1, g_2 \in G$:

1. $\varepsilon'(g_1 g_2) = \varepsilon'(g_1)\varepsilon'(g_2)$
2. $\varepsilon'(g_1 g_2 h) = \varepsilon'(g_1)\varepsilon'(g_2 h)$
3. $\varepsilon'(g_1 h g_2) = \varepsilon'(g_1 h)\varepsilon'(g_2)$
4. $\varepsilon'(g_1 h g_2 h) = \varepsilon'(g_1 h)\varepsilon'(g_2 h)$

Here, we will check the condition 3., the other cases are left to the reader. On the one hand we have that

$$\begin{aligned} \varepsilon'(g_1 h g_2) &= \varepsilon'(g_1 h g_2 h^{-1} h) \\ &= \tilde{\varepsilon}(g_1 h g_2 h) s \tilde{\varepsilon}(h^{-1} g_1 h g_2) \end{aligned}$$

While on the other hand we compute (where we will use that the two underlined parts commute)

$$\begin{aligned} \varepsilon'(g_1 h)\varepsilon'(g_2) &= \tilde{\varepsilon}(g_1 h^2)\ \underline{s\tilde{\varepsilon}(h^{-1}g_1 h)\tilde{\varepsilon}(g_2) s}\ \underline{\tilde{\varepsilon}(h^{-1} g_2 h)}\ s \\ &= \tilde{\varepsilon}(g_1 h^2)\tilde{\varepsilon}(h^{-1} g_2 h) s \tilde{\varepsilon}(h^{-1} g_1 h)\tilde{\varepsilon}(g_2) s^2 \\ &= \tilde{\varepsilon}(g_1 h g_2 h) s \tilde{\varepsilon}(h^{-1} g_1 h g_2) \end{aligned}$$

which shows that the condition is satisfied. □

Using the previous lemma in the case of the quaternion groups, we obtain the following.

Lemma 3 *Let $\alpha \geq 3$ and let $\psi_\alpha : Q_{2^\alpha} \to GL(2^{\alpha-1}, \mathbb{Q})$ denote the unique irreducible and faithful representation of Proposition 5. If we consider ψ_α as being a representation into $SO(n)$, then the pair $(\psi_\alpha, Q_{2^\alpha})$ admits a spin structure.*

Proof: We will use induction on α. For $\alpha = 3$, one can check (or consult [1, page 245]) that $\psi_3 : Q_8 \to SO(4)$ is determined by

$$\psi_3(x) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \psi_3(y) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

We leave to the reader to check that a spin structure $\varepsilon : Q_8 \to \mathrm{Spin}(4)$ is given by

$$\varepsilon(x) = \frac{1}{2} e_2 e_3 (e_1 - e_3)(e_2 - e_4) \text{ and } \varepsilon(y) = \frac{1}{2} e_3 e_4 (e_1 - e_4)(e_3 - e_2).$$

Now, assume by induction that there is a spin structure for $(\psi_{\alpha-1}, Q_{2^{\alpha-1}})$. The group Q_{2^α} contains $Q_{2^{\alpha-1}}$ as a subgroup of finite index, namely the subgroup generated by x^2 and y. It follows that (using the notations of Proposition 5)

$$\psi_\alpha = \mathrm{Ind}_{\{1,y^2\}}^{Q_{2^\alpha}} \phi = \mathrm{Ind}_{Q_{2^{\alpha-1}}}^{Q_{2^\alpha}} \mathrm{Ind}_{\{1,y^2\}}^{Q_{2^{\alpha-1}}} \phi.$$

¿From the induction hypothesis and Lemma 2, we find that also the pair $(\psi_\alpha, Q_{2^\alpha})$ admits a spin structure. □

In order to construct flat manifolds with holonomy group Q_{2^α}, we need to find a faithful Q_{2^α}–module structure ϕ on some $\mathbb{Z}^n$ such that there exists at least one special element in $H^2_\phi(Q_{2^\alpha}, \mathbb{Z}^n)$. The next proposition explains what we should look for.

Proposition 6 *Let $\phi : Q_{2^\alpha} \to GL(n, \mathbb{Z})$ denote a faithful Q_{2^α}-module structure of $\mathbb{Z}^n$. Then there exists a Q_{2^α}- submodule $T \subseteq \mathbb{Z}^n$ such that $T \otimes \mathbb{Q}$ is the unique faithful and irreducible rational Q_{2^α}-module (from Proposition 5) and $\mathbb{Z}^n/T$ is a torsion free.*

Moreover, let $p : \mathbb{Z}^n \to \mathbb{Z}^n/T$ *denotes the natural projection and* $C = \{1, y^2\}$ *be the centre of* Q_{2^α}*.*
Then, for all $\beta \in H^2(Q_{2^\alpha}, \mathbb{Z}^n)$ *we have that* $\mathrm{res}(\beta) \in H^2(C, \mathbb{Z}^n)$ *is zero if and only if* $\mathrm{res}(p_*(\beta)) \in H^2(C, \mathbb{Z}^n/T)$ *is zero.*

Proof: The module $\mathbb{Z}^n \otimes \mathbb{Q} \cong \mathbb{Q}^n$ is a faithful module and hence it contains a submodule $T_\mathbb{Q}$ which is (isomorphic to) the unique irreducible and faithful rational Q_{2^α}–module (mentioned in Proposition 5). By taking $T = \mathbb{Z}^n \cap T_\mathbb{Q}$, we immediately find the submodule satisfying the properties stated above.

Note that y^2 acts as multiplication by -1 on $T_\mathbb{Q}$ and hence also as multiplication by -1 on T. It follows at once that $H^2(C, T) = 0$. The short exact sequence $0 \to T \to \mathbb{Z}^n \to \mathbb{Z}^n/T \to 0$ of Q_{2^α}–modules gives rise to the following commutative diagram in which the rows are the long exact sequences associated to the above short exact sequence of modules:

$$\begin{tikzcd}
\cdots \arrow[r] & H^2(Q_{2^\alpha}, T) \arrow[r, "i_*"'] \arrow[d, "\mathrm{res}"'] & H^2(Q_{2^\alpha}, \mathbb{Z}^n) \arrow[r, "p_*"'] \arrow[d, "\mathrm{res}"'] & H^2(Q_{2^\alpha}, \mathbb{Z}^n/T) \arrow[r] \arrow[d, "\mathrm{res}"'] & \cdots \\
\cdots \arrow[r] & H^2(C, T) = 0 \arrow[r, "i_*"'] & H^2(C, \mathbb{Z}^n) \arrow[r, "p_*"'] & H^2(C, \mathbb{Z}^n/T) \arrow[r] & \cdots
\end{tikzcd}$$

As $H^2(C, T) = 0$, we have that p_* in the bottom row is injective. Therefore, for all $\beta \in H^2(Q_{2^\alpha}, \mathbb{Z}^n)$, we obtain that

$$\mathrm{res}(\beta) = 0 \Leftrightarrow p_*(\mathrm{res}(\beta)) = 0 \Leftrightarrow \mathrm{res}(p_*(\beta)) = 0.$$

which was to be shown. $\square$

Lemma 4 *For any* Q_{2^α}*-module structure on* $\mathbb{Z}$ *(with* $\alpha \geq 3$*), the restriction map* $\mathrm{res} : H^2(Q_{2^\alpha}, \mathbb{Z}) \to H^2(C, \mathbb{Z})$ *is trivially 0. Here* $C = \{1, y^2\}$ *denotes the centre of* Q_{2^α}*.*

Proof: Suppose first that Q_{2^α} acts non trivially on $\mathbb{Z}$. Then at least one of the generators x or y has to act as -1 on $\mathbb{Z}$ and hence at least one of the cohomology groups $H^2(\langle x \rangle, \mathbb{Z})$ or $H^2(\langle y \rangle, \mathbb{Z})$ is zero. As $y^2 = x^{2^{\alpha-1}}$, the restriction map $\mathrm{res} : H^2(Q_{2^\alpha}, \mathbb{Z}) \to H^2(C, \mathbb{Z})$ can be thought of as a composition of restrictions maps in the following two ways:

$$H^2(Q_{2^\alpha}, \mathbb{Z}) \longrightarrow H^2(\langle x \rangle, \mathbb{Z}) \longrightarrow H^2(C, \mathbb{Z}) \text{ and}$$

$$H^2(Q_{2^\alpha},\mathbb{Z}) \longrightarrow H^2(\langle y\rangle,\mathbb{Z}) \longrightarrow H^2(C,\mathbb{Z}).$$

As at least in one case one of the middle groups is trivial, it follows that the restriction map res : $H^2(Q_{2^\alpha},\mathbb{Z}) \to H^2(C,\mathbb{Z})$ is trivial.

In case the action is trivial, The short exact sequence of trivial Q_{2^α}-modules $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ induces a commutative square, where the horizontal arrows are isomorphisms (use the fact that $H^i(Q_{2^\alpha},\mathbb{Q}) = 0$ for $i > 0$):

$$\begin{tikzcd}
H^1(Q_{2^\alpha},\mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, "\mathrm{res}"'] & H^2(Q_{2^\alpha},\mathbb{Z}) \arrow[d, "\mathrm{res}"] \\
H^1(C,\mathbb{Q}/\mathbb{Z}) \arrow[r] & H^2(C,\mathbb{Z})
\end{tikzcd}$$

Now, a 1-cocycle in $H^1(Q_{2^\alpha},\mathbb{Q}/\mathbb{Z})$ is just a morphism $f : \mathbb{Q}_{2^\alpha} \to \mathbb{Z}$. As y^2 belongs to the commutator subgroup $[\mathbb{Q}_{2^\alpha},\mathbb{Q}_{2^\alpha}]$, $f(y^2) = 0$. This immediatly implies that res : $H^1(Q_{2^\alpha},\mathbb{Q}/\mathbb{Z}) \to H^1(C,\mathbb{Q}/\mathbb{Z})$ and hence also res : $H^2(Q_{2^\alpha},\mathbb{Z}) \to H^2(C,\mathbb{Z})$ is trivially 0. $\square$

We are now ready to prove the main theorem of this section

Theorem 2 *If $\alpha \geq 3$, then $s(Q_{2^\alpha}) = 2^{\alpha-1} + 3$.*

Proof: We will first prove that the minimal dimension of an orientable flat manifold with holonomy Q_{2^α} is greater than or equal to $2^{\alpha-1}+3$ and afterwards, we will show that in this dimension, there exists an orientable flat manifold with holonomy Q_{2^α} admitting a Spin structure.

So assume that Γ is a Bieberbach group with holonomy Q_{2^α} and Γ is in fact the fundamental group of an orientable flat manifold. Then Γ fits in a short exact sequence

$$1 \to \mathbb{Z}^n \to \Gamma \to Q_{2^\alpha} \to 1$$

inducing a faithful Q_{2^α}-module structure on $\mathbb{Z}^n$ and Γ is determined by a special element $\beta \in H^2(Q_{2^\alpha},\mathbb{Z}^n)$. In particular, for $C = \{1, y^2\}$, we have that the restriction map res : $H^2(Q_{2^\alpha},\mathbb{Z}^n) \to H^2(C,\mathbb{Z}^n)$ is such that $\mathrm{res}(\beta) \neq 0$. Note that the fact that the corresponding manifold is orientable is equivalent to the fact the image of the holonomy representation $\phi : Q_{2^\alpha} \to GL(n,\mathbb{Z})$ lies inside $SL(n,\mathbb{Z})$.

Let T denote the Q_{2^α}-submodule of $\mathbb{Z}^n$ obtained in the Proposition 6. We know that $T \cong \mathbb{Z}^{2^{\alpha-1}}$, and that $\mathbb{Z}^n/T \cong \mathbb{Z}^k$ for some k. We will first show that $k > 2$. Recall that by Proposition 6, we must have that $0 \neq$

$\mathrm{res}(p_*(\beta)) \in H^2(C, \mathbb{Z}^k)$. This immediately excludes the case $k = 0$. The situation for $k = 1$ can be excluded by Lemma 4.

For $k = 2$, we need to consider all possible representations of Q_{2^α} in $SL(2, \mathbb{Z})$ (remember that the manifold has to be orientable). Any finite subgroup of $SL(2, \mathbb{Z})$ is cyclic (of order 1,2,3,4 or 6) and hence abelian. It follows that any representation of Q_{2^α} factors through $Q_{2^\alpha}/[Q_{2^\alpha}, Q_{2^\alpha}] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. This implies that the image of any representation of Q_{2^α} in $SL(2, \mathbb{Z})$ is either trivial or $\mathbb{Z}_2$. In this latter case, the image consists of plus and minus the identity matrix.

It follows that any representation of Q_{2^α} in $SL(2, \mathbb{Z})$ is in fact the sum of two 1-dimensional representations and therefore, again using Lemma 4, we can also exclude the case $k = 2$, from which we conclude that $s(Q_{2^\alpha}) \geq 2^{\alpha-1} + 3$.

To prove that $s(Q_{2^\alpha}) \leq 2^{\alpha-1} + 3$, we consider the Bieberbach groups constructed in [11]. As a Q_{2^α}-module structure of $\mathbb{Z}^{2^{\alpha-1}+3}$, one considers the direct sum of two representations. As a first module, one chooses any integral representation $\phi_1 : Q_{2^\alpha} \to SL(2^{\alpha-1}, \mathbb{Z})$, such that the corresponding rational representation is the unique faithful and irreducible representation of Proposition 5. As a second representation, one considers a representation $\phi_2 : Q_{2^\alpha} \to SL(3, \mathbb{Z})$ given by

$$\phi_2(x) = \begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \text{ and } \phi_2(y) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

We will use the representation $\phi = \phi_1 \oplus \phi_2$.
In [11], it is shown that there exists a special element $\beta_2 \in H^2_{\phi_2}(Q_{2^\alpha}, \mathbb{Z}^3)$. Of course, the element $\beta = 0 \oplus \beta_2 \in H^2_{\phi_1}(\mathbb{Q}_{2^\alpha}, \mathbb{Z}^{2^{\alpha-1}}) \oplus H^2_{\phi_2}(\mathbb{Q}_{2^\alpha}, \mathbb{Z}^3) = H^2_{\phi}(\mathbb{Q}_{2^\alpha}, \mathbb{Z}^{2^{\alpha-1}+3})$ is then also special and determines a Bieberbach group Γ with holonomy Q_{2^α} (and where the corresponding manifold is orientable).

As before, we use Γ_1 and Γ_2 to denote the groups corresponding to $0 \in H^2_{\phi_1}(\mathbb{Q}_{2^\alpha}, \mathbb{Z}^{2^{\alpha-1}})$ and $\beta_2 \in H^2_{\phi_2}(\mathbb{Q}_{2^\alpha}, \mathbb{Z}^3)$ respectively. It follows from Lemma 3 that the pair (ϕ_1, Γ_1) has a spin structure (factoring through the holonomy group). On the other hand, the group Γ_2 is torsion free and hence it is the fundamental group of a three dimensional orientable flat manifold (in fact the Hantsche-Wendt manifold, cf. [18] and [21]). It follows that also the pair (ϕ_2, Γ_2) has a spin structure. As a consequence of Proposition 2, we obtain that (ϕ, Γ) has a spin structure, which was to be shown. $\square$

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