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Intersection forms of almost-flat 4-manifolds

A. Szczepański

Abstract. We calculate intersection forms of all 4-dimensional almost-flat manifolds.

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1. Introduction. Let M be a smooth, closed, oriented 4-manifold. We define a symmetric and bilinear form (the intersection form),

$$Q_M : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \to \mathbb{Z},$$

as the evaluation of the cup product $a \cup b$ on the fundamental homology class of $[M]$, that is, $Q_M(a,b) = \langle a \cup b, [M]\rangle, a, b \in H^2(M,\mathbb{Z})$. It remains a central question in 4-manifold topology: which quadratic forms occur as the intersection forms of an orientable 4-manifold? In the topological category, there is the celebrated result of Freedman which asserts that for each quadratic form Q there exists an oriented simply-connected 4-manifold with Q as its intersection form. However, in the smooth category Donaldson proved that among the definite quadratic forms only the diagonalizable ones can be realized (see [3,4,12]). In this note we are interested in a classification of intersection forms of oriented 4-dimensional *almost-flat* manifolds. They are given by almost Bieberbach groups over 2- and 3-step nilpotent groups. We prove that if M is not the torus T^4, then $Q_M = nH$ with $n = b_1(M) - 1$, where H is the hyperbolic form with intersection matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ of rank 2 and $b_1(M)$ is the first Betti number of M.

An *almost-flat* manifold is a closed manifold M such that for any $\epsilon > 0$ there exists a Riemannian metric g_ϵ on M with $|K_\epsilon|\mathrm{diam}(M, g_\epsilon)^2 < \epsilon$ where K_ϵ is the sectional curvature and $\mathrm{diam}(M, g_\epsilon)$ is the diameter of M. When $K = 0$, M is a flat manifold. Here K denotes the sectional curvature. The fundamental group

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of an almost-flat manifold is called an almost-Bieberbach group. Two almost-flat manifolds with isomorphic fundamental groups are affinely diffeomorphic, see [2, page 16]. From now on M stands for an oriented 4-dimensional almost-flat manifold. It is well known that the Euler characteristic of M is zero (cf. [2, page 134]). Hence, from Poincaré duality we have the following relation involving the Betti numbers of M :

$$\chi(M) = 2 - 2b_1(M) + b_2(M) = 0 \tag{1}$$

and hence as $b_2(M) \geq 0$, it follows from (1) that $b_1(M) \geq 1$. Moreover we have:

Lemma 1. *For any oriented 4-dimensional almost-flat manifold M, the intersection form Q_M is even.*

Proof. From [1, Lemma 1] (see also Wu's formula [9, Theorem 11.14]), any closed, oriented 4-dimensional spin-manifold has even intersection form. Let us assume that M has not a spin-structure. In [8] and [10] all such manifolds are classified. That means there is a list of their fundamental groups. We claim that the first Betti number of any such M or equivalently the rank of the abelianization of $\pi_1(M)$ is equal to 1. For the proof we can use two methods. The first one uses presentations of $\pi_1(M)$ and the computer algebra system GAP [5]. The second one uses properties of fundamental groups and is as follows. There are 3 flat manifolds without spin structure. All of them are presented in [10]. Let us recall (see [13]) that the fundamental group Γ of a four dimensional flat manifold is the middle term in the short exact sequence

$$0 \to \mathbb{Z}^4 \to \Gamma \xrightarrow{p} G \to 0, \tag{2}$$

where $\mathbb{Z}^4$ is a free abelian group of rank 4 and G is a finite group. It is easy to see (cf. [13, page 51]) that the first Betti number of the group Γ is equal to the rank $(\mathbb{Z}^4)^G$, where the action of G on $\mathbb{Z}^4$ is as follows

$$\forall g \in G, \forall z \in \mathbb{Z}^4, \quad gz = \bar{g}z\bar{g}^{-1}.$$

Here $\bar{g} \in \Gamma$ is such that $p(\bar{g}) = g$.

In the almost-flat case we have a classification of all four dimensional oriented manifolds without spin structure [8, page 12]. Let E be a fundamental group of such a manifold. It is well known [2] that there is the following generalization of the short exact sequence (2)

$$0 \to N \to E \to G \to 0,$$

where N is nilpotent group and G has a finite order. In our situation, we can restrict to the case where N is 2- or 3-step nilpotent. In our consideration, (see [2, Theorem 6.4.11]) the first Betti number of E is equal to the first Betti number of some 3-dimensional crystallographic group $Q = E/\sqrt[N]{[N,N]}$, where

$$\sqrt[N]{[N,N]} = \{n \in N \mid n^k \in [N,N] \text{ for some } k \geq 1\}.$$

The group Q is called the underlying crystallographic group of E. In order to obtain our results, we can apply [2, Remark 6.4.16] to the matrices on pages 225–230 of [2]. For example, let us consider the almost Bieberbach group E with holonomy group $\mathbb{Z}_2$ (case 5 from [2, page 171]) with the underlying crystallographic group $Q = C2$. From the definition N is 2-step nilpotent. Hence $C2$ denotes a 3-dimensional crystallographic group. It is easy to see (cf. [11, Table 3]) that $H_1(Q, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}_2^2$ and $b_1(E) = 1$.[1] Hence from (1) $Q_M = 0$. □

Theorem 1. *Let M be any almost-flat oriented 4-manifold different from the torus with intersection form Q_M. Then*

$$Q_M = \begin{cases} 0 & \text{for } b_1(M) = 1 \\ H & \text{for } b_1(M) = 2 \\ 2H & \text{for } b_1(M) = 3 \end{cases} .$$

Proof. From Lemma 1, Q_M is even and from [2], $b_1(M) \leq 3$. Hence it follows from [7, Theorem 3] that $Q_M = nH$. An application of the formula (1) gives us the equation $n = b_1(M) - 1$. □

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A. Szczepański
Institute of Mathematics
University of Gdańsk
ul. Wita Stwosza 57
80-952 Gdańsk
Poland
e-mail: matas@ug.edu.pl

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