Flat manifolds with only finitely many affinities

Gerhard Hiss and Andrzej Szczepański *

1 Introduction

Let X denote a compact, connected, flat Riemannian manifold (flat manifold for short) of dimension n with fundamental group Γ. Then Γ is a Bieberbach group of rank n, i.e., Γ is torsion free and there is a short exact sequence of groups

$$0 \to L \to \Gamma \to G \to 1, \tag{1}$$

where G is finite, the so-called holonomy group of Γ, and L is a free abelian group of rank n. Moreover, L is a maximal abelian subgroup of Γ.

It is known that X is determined by Γ up to affine equivalence [2]. The set $\mathrm{Aff}(X)$ of affine self equivalences of X is a Lie group. Let $\mathrm{Aff}_0(X)$ denote its identity component. Then $\mathrm{Aff}_0(X)$ is a torus whose dimension equals the first Betti number of X, and $\mathrm{Aff}(X)/\mathrm{Aff}_0(X)$ is isomorphic to $\mathrm{Out}(\Gamma)$, the group of outer automorphisms of Γ [2, Chapter V]. In this note we investigate some flat manifolds X for which $\mathrm{Aff}(X)$ is finite. It is natural to ask which finite groups can occur as $\mathrm{Aff}(X)$ for some flat manifold X. In particular, is there a flat manifold whose group of affinities is trivial? Related questions for a larger class of manifolds have been investigated by Malfait in [9].

By the remarks above, $\mathrm{Aff}(X)$ is finite, if and only if $\mathrm{Out}(\Gamma)$ is finite and the first Betti number of X is zero. If some (non-trivial) cyclic Sylow subgroup of G has a normal complement, then by [4, Theorem 0.1], the first Betti number of X is non-zero, in which case $\mathrm{Aff}(X)$ is infinite. In Section 2

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we show that $\mathrm{Aff}(X)$ is non-trivial if G is p-nilpotent, i.e., has a normal p-complement, for some prime p dividing the order of G.

In [6] we gave the following criterion, due independently also to Brown, Neubüser, and Zassenhaus, for a Bieberbach group to have a finite outer automorphism group. The conjugation action of Γ gives L the structure of a $\mathbf{Z}G$-lattice (a $\mathbf{Z}G$-module which is free and finitely generated as abelian group), the so-called translation lattice of Γ. Since Γ is torsion free and L is maximal abelian, G is faithfully represented on L (see [2, III.1]). In other words, there is an injective group homomorphism of G into $\mathrm{GL}(L)$. Put $L^{\mathbf{Q}} := \mathbf{Q} \otimes_{\mathbf{Z}} L$. Then $\mathrm{Out}(\Gamma)$ is finite if and only if $L^{\mathbf{Q}}$ is multiplicity free as a $\mathbf{Q}G$-module, and $\mathbf{R} \otimes_{\mathbf{Q}} V$ is irreducible for every irreducible constituent V of $L^{\mathbf{Q}}$ [6]. The first Betti number of X equals the number of trivial constituents of $L^{\mathbf{Q}}$ (see, e.g., [4, Proposition 1.4]).

It follows from [6, Lemma 2.3] that if X_1 and X_2 are flat manifolds with $\mathrm{Aff}(X_i)$ finite, $i = 1, 2$, then the group of affinities of the product manifold $X_1 \times X_2$ is also finite. In particular, taking the m-fold product of such a flat manifold with itself, we obtain a flat manifold with a finite group of affinities containing the symmetric group on m letters as a subgroup.

To give a more explicit description of $\mathrm{Out}(\Gamma)$ let N denote the normalizer in $\mathrm{GL}(L)$ of G (viewed as a subgroup of $\mathrm{GL}(L)$ via the monomorphism discussed above). Then N acts in a natural way on $H^2(G, L)$. Let $\alpha \in H^2(G, L)$ denote the cohomology class giving rise to the extension (1), and let N_α denote its stabilizer in N. Then G is a normal subgroup of N_α and we have a short exact sequence (see [2, Theorem V.1.1])

$$0 \to H^1(G, L) \to \mathrm{Out}(\Gamma) \to N_\alpha/G \to 1. \tag{2}$$

In order to construct a flat manifold X with trivial group of affinities, one must find a Bieberbach group Γ with $H^1(G, L) = 0$ and $N_\alpha = G$. In Section 2 we shall construct one example where $H^1(G, L) = 0$ and N_α/G is a group with two elements, and another example where $H^1(G, L)$ has two elements and N_α/G is trivial. So far we have not been able to construct a flat manifold X with a trivial group of affinities.

In Section 3 of our paper we compute the group of affinities of the generalized Hantzsche-Wendt manifolds introduced in [10, 13], thus giving, for the first time, examples of $\mathrm{Aff}(X)$ for flat manifolds X with non-cyclic holonomy in any dimension.

Finally, in Section 4, we use the generalized Hantzsche-Wendt manifolds to construct a family of flat manifolds X_n of dimension $2n+1$, whose holonomy groups are certain extraspecial 2-groups of order 2^{2n+1} and with $\mathrm{Aff}(X_n)$ finite for all n.

We close the introduction with a few words on notation. If M is a finite set, $|M|$ denotes the number of its elements. If G is a group acting on a set M, we write M^G for the set of G-fixed points of M. Finally, C_n denotes the cyclic group of order n, and $(C_n)^m$ is the direct product of m copies of C_n.

2 Flat manifolds with few symmetries

Let G be a finite group and M a $\mathbf{Z}G$-lattice of finite rank. We start with the following observation.

Lemma 2.1 *Suppose that $H^0(G, M) = 0$. Let p be a prime. Then p divides $|H^1(G, M)|$ if and only if the $\mathbf{F}_pG$-module M/pM has a trivial submodule.*

In particular $H^1(G, M) = 0$, if and only if M/qM has no trivial submodule for all primes q dividing $|G|$.

Proof: The hypothesis $H^0(G, M) = 0$ yields

$$H^1(G, M) \cong H^0(G, \mathbf{Q} \otimes_{\mathbf{Z}} M/M) \cong (\mathbf{Q} \otimes_{\mathbf{Z}} M/M)^G.$$

Suppose that p divides $|H^1(G, M)|$. Then there exists $0 \neq x \in (\mathbf{Q} \otimes_{\mathbf{Z}} M/M)^G$ such that $px = 0$. We thus have $x = x_0 + M$, $x_0 \in \mathbf{Q} \otimes_{\mathbf{Z}} M$ and $px_0 \in M$. Then $x_0 \in \frac{1}{p}M \subset \mathbf{Q} \otimes_{\mathbf{Z}} M$ and $x_0 \notin M$. Hence M/pM has a trivial submodule. The other direction is proved similarly.

This lemma has some interesting consequences.

Proposition 2.2 *Suppose that G is p-nilpotent, i.e., has a normal p-complement, for some prime p dividing $|G|$. If $H^0(G, M) = 0$ and p divides $|H^2(G, M)|$, then p also divides $|H^1(G, M)|$.*

Proof: We have $H^2(G, \mathbf{Z}_p \otimes_{\mathbf{Z}} M) \neq 0$, since p divides $|H^2(G, M)|$. Let U be an indecomposable direct summand of $\mathbf{Z}_p \otimes_{\mathbf{Z}} M$ which lies in the principal p-block of G. Then every indecomposable direct summand of U/pU lies in the principal p-block of G. Since G is p-nilpotent, U/pU contains a non-trivial vector fixed by G (see, e.g., [3, § 63A]). The result follows from the lemma.

Corollary 2.3 *Let X be a flat manifold whose holonomy group G is p-nilpotent for some prime p dividing $|G|$. Then* $\mathrm{Aff}(X)$ *is non-trivial.*

Proof: Let M denote the translation lattice of the fundamental group of X. If $H^0(G, M) \neq 0$, then $\mathrm{Aff}(X)$ contains a torus, hence is infinite. If $H^0(G, M) = 0$, then $H^1(G, M) \neq 0$ by the above proposition (p divides the order of $H^2(G, M)$, since this cohomology group contains a special element). As $H^1(G, M)$ is isomorphic to a subgroup of $\mathrm{Aff}(X)$, the result follows.

It follows in particular that flat manifolds with nilpotent holonomy have a non-trivial group of affinities. A somewhat weaker version of Corollary 2.3 has also been obtained by Malfait [9, Proposition 5.9]. Moreover, this author conjectures that the assumption on the holonomy group is in fact unnecessary, in other words that $\mathrm{Aff}(X)$ is non-trivial for every flat manifold X ([9, Conjecture 5.12]).

Example 2.4 Let $G = \mathrm{SL}_3(2)$. Then G has a presentation $G = \langle a, b \mid a^2, b^3, (ab)^7, [b, a]^4 \rangle$ (see [7, p. 290]). We use the notation of [7, Chapter 6, Section 11]. In particular, we consider right $\mathbf{Z}G$-modules to make it easier to match our results with the tables in [7].

Let

$$L := L_3^6 \oplus L_8^7 \oplus L_4^8,$$

where L_i^m is the lattice of dimension m (and, in case $m = 6$, with character χ_6), which is denoted by L_i in [7]. It can easily be checked with Lemma 2.1 that $H^1(G, L) = 0$.

We identify G with its image in $\mathrm{GL}(L) \cong \mathrm{GL}_{21}(\mathbf{Z})$. Since L_8^7 is not invariant under the outer automorphism of G, it follows that $N := N_{\mathrm{GL}(L)}(G) = GC_{\mathrm{GL}(L)}(G)$. We have $C_{\mathrm{GL}(L)}(G) \cong (C_2)^3$.

Let $\alpha \in H^2(G, L_3^6)$ be defined by the 1-cocycle $\delta \in Z^1(G, \mathbf{Q} \otimes_{\mathbf{Z}} L_3^6/L_3^6)$ determined by

$$\delta(a) = \frac{1}{2}[0, 1, 0, 0, 0, 0], \qquad \delta(b) = 0.$$

Then α has order 2.

Let $\beta \in H^2(G, L_8^7)$ be defined by the 1-cocycle $\delta \in Z^1(G, \mathbf{Q} \otimes_{\mathbf{Z}} L_8^7/L_8^7)$ determined by

$$\delta(a) = \frac{1}{3}[2, 0, 0, 0, 0, 0, 0], \quad \delta(b) = \frac{1}{3}[0, 0, 0, 0, 0, 0, 1].$$

Then β has order 3.

Finally, let $\gamma \in H^2(G, L_4^8)$ be defined by the 1-cocycle $\delta \in Z^1(G, \mathbf{Q} \otimes_{\mathbf{Z}} L_4^8/L_4^8)$ determined by

$$\delta(a) = \frac{1}{7}[0,2,0,5,0,0,0,5], \quad \delta(b) = \frac{1}{7}[6,0,0,0,0,0,0,0].$$

Then γ has order 7.

Put $\sigma := \alpha + \beta + \gamma \in H^2(G, L)$. Then σ is a special cocycle since $\mathrm{res}^G_{\langle a \rangle}(\alpha) \neq 0$, $\mathrm{res}^G_{\langle b \rangle}(\beta) \neq 0$ and $\mathrm{res}^G_{\langle ab \rangle}(\gamma) \neq 0$. It is clear that $N_\sigma = GC$, where $C = \langle -\mathrm{id}_{L_3^6} + \mathrm{id}_{L_8^7} + \mathrm{id}_{L_4^8} \rangle \leq C_{\mathrm{GL}(L)}(G)$.

Let Γ denote the Bieberbach group obtained by extending L by G with the cocycle σ, and let X be the flat manifold with fundamental group Γ. Then $\mathrm{Aff}(X) \cong C$ is a group of order 2.

Example 2.5 Let $\tilde{G} = \mathrm{SL}_3(2).2$, the automorphism group of G. Then $\tilde{G}$ has a presentation $\tilde{G} = \langle a, b, c \mid a^2, b^3, c^2, (ab)^7, [b,a]^4, [a,c], (acb^2)^2 \rangle$.

Let

$$\tilde{L} := \tilde{L}_4^7 \oplus \tilde{L}_{10}^7 \oplus \tilde{L}_4^8,$$

where $\tilde{L}_i^m$ is an extension to $\tilde{G}$ of the $\mathbf{Z}G$-lattice L_i^m (with the notation of Example 2.4). More precisely, the trace of c on these lattices equals -1, 1, -2, in the respective cases.

We find $H^1(\tilde{G}, \tilde{L}_4^7) = C_2$ and $H^1(\tilde{G}, \tilde{L}_{10}^7) = 0 = H^1(\tilde{G}, \tilde{L}_4^8)$. Thus $H^1(\tilde{G}, \tilde{L}) = C_2$.

Let $\alpha \in H^2(\tilde{G}, L_4^7)$ be defined by the 1-cocycle $\delta \in Z^1(\tilde{G}, \mathbf{Q} \otimes_{\mathbf{Z}} L_4^7/L_4^7)$ determined by

$$\delta(a) = \frac{1}{6}[3,5,3,3,3,3,0,1],$$

$$\delta(b) = \frac{1}{6}[0,4,2,2,0,0,0],$$

$$\delta(c) = \frac{1}{6}[3,3,0,3,0,0,0].$$

Then α has order 6.

Let $\beta \in H^2(\tilde{G}, L_{10}^7)$ be defined by the 1-cocycle $\delta \in Z^1(\tilde{G}, \mathbf{Q} \otimes_{\mathbf{Z}} L_{10}^7/L_{10}^7)$ determined by

$$\delta(a) = \frac{1}{12}[6,0,0,6,6,0,6],$$

$$\delta(b) = \frac{1}{12}[0, 8, 0, 0, 0, 1, 1],$$

$$\delta(c) = \frac{1}{12}[2, 8, 8, 2, 6, 9, 9].$$

Then β has order 12.

Let $\gamma \in H^2(\tilde{G}, L_4^8)$ be defined by the 1-cocycle $\delta \in Z^1(\tilde{G}, \mathbf{Q} \otimes_{\mathbf{Z}} L_4^8/L_4^8)$ determined by

$$\delta(a) = \frac{1}{7}[0, 3, 0, 4, 0, 0, 5, 2],$$

$$\delta(b) = \frac{1}{7}[0, 5, 0, 0, 5, 0, 0, 0],$$

$$\delta(c) = \frac{1}{7}[0, 0, 0, 0, 3, 3, 0, 0].$$

Then γ has order 7.

Let $\sigma := \alpha + \beta + \gamma$. Then σ is special in $H^2(\tilde{G}, \tilde{L})$, since $\operatorname{res}^{\tilde{G}}_{\langle a \rangle}(\alpha) \neq 0$, $\operatorname{res}^{\tilde{G}}_{\langle b \rangle}(\alpha) \neq 0$, $\operatorname{res}^{\tilde{G}}_{\langle c \rangle}(\beta) \neq 0$ and $\operatorname{res}^{\tilde{G}}_{\langle ab \rangle}(\gamma) \neq 0$.

Since $\operatorname{Aut}(\tilde{G}) = \tilde{G}$, we have $N_{\mathrm{GL}(\tilde{L})}(\tilde{G}) = \tilde{G} C_{\mathrm{GL}(\tilde{L})}(\tilde{G})$. No non-trivial element of $C_{\mathrm{GL}(\tilde{L})}(\tilde{G}) \cong (C_2)^3$ fixes σ. Hence $N_\sigma = \tilde{G}$.

Let $\tilde{\Gamma}$ denote the extension of $\tilde{L}$ by $\tilde{G}$ determined by σ, and let $\tilde{X}$ be the flat manifold with fundamental group $\tilde{\Gamma}$. Then $\operatorname{Aff}(\tilde{X})$ is a group of order 2.

The computations in these examples have been performed with Maple [1] and GAP [12].

3 Generalized Hantzsche-Wendt manifolds

In this section we shall calculate the group $\operatorname{Out}(\Gamma_{2n})$ where Γ_{2n} is the fundamental group of the generalized Hantzsche-Wendt manifold of dimension $2n+1$ introduced in [10, 13].

Let us recall the definition of Γ_{2n}, which we shall call Hantzsche-Wendt group for short. We denote by a_i, $1 \le i \le 2n+1$, the $(2n+1) \times (2n+1)$-diagonal matrices over $\mathbf{Z}$ with diagonal entries 1 on position i and -1 on the other positions (see [13, p. 292]). Let A be the subgroup of $\mathrm{GL}_{2n+1}(\mathbf{Z})$ generated by a_i, $i = 1, 2, \ldots, 2n$. Let $L = \mathbf{Z}^{2n+1}$ with standard (column) basis vectors $u_1, u_2, \dots, u_{2n+1}$. We view L as a left $\mathbf{Z}A$-lattice. The Hantzsche-Wendt group Γ_{2n} is an extension of L by A. This extension is given by the element

$$[s] \in H^1(A, \mathbf{Q} \otimes_{\mathbf{Z}} L/L) \cong H^2(A, L),$$

represented by the 1-cocycle $s \in Z^1(A, \mathbf{Q} \otimes_{\mathbf{Z}} L/L)$ with

$$s(a_i) = \frac{1}{2}(u_i + u_{i+1}) + L, \qquad 1 \leq i \leq 2n \tag{3}$$

(see [13, p. 294]). We remark that Maxwell has also defined a Bieberbach group with holonomy group A and $\mathbf{Z}A$-lattice L in Part (e) of the proof of [10, Proposition 6]. It is not difficult to see that Maxwell's cocycle is cohomologous to the one in (3).

We shall use (2) to calculate $\mathrm{Out}(\Gamma_{2n})$. First we must find the normalizer $N = N_{\mathrm{GL}_{2n+1}(\mathbf{Z})}(A)$ of the holonomy group $A \cong (C_2)^{2n}$ in $\mathrm{GL}_{2n+1}(\mathbf{Z})$. By immediate calculations one can prove the following lemma.

Lemma 3.1 *Let $S_{2n+1} \leq \mathrm{GL}_{2n+1}(\mathbf{Z})$ denote the group of permutation matrices, and let $\tilde{A} = \langle -I_{2n+1}, A \rangle$, where I_{2n+1} denotes the identity matrix. Then $N = \tilde{A}S_{2n+1}$. In particular, $N \cong C_2 \wr S_{2n+1}$, the wreath product of C_2 and S_{2n+1}.*

Example 3.2 If $n = 1$, then the cohomology class defined by (3) is the only (up to conjugation by N) special element of $H^2(A, L)$.

However, for $n = 2$, there are exactly two N-orbits of special elements in $H^2(A, L)$. Let $t : A \to \mathbf{Q} \otimes_{\mathbf{Z}} L/L$ be the 1-cocycle defined by

$$t(a_1) = \frac{1}{2}(u_1 + u_3) + L,$$

$$t(a_2) = \frac{1}{2}u_2 + L,$$

$$t(a_3) = \frac{1}{2}(u_3 + u_4) + L,$$

$$t(a_4) = \frac{1}{2}(u_4 + u_5) + L,$$

and let $[t]$ denote the corresponding cohomology class in $H^2(A, L)$. Then it is easily checked that $[t]$ is special. Moreover, the stabilizer of $[t]$ in S_5 is trivial.

We shall show below that this is not the case for $[s]$, so that $[s]$ and $[t]$ are not in the same N-orbit. Finally, it is not hard to prove that there are no other N-orbits containing special cocycles.

We expect that the number of N-orbits of special cocycles in $H^2(A, L)$ grows as n grows. We have not attempted to classify these special orbits.

Let us recall the definition of the action $*$ of N on $H^1(A, \mathbf{Q} \otimes_{\mathbf{Z}} L/L)$. Let $n \in N$ and let $c : A \to \mathbf{Q} \otimes_{\mathbf{Z}} L/L$ represent a cohomology class from $H^1(A, \mathbf{Q} \otimes_{\mathbf{Z}} L/L)$. Then

$$(n * c)(x) = nc(n^{-1}xn)$$

for $x \in A$. We shall prove:

Lemma 3.3 *For $n \geq 2, N_{[s]} := \{n \in N \mid n * [s] = [s]\} = \tilde{A}F$, where F is the cyclic subgroup of S_{2n+1} generated by the $(2n+1)$-cycle $(1, 2, \ldots, 2n, 2n+1)$. For $n = 1$, $N_{[s]} = N$.*

Proof: It is obvious that $n * s = s$ for $n \in \tilde{A}$. Let $\sigma \in S_{2n+1}$. From $\sigma^{-1} a_i \sigma = a_{\sigma^{-1}(i)}$ we obtain

$$\sigma * s(a_i) = \sigma(s(a_{\sigma^{-1}(i)})) = \frac{1}{2}(u_i + u_{\sigma(\sigma^{-1}(i)+1)}). \tag{4}$$

(Here, and in the remainder of the proof, the subscripts have to be read modulo $2n + 1$.) Note that $\sigma \in F$ if and only if $\sigma(i) + 1 = \sigma(i + 1)$ for all $1 \leq i \leq 2n + 1$. Thus (4) shows that $F \leq N_{[s]}$.

We shall prove that $\sigma \notin N_{[s]}$ for $\sigma \notin F$ and $n \geq 2$. Let $\sigma \notin F$ and suppose that $\sigma \in N_{[s]}$. Then there is an element $v = \sum_{i=1}^{2n+1} v_i u_i \in \mathbf{Q} \otimes_{\mathbf{Z}} L/L \cong (\mathbf{Q}/\mathbf{Z})^{2n+1}$ such that

$$(\sigma * s - s)(a_i) = \frac{1}{2}(u_{\sigma(\sigma^{-1}(i)+1)} - u_{i+1}) = (a_i - I_{2n+1})v, \qquad 1 \leq i \leq 2n+1. \tag{5}$$

Since $\sigma^{-1} \notin F$, there is an i such that $\sigma(\sigma^{-1}(i) + 1) \neq i + 1$. Therefore, $(\sigma * s - s)(a_i) \neq 0$ and $v_{i+1} = 1/4 \in \mathbf{Q}/\mathbf{Z}$. Let $1 \leq j \leq 2n + 1$, $j \neq i + 1$. Then the $(i+1)$st component of $(a_j - I_{2n+1})v$ equals $1/2 \in \mathbf{Q}/\mathbf{Z}$. Equation (5) implies that $\sigma(\sigma^{-1}(j) + 1) = i + 1$ or $j + 1 = i + 1$. Thus $2n + 1 \leq 3$, i.e., $n \leq 1$. It is easy to see that $N_{[s]} = N$ if $n = 1$.

From the definition and the properties of the holonomy representation we conclude that

$$H^1(A, L) \simeq (C_2)^{2n+1}.$$

We describe the action of $N_{[s]}/A = \langle -I_{2n+1} \rangle \times F$ on $H^1(A, L)$ following [2, Example V.6.1]. F acts by permuting the direct factors, and $\langle -I_{2n+1} \rangle$ acts trivially. We can now formulate the main result of this section. It follows from Lemmas 3.1 and 3.3. (For $n = 1$, see [5, pp. 128,129] and [15, pp. 321–323].)

Theorem 3.4 *Let M^{2n+1}, $n \geq 2$, be the generalized Hantzsche-Wendt flat manifold of dimension $2n + 1$. Then $\mathrm{Aff}(M^{2n+1})$ is a split extension of $H^1(A, L)$ and $\langle -I_{2n+1} \rangle \times F$, i.e., it is isomorphic to $C_2 \times (C_2 \wr F)$. The subgroup of $\mathrm{Aff}(M^{2n+1})$ which preserves orientation is isomorphic to $C_2 \wr F$.*

Proof: Lemma 3.3 and the sequence (2) show that $\mathrm{Aff}(M^{2n+1})$ is an extension of $H^1(A, L)$ by $\langle -I_{2n+1} \rangle \times F$. Since $|F|$ is odd, the extension $H^1(A, L)$ by F splits, so $\mathrm{Aff}(M^{2n+1})$ has a subgroup B of index 2 isomorphic to $C_2 \wr F$.

We embed Γ_{2n} in the usual way into the group $\mathcal{A}_{2n+1}$ of affine motions of $\mathbf{R}^{2n+1}$. Then $M^{2n+1} = \mathbf{R}^{2n+1}/\Gamma_{2n}$ and $\mathrm{Aff}(M^{2n+1}) \cong N_{\mathcal{A}_{2n+1}}(\Gamma_{2n})/\Gamma_{2n}$ (see [2, Lemma V.6.1]). It is clear that B can be represented by elements of $\mathcal{A}_{2n+1}$ whose linear part has determinant 1. Thus $-I_{2n+1}$, having determinant -1, gives rise to a non-trivial element of $\mathrm{Aff}(M^{2n+1})$ not lying in B. It also follows that B is equal to the subgroup of orientation preserving elements of $\mathrm{Aff}(M^{2n+1})$.

4 Extra-special 2-groups

We continue with the notation of the previous section. Let T_{2n} be the sublattice of L generated by

$$2u_1, u_1 - u_2, u_2 - u_3, \ldots, u_{2n} - u_{2n+1}.$$

Then T_{2n} has index 2 in L and the group $\mathcal{G}_{2n} := \Gamma_{2n}/T_{2n}$ is a finite group of order 2^{2n+1}. It is proved in [11] that $\mathcal{G}_{2n}$ is extraspecial (of type varying with n). We can use the construction of the generalized Hantzsche-Wendt manifolds to prove that the extraspecial groups $\mathcal{G}_{2n}$ occur as holonomy groups of Bieberbach groups with finite outer automorphism groups. Such groups were called $\mathfrak{R}_1$-groups in [6].

Proposition 4.1 *For each $n \geq 1$, $\mathcal{G}_{2n}$ is an $\Re_1$-group.*

Proof: It is easy to give a list of all irreducible complex representations of the group $\mathcal{G}_{2n}$. There are 2^{2n} 1-dimensional representations arising from the commutator factor group and one faithful representation of dimension 2^n (see [8, Problem 2.13]). If $\mathcal{G}_{2n}$ is a central product of dihedral groups of order 8, the complex irreducible representation of degree 2^n can be realized over $\mathbf{Q}$. Otherwise, it has Frobenius-Schur indicator -1, and thus can not be realized over $\mathbf{R}$. It follows that any $\mathbf{Q}$-irreducible representation of $\mathcal{G}_{2n}$ is $\mathbf{R}$-irreducible.

Let K be a faithful integral representation of $\mathcal{G}_{2n}$ which, as a rational representation, is multiplicity free and does not contain a 1-dimensional direct summand. Then the $\mathbf{Z}\mathcal{G}_{2n}$-module $T_{2n} \oplus K$ is a translation lattice for a Bieberbach group with finite outer automorphism group and holonomy $\mathcal{G}_{2n}$.

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IWR der Universität Heidelberg
Im Neuenheimer Feld 368
69120 Heidelberg
Germany
e-mail: hiss@euterpe.iwr.uni-heidelberg.de

and

Institute of Mathematics
University of Gdańsk
ul. Wita Stwosza 57
80 - 952 Gdańsk
Poland
e-mail: matas@univ.gda.pl