Kähler flat manifolds

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October 15, 2008

Abstract

Using a criterion of Johnson-Rees [9] we give a list of all four and six dimensional flat Kähler manifolds. We calculate their $\mathbb{R}$–cohomology, including the Hodge numbers. As a corollary, we classify all flat complex manifolds of dimension 3 whose holonomy groups are subgroups of $SU(3)$. Moreover, we define a family of flat Kähler manifolds which are generalizations of the oriented Hantzsche-Wendt Riemannian manifolds [14].

Key Words and Phrases: Bieberbach group, flat manifold, Kähler manifold, Hantzsche - Wendt manifold, Hodge diamond, hyperelliptic variety.

2000 Mathematics Subject Classification. Primary 57N16; Secondary 20H15, 14J32.

1 Introduction

We shall present a complete list of flat manifolds of $\mathbb{R}$-dimension 4 and 6 which have a complex Kähler structure of dimension 2 and 3 correspondingly. We compare it with the classification of hyperelliptic varietes of low dimensions. Moreover, we shall calculate the real cohomology of such objects, including their Hodge numbers. Hence we obtain the flat Kähler manifolds with holonomy groups enclosed in the special unitary group. Inspired by the recently introduced and in the meantime well studied class of generalized Hantzsche-Wendt manifolds we define an infinite family of complex flat Kähler manifolds, compute their Hodge numbers and answer positively the question about the existence of a spin structure on such a manifold.
This work is an extended and modified version of [16].

Let us introduce the basic definitions and conventions. A closed flat Riemannian manifold M is isometric to one of the form $M = \Gamma\backslash E(n)/O(n)$ where $E(n) = O(n) \ltimes \mathbb{R}^n$ is the group of Euclidean motions of $\mathbb{R}^n$ and Γ is a cocompact, discrete and torsion free subgroup of $E(n)$. ¿From the Bieberbach theorems it is well known that (cf. [1]) $\pi_1(M) = \Gamma$ and the subgroup T of Γ consisting of all pure translations is of finite index and the quotient group Γ/T is isomorphic to holonomy group of M. Hence we have a short exact sequence of groups

$$0 \to T \to \Gamma \to H \to 0,$$

where T is a torsion free maximal abelian group $\mathbb{Z}^n$. Conjugation inside Γ, the above short exact sequence defines a faithful (cf. [1]) holonomy representation $\varphi : H \to GL(n, \mathbb{Z})$. We shall call such a group Γ a Bieberbach group.

*Supported by Polish grant number 20100831/0524

Now, let us assume that n is an even number. We say that φ is essentially complex if there exist a real vector space isomorphism $i : \mathbb{R}^n \to \mathbb{C}^{\frac{n}{2}}$ and a representation $\phi_{\mathbb{C}} : H \to GL(\frac{n}{2}, \mathbb{C})$ such that the following diagram commutes for each $h \in H$:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\cong", "i"'] \arrow[d, "\varphi(h)"'] & \mathbb{C}^{\frac{n}{2}} \arrow[d, "\phi_{\mathbb{C}}(h)"] \\
\mathbb{R}^n \arrow[r, "i", "\cong"'] & \mathbb{C}^{\frac{n}{2}}
\end{tikzcd}$$

Equivalently, this means that there exist a φ–invariant linear map $t : \mathbb{R}^n \to \mathbb{R}^n$, such that $t^2 = -id$, (see [10, part 3]). In [9, Theorem 3.1] the following is proved

Theorem 1.1 *The following conditions on the group $\Gamma \subset E(n)$ are equivalent*
(i) Γ is the fundamental group of Kähler flat manifold
(ii) Γ is a Bieberbach group and its holonomy representation is essentially complex
(iii) Γ is a discrete cocompact torsion-free subgroup of $U(\frac{n}{2}) \ltimes \mathbb{C}^{\frac{n}{2}}$.

The following characterization of an essentially complex representation is given in ([9, Proposition 3.2], [10, Proposition 3.1])

Proposition 1.2 *Let H be a finite group and $\varphi : H \to GL(m, \mathbb{R})$ be some representation. Then φ is essentially complex if and only if m is even and each $\mathbb{R}$-irreducible summand of φ which is also $\mathbb{C}$-irreducible occurs with even multiplicity.*

Definition 1.3 *([12, page 495]) A hyperelliptic variety is a complex projective variety, not isomorphic to an abelian variety, but admitting an abelian variety as a finite covering.*

It is proved in [10] that the class of fundamental groups of complex flat manifolds (with exception of the complex torus) and hyperelliptic varieties coincide. However, in dimension three there are nonalgebraic Kähler flat manifolds. An example of such manifold is given in [12, page 495, page 501 Remark 3.9].

In the next part, using the above results, we shall give a list of the Kähler flat manifolds of $\mathbb{R}$-dimension 4 and 6.

2 Kähler flat manifolds in low dimensions

Before we start our investigation of Kähler flat manifolds in low dimensions, we first prove a lemma providing constraints on the possible holonomy groups of such manifolds.

Lemma 2.1 *Let $\mathbb{Z}_2^k$ be the holonomy group of an n–dimensional, complex flat Kähler manifold. Then $k \leq n - 1$.*

Proof: Let $\varphi : \mathbb{Z}_2^k \to GL(2n, \mathbb{R})$ denote the realization of the holonomy representation. Then, seen as a $\mathbb{Z}_2^k$–module, $\mathbb{R}^{2n}$ can be written as a direct sum

$$\mathbb{R}^{2n} = V_1^{l_1} \oplus V_2^{l_2} \oplus \cdots \oplus V_m^{l_m},$$

where each V_i is an $\mathbb{R}$-irreducible $\mathbb{Z}_2^k$–module and V_i and V_j are not equivalent if $i \neq j$. Of course, as a vector space, each of the $V_i = \mathbb{R}$ and the corresponding representation $\mathbb{Z}_2^k \to$

$GL(\mathbb{R})$ has its image lying inside $\{1,-1\} \cong \mathbb{Z}_2$. Note that the $\mathbb{R}$-irreducible components are also $\mathbb{C}$–irreducible and hence all l_i are even numbers (Proposition 1.2).

It follows that, because the holonomy representation φ is faithful, m must be at least k. We can also exclude the case where $m = k$. For if $m = k$, the case where V_i is the trivial module does not occur. It follows that we can find for any i an element $a_i \in \mathbb{Z}_2^k$ acting as -1 on V_i and as $+1$ on the other components. This would imply that the element $a_1 a_2 \cdots a_m$ acts as -1 on the total space, which is impossible. So we have that $k+1 \leq m$.

Finally, as each of l_i is even, we have that the real dimension of the manifold is at least $2m$ or the complex dimension $n \geq m$, which finishes the proof. □

Lemma 2.2 *In complex dimension 2, the only groups appearing as a holonomy group of a Kähler flat manifold are* 1, $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ *and* $\mathbb{Z}_6$. *In fact there are exactly eight Kähler flat manifolds in dimension 2.*

Proof: Looking at the classification (cf. [2], [3] and [6]) of flat manifolds in real dimension 4, one sees that the groups occuring as a holonomy group of a 4-dimensional flat manifold are

$$1,\ \mathbb{Z}_2,\ \mathbb{Z}_2^2,\ \mathbb{Z}_2^3,\ \mathbb{Z}_3,\ \mathbb{Z}_6,\ \mathbb{Z}_2\times\mathbb{Z}_6,\ \mathbb{Z}_4,\ \mathbb{Z}_2\times\mathbb{Z}_4,\ D_8,\ D_6,\ \mathbb{Z}_2\times D_6,$$

where D_n is a finite dihedral group of order n. As all of the groups $\mathbb{Z}_2^2$, $\mathbb{Z}_2^3$, $\mathbb{Z}_2\times\mathbb{Z}_6, \mathbb{Z}_2\times\mathbb{Z}_4$, D_8 and $\mathbb{Z}_2 \times D_6$ contain a subgroup which is isomorphic to $\mathbb{Z}_2^2$, we deduce from Lemma 2.1 that those groups cannot occur as the holonomy group of a 2-dimensional Kähler flat manifold.

There are three flat manifolds in dimension 4 having D_6 as their holonomy group. It is however easy to see that all of them have first Betti number one, so that we can exclude this group too. For the rest of the possible holonomy groups, there are flat manifolds supporting a Kähler structure. Going through the list of all such groups, we find the following table of 2 dimensional Kähler flat manifolds.

holonomy	CARAT symbols
1	15.1.1
$\mathbb{Z}_2$	18.1.1; 18.1.2
$\mathbb{Z}_3$	35.1.1; 35.1.2
$\mathbb{Z}_4$	25.1.2; 27.1.1
$\mathbb{Z}_6$	70.1.1

□

In the case of $\mathbb{R}$-dimension 6 we have more cases.

Lemma 2.3 *The following finite groups occur as holonomy groups of a three - dimensional Kähler flat manifold:* 1, $\mathbb{Z}_n$, *for* $n = 2,3,4,5,6,8,10,12$, $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_4$, $\mathbb{Z}_3\times\mathbb{Z}_3$, $\mathbb{Z}_6\times\mathbb{Z}_2$, $\mathbb{Z}_4\times\mathbb{Z}_4$, $\mathbb{Z}_6\times\mathbb{Z}_3, \mathbb{Z}_6\times\mathbb{Z}_4, \mathbb{Z}_6\times\mathbb{Z}_6, D_8$.

Proof: In [6], [3] a list of all holonomy groups of six-dimensional flat manifolds is given. We use the notation of [3].

We shall now go through the list of all finite groups appearing as the holonomy group of a 6-dimensional flat manifold. We first remark that all the groups

$$[64,250], [32,47], [32,46], [32,36], [32,33], [24,11], \mathbb{Z}_2\times[16,9], [16,9],$$

$$\mathbb{Z}_2 \times D_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times D_8, \mathbb{Z}_4 \times D_8, \mathbb{Z}_2 \times \mathbb{Z}_2 \times A_4, \mathbb{Z}_2 \times A_4,$$
$$\mathbb{Z}_2^2 \times \mathbb{Z}_4, \mathbb{Z}_2^3 \times \mathbb{Z}_4, \mathbb{Z}_2^3 \times \mathbb{Z}_3, \mathbb{Z}_2^4 \times \mathbb{Z}_3, \mathbb{Z}_2^n, n = 3, 4, 5,$$

have a group $(\mathbb{Z}_2)^3$ as a subgroup, hence by Lemma 2.1 they can be eliminated.
Moreover the finite groups

$$\mathbb{Z}_2 \times [80, 52], [80, 52], [32, 31], [16, 11], \mathbb{Z}_2 \times D_{10}, D_{10}, \mathbb{Z}_6 \times D_8, \mathbb{Z}_3 \times D_8,$$
$$\mathbb{Z}_2 \times (\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2), \mathbb{Z}_3^2 \rtimes \mathbb{Z}_2, \mathbb{Z}_3 \rtimes \mathbb{Z}_8, \mathbb{Z}_3 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4), \mathbb{Z}_2 \times (\mathbb{Z}_3 \rtimes \mathbb{Z}_4),$$
$$\mathbb{Z}_3 \rtimes \mathbb{Z}_4, \mathbb{Z}_3 \times Q_8, Q_8, \mathbb{Z}_2 \times \mathbb{Z}_4 \times A_4, \mathbb{Z}_6 \times A_4, \mathbb{Z}_3 \times A_4, \mathbb{Z}_2^3 \times A_4,$$
$$\mathbb{Z}_3 \times S_4, \mathbb{Z}_2 \times S_4, D_6^2, \mathbb{Z}_2 \times \mathbb{Z}_4 \times D_6, \mathbb{Z}_4 \times D_6, \mathbb{Z}_2^3 \times D_6, \mathbb{Z}_3 \times D_8,$$
$$\mathbb{Z}_6 \times D_8, \mathbb{Z}_2 \times \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_{10}, \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_6, \mathbb{Z}_4 \times A_4, \mathbb{Z}_2 \times D_{24}$$

only occur as holonomy groups of flat manifolds with first Betti number one. By Proposition 1.2 these can be eliminated too.

Let $M \cong \mathbb{Z}^6$ be any faithful D_6–module which is essentialy complex and where the D_6-sublattice M^{D_6} is of rank 2. Then using similar methods as in the proof of Proposition 1 in [15, p. 192] and properties of the group $H^2(D_6, \mathbb{Z})$ we can show that any cocycle $\alpha \in H^2(D_6, M)$ is mapped to zero by homomorphism $res^{D_6}_{<x>} : H^2(D_6, M) \to H^2(< x >, M)$, where x is an element of order three. Hence we can eliminate the groups:

$$D_6, \mathbb{Z}_2 \times D_6, \mathbb{Z}_3 \times D_6, D_{24}, S_4, \mathbb{Z}_6 \times D_6, \mathbb{Z}_2 \times \mathbb{Z}_2 \times D_6.$$

There exists one Kähler flat manifold with holonomy group D_8 and first Betti number equal to zero. It has in CARAT notations symbol 207.1.1. As a subgroup of $E(6)$, it is generated by the following elements

$$(I, (0, 0, 1, 0, 0, 0)),\ (I, (0, 0, 0, 1, 0, 0)),\ (I, (0, 0, 0, 0, 1, 0)),$$
$$(I, (0, 0, 0, 0, 0, 1)),\ (A_1, (1/2, 0, 0, 0, 1/4, 0)),\ (A_2, (0, 1/2, 0, 0, 0, 0)),$$

where I denotes identity 6×6 matrix and

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

Moreover it is known [7] that any flat manifold with holonomy group D_6 has a non zero first Betti number.

The group A_4 has one absolutely irreducible faithful representation of rank 3. Hence, by Proposition 1.2 it cannot be on our list (there are no 6-dimensional flat manifolds with holonomy group A_4 and with first Betti number 0).

Let us now consider the group of order sixteen which is refered to as $[16, 10]$ in the notations of [3] and which we have not yet considered. It is the holonomy group of 31 Bieberbach groups of rank 6 with non trivial center. We can prove that all of them have the first Betti number one (cf. [15, Lemma 1, page 194].) Moreover it is also the holonomy group of 3 Bieberbach groups of rank 6 with trivial center. In this case it is easy to see, for example from elementary representation theory, that the conditions of Proposition 1 are not satisfied. By an analogous procedure we can eliminate the groups $[16.8], [16, 13]$ and D_{16} completing the proof. $\square$

Finally, we have:

Theorem 2.4 *There are 174 3 dimensional Kähler flat manifolds.*

Proof: We shall use the results about the holonomy groups proved in Lemma 2.3 and the list of six dimensional Bieberbach group from CARAT, [6], [3]. To prepare the final list we shall use mainly Proposition 1.2. $\square$

Let us present a final table.

holonomy	number	CARAT symbols and β_1
1	1	$\beta_1 = 6$, 170.1.1,
$\mathbb{Z}_2$	5	$\beta_1 = 2$, 174.1.1, 174.1.2, $\beta_1 = 4$, 173.1.1, 173.1.2, 173.1.3,
$\mathbb{Z}_3$	4	$\beta_1 = 2$, 291.1.1, 291.1.2, $\beta_1 = 4$, 311.1.1, 311.1.2,
$\mathbb{Z}_4$	22	$\beta_1 = 2$, 202.1.1, 202.1.2, 225.1.1, 225.1.10, 225.1.11, 225.1.12 (2 groups), 225.1.13, 225.1.2, 225.1.3, 225.1.4 (2 groups), 225.1.5, 225.1.6 (2 groups), 225.1.7 (2 groups), 225.1.8 (2 groups), 225.1.9, $\beta_1 = 4$, 219.1.1, 219.1.2
$\mathbb{Z}_5$	2	$\beta_1 = 2$, 626.1.1, 626,1,2
$\mathbb{Z}_6$	14	$\beta_1 = 2$, 1611.1.1, 318.1.1, 318.1.2, 318.1.3, 318.1.5, 319.1.1, 319.1.2, 319.1.3, 319.1.5, 404.1.1, 404.1.2, 404.1.3, 404.1.4 $\beta_1 = 4$, 1694.1.1
$\mathbb{Z}_8$	1	$\beta_1 = 2$, 468.1.1
$\mathbb{Z}_{10}$	1	$\beta_1 = 2$, 7093.1.1
$\mathbb{Z}_{12}$	6	$\beta_1 = 2$, 359.1.1, 359.1.3, 359.1.4, 361.1.1, 361.1.2, 554.1.1
$\mathbb{Z}_2 \times \mathbb{Z}_2$	33	$\beta_1 = 0$, 185.1 (4 groups), $\beta_1 = 2$, 186.1 (29 groups)
$\mathbb{Z}_2 \times \mathbb{Z}_4$	45	$\beta_1 = 2$, 257.1 (19 groups), 1135.1 (26 groups)
$\mathbb{Z}_3 \times \mathbb{Z}_3$	13	$\beta_1 = 2$, 405.1.1 (5 groups), 405.1.2 (3 groups), 405.1.3 (3 groups), 405.1.4, 405.1.5
$\mathbb{Z}_6 \times \mathbb{Z}_2$	7	$\beta_1 = 2$, 1732.1 (5 groups), 2701.1 (2 groups)
$\mathbb{Z}_4 \times \mathbb{Z}_4$	8	$\beta_1 = 2$, 1264.1 (8 groups)
$\mathbb{Z}_6 \times \mathbb{Z}_3$	6	$\beta_1 = 2$, 2719.1 (2 groups), 2720.1 (4 groups)
$\mathbb{Z}_6 \times \mathbb{Z}_4$	4	$\beta_1 = 2$, 1920.1 (4 groups)
$\mathbb{Z}_6 \times \mathbb{Z}_6$	1	$\beta_1 = 2$, 2752.1
D_8	1	$\beta_1 = 0$, 207.1.1

It is interesting to compare the above classification with the classification of the hyperelliptic varieties, [12]. There is crucial difference. It is the case of the group D_8 which is not present on the list in [12, Theorem 6.1].

Lemma 2.3 should be also compared with [17]. The main theorem of this work contains a list of possible finite quotients G/G_0 where G is cocompact group of affine transformations acting freely and properly discontinuously on $\mathbb{C}^3$, and G_0 is its normal subgroup consisting of translations. In fact, it is a list of holonomy groups of Kähler flat manifolds and contains all groups occuring in Lemma 2.3. It contains also the dihedral group D_8 of order 8.

We want to say that the methods in [12] and [17] are different from ours.

Finally we would like to mention that in [10] it has been observed that using "the double" construction it is possible to construct for any finite group G, a Kähler flat manifold with holonomy group G.

3 The Hodge diamond for Kähler flat manifolds

In this section we shall show how to compute the real cohomology and Hodge numbers for any flat Kähler manifold. We shall explicitely list all possible Hodge diamonds up to complex dimension 3. We shall continue this study in the next section where we shall be dealing with a general class of flat Kähler manifolds in arbitrary high dimensions.

Any flat Kähler complex n-dimensional manifold M is a quotient of the form T^{2n}/H, where T^{2n} is a real $2n$-dimensional torus and $H \subset U(n)$ is a finite group. From the standard observations we have:

$$H^{p.q}(M) = (\Lambda^{p,q}(\mathbb{C}^n \oplus (\mathbb{C}^n)^*))^H,$$

where $H^{*,*}$ denotes the Hodge cohomology. Recall that $\Lambda^{p,q}(\mathbb{C}^n \oplus (\mathbb{C}^n)^*)$ is the vector space with basis elements

$$dz_{i_1} \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \wedge d\bar{z}_{j_q}, \quad \begin{array}{l} 1 \le i_1 < i_2 < \cdots < i_p \le n, \\ 1 \le j_1 < j_2 < \cdots < j_q \le n \end{array} \tag{1}$$

on which the action of H is induced by the holonomy representation $H \to U(n)$. For a given flat manifold M, such a representation is unique up to conjugation in $GL(2n, \mathbb{R})$, however, sometimes it is not unique up to conjugation in $GL(n, \mathbb{C})$. When this is the case, such a flat Riemannian manifold M carries different kinds of complex structures, with possibly different Hodge numbers. For example, we will see that every complex 3-dimensional Kähler flat manifold with $\beta_1 = 2$ and $\beta_2 = 5$ has two different complex structures leading to different Hodge numbers, (see the example below). We can also calculate the Betti numbers directly from the holonomy representation $G \to GL(2n, \mathbb{R})$, using the equation: $\beta_i(M) = \dim(\Lambda^i(\mathbb{R}^{2n}))^G$. Let us present the table of Kähler flat manifolds from section 2 with their Betti numbers. [1]

[1] Note that $\beta_4 = \beta_2,\ \beta_5 = \beta_1,\ \beta_6 = \beta_0 = 1$, moreover as the Euler characteristic of such a manifold is 0, we also have the relation $\beta_3 = 2 - 2\beta_1 + 2\beta_2$.

β_1	β_2	β_3	Holonomy	CARAT symbol
0	3	8	$\mathbb{Z}_2 \times \mathbb{Z}_2$	185.1
0	2	6	D_8	207.1
2	3	4	$\mathbb{Z}_4$	225.1
			$\mathbb{Z}_5$	all
			$\mathbb{Z}_6$	318.1, 319.1, 404.1
			$\mathbb{Z}_8$	all
			$\mathbb{Z}_{10}$	all
			$\mathbb{Z}_{12}$	all
			$\mathbb{Z}_2 \times \mathbb{Z}_2$	186.1
			$\mathbb{Z}_2 \times \mathbb{Z}_4$	all
			$\mathbb{Z}_3 \times \mathbb{Z}_3$	all
			$\mathbb{Z}_6 \times \mathbb{Z}_2$	all
			$\mathbb{Z}_4 \times \mathbb{Z}_4$	all
			$\mathbb{Z}_6 \times \mathbb{Z}_3$	all
			$\mathbb{Z}_6 \times \mathbb{Z}_4$	all
			$\mathbb{Z}_6 \times \mathbb{Z}_6$	all
2	5	8	$\mathbb{Z}_3$	291.1
			$\mathbb{Z}_4$	202.1
			$\mathbb{Z}_6$	1611.1
2	7	12	$\mathbb{Z}_2$	174.1
4	7	8	$\mathbb{Z}_2$	173.1
			$\mathbb{Z}_3$	311.1
			$\mathbb{Z}_4$	219.1
			$\mathbb{Z}_6$	1694.1
6	15	20	1	170.1.1 $(M = T^6)$

Below we present some calculations of the Hodge numbers $\{h^{p,q}\}$, for some of the manifolds above. The case $\beta_1 = 0$ and holonomy $\mathbb{Z}_2 \times \mathbb{Z}_2$ is also being considered in the next section. As we are working in complex dimension 3, we have that $p, q \in \{0, 1, 2, 3\}$.

Example To illustrate several possibilities, we consider as an example what happens in case the holonomy group G is isomorphic to $\mathbb{Z}_6$. If t denotes the generator of $\mathbb{Z}_6$, we can distinguish, up to conjugation inside $GL(3, \mathbb{C})$, 4 possibilities for the representation $\mathbb{Z}_6 \to U(3)$ given by the following possible images for t:

$$t \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & w \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & z \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & \bar{z} \end{pmatrix}, \text{ or } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z \end{pmatrix},$$

where z denotes a primitive 6-th root of unity and $1 \neq w$ denotes a non-primitive 6-th root of unity.
One can easily check that the second and third possibility, when regarded as representations in $GL(6, \mathbb{R})$, are conjugate to each other.

When we make these computations for all possible holonomy groups, we find the following table of Hodge diamonds, where in the case of manifolds with $\beta_1 = 2$ and $\beta_2 = 5$, there are always 2 possibilities, depending on a choice of complex structure.

<table>
<tr>
<td>Manifold with $\beta_1 = 6$ (T^6):
$$\begin{array}{ccccccc} &&&1&&& \\ &&3&&3&& \\ &3&&9&&3& \\ 1&&9&&9&&1 \\ &3&&9&&3& \\ &&3&&3&& \\ &&&1&&& \end{array}$$</td>
<td>Manifolds with $\beta_1 = 2$ and $\beta_2 = 3$:
$$\begin{array}{ccccccc} &&&1&&& \\ &&1&&1&& \\ &0&&3&&0& \\ 0&&2&&2&&0 \\ &0&&3&&0& \\ &&1&&1&& \\ &&&1&&& \end{array}$$</td>
</tr>
<tr>
<td colspan="2">Manifolds with $\beta_1 = 2$ and $\beta_2 = 5$
$$\begin{array}{ccccccc} &&&1&&& \\ &&1&&1&& \\ &0&&5&&0& \\ 0&&4&&4&&0 \\ &0&&5&&0& \\ &&1&&1&& \\ &&&1&&& \end{array} \quad \text{or} \quad \begin{array}{ccccccc} &&&1&&& \\ &&1&&1&& \\ &1&&3&&1& \\ 1&&3&&3&&1 \\ &1&&3&&1& \\ &&1&&1&& \\ &&&1&&& \end{array}$$</td>
</tr>
<tr>
<td>Manifolds with $\beta_1 = 2$ and $\beta_2 = 7$:
$$\begin{array}{ccccccc} &&&1&&& \\ &&1&&1&& \\ &1&&5&&1& \\ 1&&5&&5&&1 \\ &1&&5&&1& \\ &&1&&1&& \\ &&&1&&& \end{array}$$</td>
<td>Manifolds with $\beta_1 = 4$
$$\begin{array}{ccccccc} &&&1&&& \\ &&2&&2&& \\ &1&&5&&1& \\ 0&&4&&4&&0 \\ &1&&5&&1& \\ &&2&&2&& \\ &&&1&&& \end{array}$$</td>
</tr>
<tr>
<td>Manifold with holonomy $\mathbb{Z}_2 \times \mathbb{Z}_2$:
$$\begin{array}{ccccccc} &&&1&&& \\ &&0&&0&& \\ &0&&3&&0& \\ 1&&3&&3&&1 \\ &0&&3&&0& \\ &&0&&0&& \\ &&&1&&& \end{array}$$</td>
<td>Manifolds with holonomy D_8:
$$\begin{array}{ccccccc} &&&1&&& \\ &&0&&0&& \\ &0&&2&&0& \\ 1&&2&&2&&1 \\ &0&&2&&0& \\ &&0&&0&& \\ &&&1&&& \end{array}$$</td>
</tr>
</table>

In the same way we can compute the Hodge diamond of all hyperelliptic surfaces in which case we always find:

$$\begin{array}{ccccc} &&1&& \\ &1&&1& \\ 0&&2&&0 \\ &1&&1& \\ &&1&& \end{array}$$

A **Calabi-Yau manifold** is a Kähler manifold with holonomy group group contained in $SU(n)$.

There are also other definitions of Calabi-Yau manifolds. In [11] the author mentions five non-equivalent definitions. For example one definition requires that such a manifold be projective. Moreover there are two definitions which are not interesting in case of flat manifolds: the first defines Calabi-Yau manifolds as Ricci-flat Kähler manifolds, while the second requires that the holonomy group be the full $SU(n)$. These two definitions are out of our interest, since any flat manifold is Ricci-flat and the holonomy group of a flat manifold is always finite. We have.

Proposition 3.1 *([11, Corollary 6.2.5]) Let M be a flat Kähler manifold of complex dimension n with induced holonomy representation $\varphi : H \to U(n)$. Then $h^{n,0} = 1$ if and only if $\varphi(H) \subseteq SU(n)$.*

Proof: $h^{n,0}$ is the dimension of $(\Lambda^{n,0}(\mathbb{C}^n \oplus (\mathbb{C}^n)^*)^H$. Therefore, $h^{n,0}$ is 1 if and only if $dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$ is fixed under the action of any element $h \in H$. However as the action of h on this basis vector is given by

$$^h(dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n) = \mathrm{Det}(\varphi(h)) dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n,$$

we have that h fixes this basis vector if and only if $\mathrm{Det}(\varphi(h)) = 1$. Therefore $h^{n,0} = 1 \Leftrightarrow \varphi(H) \subseteq SU(n)$. $\square$

Corollary 3.2 *There are no Calabi-Yau hyperelliptic surfaces. In complex dimension three, there are twelve Calabi-Yau flat Kähler manifolds with non-trivial holonomy:*

1. *five manifolds with the first Betti number equal to zero, where four manifolds have holonomy $\mathbb{Z}_2 \times \mathbb{Z}_2$ and one has holonomy D_8;*

2. *two manifolds with the first Betti number equal to 2 and holonomy $\mathbb{Z}_2$;*

3. *five manifolds with the following Betti numbers: $\beta_1 = 2$, $\beta_2 = 5$, where two manifolds have holonomy $\mathbb{Z}_3$, two have holonomy $\mathbb{Z}_4$ and one has holonomy $\mathbb{Z}_6$.*

4 Complex Hantzsche-Wendt manifolds

The original Hantzsche-Wendt manifold is the unique flat manifold in dimension 3 with vanishing first betti number, (cf. [8]). It is an orientable flat manifold, with holonomy group $\mathbb{Z}_2^2$. Several generalizations were given: a Hantzsche-Wendt manifold is an orientable flat manifold of dimension n and with holonomy group $\mathbb{Z}_2^{n-1}$, while a generalized Hantzsche Wendt manifold of dimension n is an non-necessarily orientable manifold of dimension n with holonomy group $\mathbb{Z}_2^{n-1}$. In each dimension $n \geq 2$, there are generalized Hantzsche-Wendt manifolds, while orientable Hantzsche-Wendt manifolds only occur in each odd dimension $n \geq 3$. Remark that n is the minimal dimension in which a flat manifold with holonomy $\mathbb{Z}_2^{n-1}$ exists. Lemma 2.1 shows that also in the complex case we cannot expect to find a Kähler flat manifold with holonomy group $\mathbb{Z}_2^{n-1}$ below dimension n. For this reason we introduce analogously as in the real case a concept of complex (generalized) Hantzsche-Wendt manifold.

Definition 4.1 *A flat Kähler n-manifold of holonomy $\mathbb{Z}_2^{n-1}$ is called a complex generalized Hantzsche-Wendt manifold (abbreviated as complex GHW). It will be called complex Hantzsche-Wendt manifold if, in addition the holonomy representation $\mathbb{Z}_2^{n-1} \to U(n)$ has its image lying inside $SU(n)$ (complex HW in short).*

Lemma 4.2 *For each $n \geq 2$, there exists a complex GHW of complex dimension n. Complex HW only exist in odd dimensions and for each odd $n \geq 3$, there exists a complex HW of dimension n.*

Proof: First we show that complex HW only exist in odd dimensions. Let M be a complex HW of dimension n and with holonomy representation $\varphi : \mathbb{Z}_2^{n-1} \to SU(n)$. After conjugation inside $GL(n, \mathbb{C})$ we may assume that the image of φ consists of diagonal $n \times n$ matrices with ± 1's on the diagonal. As the total subgroup of $SU(n)$ consisting of diagonal matrices with ± 1's on the diagonal is isomorphic to $\mathbb{Z}_2^{n-1}$ and φ is faithful, the image of φ is completely determined. Now, if n is even $-I_n$, minus the $n \times n$-identity matrix belongs to $SU(n)$. However, this would imply that $-I_{2n}$ belongs to the image of the real holonomy representation $\mathbb{Z}_2^{n-1} \to GL(2n, \mathbb{R})$, which is a contradiction. Therefore n has to be odd.

Now, given n we show that there exist a complex (G)HW of complex dimension n. First of all, there exist a (real) GHW of real dimension n, which we take to be orientable (HW) when n is odd, and where the fundamental group $\pi_1(M)$ satisfies a short exact sequence

$$0 \to \mathbb{Z}^n \to \pi_1(M) \to \mathbb{Z}_2^{n-1} \to 0.$$

Hence $\pi_1(M)$ is given by a 2-cohomology class $\langle f \rangle \in H^2(\mathbb{Z}_2^{n-1}, \mathbb{Z}^n)$. Now, consider

$$\langle f \rangle \oplus \langle f \rangle \in H^2(\mathbb{Z}_2^{n-1}, \mathbb{Z}^n) \oplus H^2(\mathbb{Z}_2^{n-1}, \mathbb{Z}^n) \cong H^2(\mathbb{Z}_2^{n-1}, \mathbb{Z}^{2n}).$$

This 2-cohomology class determines a Bieberbach group π' and the direct sum of modules $\mathbb{Z}^n \oplus \mathbb{Z}^n$ automatically statisfies the criterion of Proposition 1.2. Therefore, π' is the fundamental group of a complex GHW, which is a complex HW in case n is odd. $\square$

As the image of the representation $\mathbb{Z}_2^{n-1} \to SU(n)$ is fixed for a complex HW, we are able to compute the Hodge diamonds for any complex HW and we prove:

Theorem 4.3 *Let $n \geq 3$ be an odd number and let M be a complex Hantzsche-Wendt flat manifold of complex dimension n. Then it is Calabi-Yau and has the following Betti numbers:*

$$\beta_1 = \beta_3 = \cdots = \beta_{n-2} = \beta_{n+2} = \beta_{2n-1} = 0 \text{ and } \beta_n = 2^n,$$

$$\beta_0 = \binom{n}{0},\ \beta_2 = \binom{n}{1},\ \ldots \beta_{2k} = \binom{n}{k},\ \ldots \beta_{2n} = \binom{n}{n}.$$

Proof: We compute the Hodge diamond for these manifolds, from which the result follows easily. As in the proof of Lemma 4.2 we may assume that the representation $\varphi : \mathbb{Z}_2^{n-1} \to SU(n)$ is diagonal and that the image consists of all diagonal matrices with ± 1 on the diagonal (and of course with determinant 1).

Let us compute the upper left corner of the Hodge diamond, this is, the entries $h^{p,q}$ with $0 \leq q \leq p \leq n$. The other terms then follow by symmetry. As the action of the holonomy group is diagonal, we have to look for those (p, q)–forms

$$dz_{i_1} \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \wedge d\bar{z}_{j_q} \tag{2}$$

which are fixed under the action of the holonomy group. As a conclusion, we have that $h^{p,q} = \binom{n}{p}$.

Summarizing all of the above situations, we find the following Hodge diamond for a complex HW of complex dimension n:

$$
\begin{array}{c}
\binom{n}{0} \\
0 \qquad\qquad 0 \\
0 \qquad\qquad \binom{n}{1} \qquad\qquad 0 \\
0 \qquad\qquad 0 \qquad\qquad 0 \qquad\qquad 0 \\
0 \qquad\qquad 0 \qquad\qquad \binom{n}{2} \qquad\qquad 0 \qquad\qquad 0 \\
\cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \\
\binom{n}{0} \qquad\qquad \binom{n}{1} \qquad\qquad \binom{n}{2} \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \binom{n}{n-1} \qquad\qquad \binom{n}{n} \\
\cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \\
0 \qquad\qquad 0 \qquad\qquad \binom{n}{n-2} \qquad\qquad 0 \qquad\qquad 0 \\
0 \qquad\qquad 0 \qquad\qquad 0 \qquad\qquad 0 \\
0 \qquad\qquad \binom{n}{n-1} \qquad\qquad 0 \\
0 \qquad\qquad 0 \\
\binom{n}{0}
\end{array}
$$

The rest of the theorem now follows easily, because $\beta_i = \sum_{p+q=1} h^{p,q}$. $\square$

There are exactly four manifolds of this type in real dimension 6 (complex dimension 3), cf. [16]. Of all 174 six dimensional flat manifolds admitting a complex structure only five of them are having a first Betti number equal to zero and four of them are complex HW.

In the real case, all generalized HW manifolds are having a holonomy representation which is diagonizable over $\mathbb{Z}$. This does no longer hold in the complex case.

Proposition 4.4 *Any complex Hantzsche-Wendt manifold has a spin structure.*

Proof: (See also [13, Example 4.6].) Let M be a complex Hantzsche-Wendt manifold of complex dimension n. There is a short exact sequence

$$0 \to \mathbb{Z}^{2n} \to \pi_1(M) \to (\mathbb{Z}_2)^{n-1} \to 0,$$

inducing a holonomy representation $\varphi : \mathbb{Z}_2^{n-1} \to GL(2n, \mathbb{Z})$. When we consider $\mathbb{R}^{2n}$ as a $\mathbb{Z}_2^{n-1}$–module via φ, we have that $\mathbb{R}^{2n}$ is the direct sum $M \oplus M$ of two identical $\mathbb{Z}_2^{n-1}$–modules, where the action on M is given via matrices belonging to $SO(n)$. Hence it is enough to apply the definition of the spin structure for the "double" construction from the proof of the Theorem 1 of [4]. $\square$

Acknowledgements

The third author is grateful to M. Kontsevich for his advise and help during preparations of the manuscript.

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