A short note about diffuse Bieberbach groups

A. Gąsior, R. Lutowski, A. Szczepański [*]

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Abstract

We consider low dimensional diffuse Bieberbach groups. In particular we classify diffuse Bieberbach groups up to dimension 6. We also answer a question from [5, page 887] about minimal dimension of a non-diffuse Bieberbach group which does not contain three-dimensional Hantzsche-Wendt group.

1   Introduction

The class of diffuse groups was introduced by B. Bowditch in [2]. By definition a group Γ is *diffuse*, if every finite non-empty subset $A \subset \Gamma$ has an extremal point, i.e. an element $a \in A$ such that for any $g \in \Gamma \setminus \{1\}$ either ga or $g^{-1}a$ is not in A. Equivalently (see [5]) a group Γ is diffuse if it does not contain a non-empty finite set without extremal points.

The interest in diffuse groups follows from the Bowditch's observation that they have the unique product property [1]. Originally unique products were introduced in the study of group rings of discrete, torsion-free groups. More precisely, it is easily seen that if a group Γ has unique product property, then it satisfies Kaplansky's unit conjecture. In simple terms this means that

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[1]The group Γ is said to have the unique product property if for every two finite non-empty subsets $A, B \subset \Gamma$ there is an element in the product $x \in A\dot{B}$ which can be written uniquely in the form $x = ab$ with $a \in A$ and $b \in B$.

the units in the group ring $\mathbb{C}[\Gamma]$ are all trivial, i.e. of the form λg with $\lambda \in \mathbb{C}^*$ and $g \in \Gamma$. For more information about these objects we send a reader to research and survey articles [1], [7, Chapter 10] and [5]. In part 3 of [5] the authors prove that any torsion-free crystallographic group (Bieberbach group) with trivial center is not diffuse. By definition a crystallographic group is a discrete and cocompact subgroup of the group $O(n) \ltimes \mathbb{R}^n$ of isometries of the Euclidean space $\mathbb{R}^n$. From Bieberbach theorem (see [10]) the normal subgroup T of all translations of any crystallographic group Γ is a free abelian of finite rank group and the quotient group (holonomy group) $\Gamma/T = G$ is finite.

In [5, Theorem 3.5] there is proved that for a finite group G:

1. If G is not solvable then any Bieberbach group the holonomy group isomorphic to G *is not* diffuse.

2. If every Sylow subgroup of G is cyclic then any Bieberbach group with the holonomy group isomorphic to G *is* diffuse.

3. If G is solvable and has a non-cyclic Sylow subgroup then there are examples of Bieberbach groups with the holonomy group isomorphic to G which *are* and which *are not* diffuse.

Using the above the authors of [5] classify non-diffuse Bieberbach groups in dimensions ≤ 4. One of the most important non-diffuse group is 3-dimensional Hantzsche-Wendt group

$$\Delta_P = \{x, y \mid x^{-1}y^2x = y^{-1}, y^{-1}x^2y = x^{-2}\},$$

see [9], [10]. At the end of the part 3.4 of [5] the authors ask the following question.

Question 1. What is the smallest dimension d_0 of a non-diffuse Bieberbach group which does not contain Δ_P ?

The answer for the above question was the main motivation for us. In fact we prove, in the next section, that $d_0 = 5$. Moreover, we extend the results of the part 3.4 of [5] and with support of computer, we present the classification of all Bieberbach groups in dimension $d \leq 6$ which are (non)diffuse.

2 (Non)diffuse Bieberbach groups in dimension ≤ 6.

We use the computer system CARAT [8] to list all Bieberbach groups of dimension ≤ 6.

Our main tools are the following observations:

1. The property of being diffuse is inherited by subgroups (see [2, page 815]).

2. If Γ is a torsion-free group, $N \lhd \Gamma$, and that N and Γ/N are both diffuse then Γ is diffuse (see [2, Theorem 1.2 (1)]).

Note that we are only interested in Bieberbach groups with non-trivial center. Let Γ be a such group. We shall use a method of E. Calabi [10, Propostions 3.1 and 4.1]. By the first Betti number $\beta_1(\Gamma)$ we mean the rank of the abelianization $\Gamma/[\Gamma, \Gamma]$. We have an epimorphism

$$f : \Gamma \to \mathbb{Z}^k, \text{ where } k = \beta_1(\Gamma). \tag{1}$$

From assumptions $\ker f$ is a Bieberbach group of dimension < 6. Since $\mathbb{Z}^k$ is a diffuse group our problem is reduced to the question about the group $\ker f$. If Γ has rank 4 we know that the only non-diffuse Bieberbach group of dimension less than or equal to 3 is Δ_P. Using the above facts we obtain 17 non-diffuse groups. Note that the list from [5, section 3.4] consists of 16 groups. The following example presents the one which is not in [5].

Example 1. *Let Γ be a crystallographic group denoted by "05/01/06/006" in [3] as a subgroup of $\mathrm{GL}(5, \mathbb{R})$. Its non-lattice generators are as follows*

$$A = \begin{bmatrix} 0 & -1 & 1 & 0 & 1/2 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & -1 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & -1 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1/2 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Conjugating the above matrices by

$$Q = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \in \mathrm{GL}(5, \mathbb{Z})$$

one gets

$$A^Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 1/2 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } B^Q = \begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now its easy to see that the rank of the center of Γ equals 1 and the kernel of the epimorphism $\Gamma \to \mathbb{Z}$ is isomorphic to a 3-dimensional Bieberbach group Γ' with the following non-lattice generators:

$$A' = \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } B' = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Clearly the center of Γ' is trivial, hence it is isomorphic to the group Δ_P.

Now we formulate our main result.

Theorem 1. *The following table summarizes the number of diffuse and non-diffuse Bieberbach groups of dimension ≤ 6.*

Dimension	Total	Non-diffuse	Diffuse
1	1	0	1
2	2	0	2
3	10	1	9
4	74	17	57
5	1060	352	708
6	38746	19256	19490

Proof: If a group has a trivial center then it is not diffuse. In other case we use the Calabi (1) method and induction. A complete list of groups was obtained using computer algebra system GAP [4] and it is available here [6].

$\square$

Before we answer Question 1 from the introduction, let us formulate the following lemma:

Lemma 1. *Let α, β be any generators of the group Δ_P. Let $\gamma = \alpha\beta, a = \alpha^2, b = \beta^2, c = \gamma^2$. Then the following relations hold:*

$$
\begin{array}{lll}
[a, b] = 1 & a^\beta = a^{-1} & a^\gamma = a^{-1} \\
[a, c] = 1 & b^\alpha = b^{-1} & b^\gamma = b^{-1} \\
[b, c] = 1 & c^\alpha = c^{-1} & c^\beta = c^{-1}
\end{array} \tag{2}
$$

where $x^y := y^{-1}xy$ denotes the conjugation of x by y.

The proof of the above lemma is omitted. Just note that the relations are easily checked if you take isomorphic to Δ_P group

$$
\left\langle
\begin{bmatrix}
1 & 0 & 0 & 1/2 \\
0 & -1 & 0 & 1/2 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1/2 \\
0 & 0 & -1 & 1/2 \\
0 & 0 & 0 & 1
\end{bmatrix}
\right\rangle \subset \mathrm{GL}(4, \mathbb{Q}).
$$

Proposition 1. *There exists an example of five dimensional non-diffuse Bieberbach group which has not the group Δ_P as a subgroup.*

Proof. Let Γ be the Bieberbach group enumerated in CARAT as "min.88.1.1.15". It generated by the elements $\gamma_1, \gamma_2, l_1, \ldots, l_5$ where

$$
\gamma_1 =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1/2 \\
0 & 0 & 0 & -1 & 0 & 1/4 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\text{ and } \gamma_2 =
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1/2 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1/2 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

and $l_1, \ldots, l_5$ generate the lattice L of Γ:

$$
l_i :=
\begin{bmatrix}
I_5 & e_i \\
0 & 1
\end{bmatrix}
$$

where e_i is the i-th column of the identity matrix I_5. Γ fits into the following short exact sequence

$$
1 \longrightarrow L \longrightarrow \Gamma \xrightarrow{\ \pi\ } D_8 \longrightarrow 1
$$

where π takes the linear part of every element of Γ:

$$
\begin{bmatrix}
A & a \\
0 & 1
\end{bmatrix}
\mapsto A
$$

5

and the image D_8 of π is the dihedral group of order 8.

Now assume that Γ' is a subgroup of Γ isomorphic to Δ_P. Let T be its maximal normal abelian subgroup. Then T is free abelian group of rank 3 and Γ' fits into the following short exact sequence
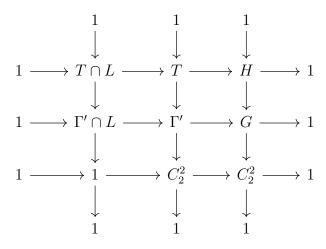
$$1 \longrightarrow T \longrightarrow \Gamma' \longrightarrow C_2^2 \longrightarrow 1,$$

where C_m is a cyclic group of order m. Consider the following commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & T \cap L & \longrightarrow & T & \longrightarrow & H = \pi(T) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \| & & \\
1 & \longrightarrow & L & \longrightarrow & \pi^{-1}(H) & \longrightarrow & H & \longrightarrow & 1
\end{array}
$$

We get that H must be abelian subgroup of $D_8 = \pi(\Gamma)$ and $T \cap L$ is free abelian group of rank 3 which lies in the center of $\pi^{-1}(H) \subset \Gamma$. Now if H is isomorphic either to C_4 or C_2^2 then the center of $\pi^{-1}(H)$ is of rank at most 2. Hence H must be the trivial group or the cyclic group of order 2. Note that as $\Gamma' \cap L$ is normal abelian subgroup of Γ' it must be a subgroup of T:

$$T \cap L \subset \Gamma' \cap L \subset T \cap L,$$

hence $T \cap L = \Gamma' \cap L$. We get the following commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & T \cap L & \longrightarrow & T & \longrightarrow & H & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \Gamma' \cap L & \longrightarrow & \Gamma' & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & 1 & \longrightarrow & C_2^2 & \longrightarrow & C_2^2 & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}
$$

where $G = \pi(\Gamma')$. Consider two cases:

1. H is trivial. In this case G is one of two subgroups of D_8 isomorphic to C_2^2. Since arguments for both subgroups are similar, we present only one of them. Namely, let

$$G = \langle \mathrm{diag}(1, -1, -1, -1, 1), \mathrm{diag}(-1, -1, 1, 1, 1).\rangle$$

In this case Γ' is generated by the matrices of the form

$$\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & x_1 - \frac{1}{2} \\ 0 & -1 & 0 & 0 & 0 & x_2 \\ 0 & 0 & -1 & 0 & 0 & x_3 \\ 0 & 0 & 0 & -1 & 0 & x_4 \\ 0 & 0 & 0 & 0 & 1 & x_5 - \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \beta = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & y_1 + \frac{1}{2} \\ 0 & -1 & 0 & 0 & 0 & y_2 - \frac{1}{2} \\ 0 & 0 & 1 & 0 & 0 & y_3 \\ 0 & 0 & 0 & 1 & 0 & y_4 + \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & y_5 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where $x_i, y_i \in \mathbb{Z}$ for $i = 1, \ldots, 5$. If $c = (\alpha\beta)^2$ then by Lemma 1 $c^\alpha = c^{-1}$, but

$$c^\alpha - c^{-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4y_5 + 4x_5 - 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Obviously solutions of the equation $4y_5 + 4x_5 - 2 = 0$ are never integral and we get a contradiction.

2. H is of order 2. Then $G = D_8$ and H is the center of G. The generators α, β of Γ' lie in the cosets $\gamma_1\gamma_2 L$ and $\gamma_2 L$, hence

$$\alpha = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & x_1 \\ 1 & 0 & 0 & 0 & 0 & x_2 - \frac{1}{2} \\ 0 & 0 & -1 & 0 & 0 & x_3 - \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & x_4 + \frac{1}{4} \\ 0 & 0 & 0 & 0 & -1 & x_5 + \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } \beta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & y_1 - \frac{1}{2} \\ 0 & -1 & 0 & 0 & 0 & y_2 \\ 0 & 0 & -1 & 0 & 0 & y_3 \\ 0 & 0 & 0 & -1 & 0 & y_4 \\ 0 & 0 & 0 & 0 & 1 & y_5 - \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where $x_i, y_i \in \mathbb{Z}$ for $i = 1, \ldots, 5$, as before. Setting $a = \alpha^2, b = \beta^2$ we get

$$ab - ba = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 2 - 4y_1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and again the equation $2 - 4y_1 = 0$ does not have an integral solution.

The above considerations show that Γ does not have a subgroup which is isomorphic to Δ_P.

$\square$

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Institute of Mathematics, Maria Curie-Skłodowska University
Pl. Marii Curie-Skłodowskiej 1
20-031 Lublin
Poland
E-mail: `anna.gasior@poczta.umcs.lublin.pl`

Institute of Mathematics, University of Gdańsk
ul. Wita Stwosza 57,
80-952 Gdańsk,
Poland
E-mail: `rlutowsk@mat.ug.edu.pl`, `matas@univ.gda.pl`