Holonomy groups of crystallographic groups with finite outer automorphism groups

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A crystallographic group Γ of dimension n is a discrete, cocompact subgroup of isometries of euclidean space $\mathbb{R}^n$. From Bieberbach's theorems any such group contains a free abelian subgroup $\mathbb{Z}^n$ of finite index. Moreover the finite group $G = \Gamma/\mathbb{Z}^n$ acts faithfully by conjugation on $\mathbb{Z}^n$. We shall call the correspond representation $G \to GL(n, \mathbb{Z})$ the holonomy representation of Γ and G a holonomy group of Γ. A Bieberbach group is a torsion free crystallographic group.

We shall use the notations from [4] and [8]. It is well known (cf. [8]) that the outer automorphism group of a crystallographic group Γ is finite if and only if in the holonomy representation of Γ all $\mathbb{Q}$-irreducible components are multiplicity free and $\mathbb{R}$-irreducible.

Definition 1 *The finite group G is an ($\mathcal{R}_1$) $\mathcal{R}'_1$-group if it is the holonomy group of a (Bieberbach) crystallographic group with finite outer automorphism group. We shall the class of ($\mathcal{R}_1$) $\mathcal{R}'_1$-groups denote by ($\mathcal{R}_1$) $\mathcal{R}'_1$.*

Let us recall the definition of the Whitehead group (cf. [5]). We start from a group ring $\mathbb{Z}[G]$. Then consider the linear group $GL(n, \mathbb{Z}[G])$. There is an inclusion $GL(n, \mathbb{Z}[G])$ into $GL(n+1, \mathbb{Z}[G])$ given by the map $A \to \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. Let $GL(\mathbb{Z}[G]) = \bigcup_{n=1}^{\infty} GL(n, \mathbb{Z}[G])$ denote the direct limit of the above linear

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groups and $[GL(\mathbb{Z}[G]), GL(\mathbb{Z}[G])] = E(G)$ denote its commutator subgroup. Hence the quotient group $K_1(G) = GL(\mathbb{Z}[G])/E(G)$ is an abelian group. The Whitehead group of G is defined by setting

$$Wh(G) = K_1(G)/\langle \pm g : g \in G\rangle.$$

The following observation is obvious:

Proposition 1 *Let G be a finite group. If the Whitehead group Wh(G) is finite then $G \in \mathcal{R'}_1$.*

Proof: It is clear that there exists a crystallographic group with holonomy group G and $\mathbb{Q}$-multiplicity free holonomy representation. Hence it is enough to use the following formula which follows from Dirichlet's unit theorem (see [5] page 49, Theorem 2.6.). Let $r = \{$number of $\mathbb{R}$-irreducible representations of $G\}$ and $q = \{$number of $\mathbb{Q}$-irreducible representations of $G\}$, then

$$rank(Wh(G)) = r - q.$$

□

Example 1 *Let $S_n, n \geq 1$ be a symmetric group on n elements. It is known (see [5]) that $Wh(S_n) = 1$. Hence $S_n \in \mathcal{R'}_1$.*

Example 2 *$A_5 \in \mathcal{R'}_1 \setminus \mathcal{R}_1$ but $Wh(A_5)$ is infinite (cf. [4] Proposition 6.1). Moreover the finite group $SL_3(2).2$ from example 2.5 of [3] belongs to $\mathcal{R}_1$ but its Whitehead group is also infinite.*

We expect that the Proposition is also true for Bieberbach groups. But to prove it we need the following.

Conjecture Let G be a finite group. There exists a Bieberbach group with holonomy group G and $\mathbb{Q}$-multiplicity free holonomy representation.

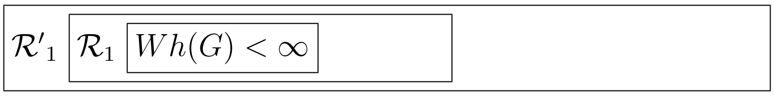
We know a proof of it for some abelian groups (cf. [4] Theorem 4.2), p-groups (cf. [4] Proposition 3.3), dihedral groups (cf. [4] Proposition 5.1) and simple groups (cf. [4] section 6). Moreover, Theorem 5 in [1] proved it in a special case of the trivial representation. That means, for any finite group G there exists a Bieberbach group with holonomy group G and holonomy representation with the trivial $\mathbb{Q}$-component of maximal dimension one.

Hence we have.

Corollary 1 *Let G be a finite group and assume that the above conjecture is true. If the group $Wh(G)$ is finite then $G \in \mathcal{R}_1$.*

$\square$

Then (under assumption of the corollary) we have the following relations between the classes of finite groups:



From example 1 we can expect that all symmetric groups are in $\mathcal{R}_1$. The following theorem is the first step in that direction.

Theorem 1 *Let $n \leq 6$ then $S_n \in \mathcal{R}_1$.*

Proof: For $n = 2, 3$ the result follows from [3]. Moreover, for $n = 4$ we refer the reader to [7] page 386. Let us start a proof for $n = 5$. We have to construct a special element $\alpha \in H^2(S_5, M)$, where M is a faithfully multiplicity free S_5 - lattice. For the prime number 5, let us define an S_5-lattice $M(5)$ of rank 6 which corresponds to a 6-dimensional irreducible representation of S_5. From CARAT[1] (see [6]) it is known that among lattices we can find one with $H^2(S_5, M(5)) = Z_5$. For the prime number 3, again from CARAT, there exists an S_5-lattice $M(3)$ of rank 4, which corresponds to a 4-dimensional irreducible representation of S_5, with $H^2(S_5, M(3)) = Z_3$. Finally, for the prime number 2 we have two conjugacy classes in S_5. For the first one, to construct a special element, we use the A_5-lattice $L(2)$ from [2] page 894. $L(2)$ corresponds to a 5-dimensional irreducible representation of A_5. For the second one, it is enough to use the trivial S_5-representation $\mathbb{Z}$. Then $M(2) = \mathbb{Z} \oplus ind_{A_5}^{S_5} L(2)$. Put $M = M(2) \oplus M(3) \oplus M(5)$. By Frobenius reciprocity lattice M is the multiplicity free. Hence from the above the group $H^2(S_5, M)$ has a special element which defines a Bieberbach group with S_5 holonomy and finite outer automorphism group.

[1]We thank Tilman Schulz for help.

For S_6 we start with the prime number 5. Let $N = ind_{S_5}^{S_6} M(5)$, where $M(5)$ is defined as above. It is not difficult to see that as $Z_5[S_6]$ lattice N is a direct sum of three lattices. The two 10-dimensional and the 16-dimensional ones. The first ones are projective (of defect 0), and hence do not contribute to the cohomology group $H^2(S_6, N)$. Then finally $N(5)$ is the 16-dimensional irreducible S_6-lattice. For the prime number 2 it is not difficult to see that the S_6-lattice $N(2) = \mathbb{Z} \oplus ind_{A_6}^{S_6} \chi_3$, where χ_3 is 9 dimensional irreducible lattice of A_6, can be used to define a 2-special element in $H^2(S_6, N(2))$. For a 3-special element let us consider a subgroup $B' \subset S_6$ of order 18 which is a normal subgroup of a Borel subgroup of $A_6 \simeq PSL(2,9)$. From the definition $H^2(B', sgn)$ has an element which restricts non trivialy to two conjugacy classes of order 3. Hence the second cohomology group $H^2(S_6, ind_{B'}^{S_6} sgn)$ contains a special 3-element. It is obvious that the coefficient lattice $N(3) = ind_{B'}^{S_6} sgn$ is multiplicity free. Moreover, from the above, the S_6-lattice $N = N(2) \oplus N(3) \oplus N(5)$ is a translation subgroup of a Bieberbach group Γ with $\Gamma/N \simeq S_6$ and finite outer automorphism group.

□

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