Symmetries of flat manifolds

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Crystallographic group

Let us denote by $E(n)$ the isometry group $Isom(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$ of the n-dimensional Euclidean space.

Definition

A crystallographic group of dimension n is a cocompact and discrete subgroup of $E(n)$.

Example

1. Z^n

2. If $(B, \left(\begin{smallmatrix} 1/2 \\ 0 \end{smallmatrix}\right)), (I, \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)) \in E(2)$, where $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then the group $\Gamma \subset E(2)$ generated by the above elements is a crystallographic group of dimension 2.

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Bieberbach theorems

The first part of the eighteenth Hilbert problem was about the description of discrete and cocompact groups of isometries of $\mathbb{R}^n$.

The answer for the above Hilbert problem was given by the German mathematician L. Bieberbach in 1913.

Theorem

(Bieberbach) *1. If $\Gamma \subset E(n)$ is a crystallographic group then the set of translations $\Gamma \cap (I \times \mathbb{R}^n)$ is a torsion free and finitely generated abelian group of rank n, and is a maximal abelian and normal subgroup of finite index.*

2. For any natural number n, there are only a finite number of isomorphism classes of crystallographic groups of dimension n.

3. Two crystallographic groups of dimension n are isomorphic if and only if they are conjugate in the group $A(n) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$.

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Flat manifold

Definition

A flat manifold M^n of dimension n is a compact connected Riemannian manifold without boundary with sectional curvature equal to zero.

Example

1. torus $\mathbb{R}^n/\mathbb{Z}^n \simeq \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n}$
2. $\mathbb{R}^n/\Gamma$, where $\Gamma \subset E(n)$ is a torsion free crystallographic group

Remark

Any flat manifold $M^n \simeq \mathbb{R}^n/\Gamma$, where $\Gamma = \pi_1(M^n)$. Γ is a torsion free crystallographic group of rank n.

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From the theorems of Bieberbach the fundamental group $\pi_1(M^n) = \Gamma$ (Bieberbach group) determines a short exact sequence

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} G \to 0,$$

where $\mathbb{Z}^n$ is a torsion free abelian group of rank n and G is a finite group with is isomorphic to the holonomy group of M^n.

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Any flat manifold $M^n \simeq \mathbb{R}^n/\Gamma \simeq \mathbb{R}^n/\mathbb{Z}^n/\Gamma/\mathbb{Z}^n \simeq T^n/G$.

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Diagram

Let Γ be a crystallographic group. We have

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} G \to 0.$$

Let $h_\Gamma : G \to GL(n, \mathbb{Z})$ be the integral holonomy representation defined by the formula

$$\forall_{g \in G} h_\Gamma(g)(e) = \bar{g} e \bar{g}^{-1},$$

where $\bar{g} \in \Gamma, p(\bar{g}) = g$ and $e \in \mathbb{Z}^n$. Since $\mathbb{Z}^n$ is a maximal abelian subgroup, h_Γ is a faithfull representation.

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Diagram

Let $\alpha \in H^2(G, \mathbb{Z}^n)$ be an element which corresponds with the above short exact sequence. Let

$$N = \{X \in GL(n, \mathbb{Z}) \mid \forall f \in h_\Gamma(G) \;\; X\, f\, X^{-1} \in h_\Gamma(G)\}.$$

N acts on $H^2(G, \mathbb{Z}^n)$ by formula

$$n * [c](g_1, g_2) = n^{-1}[c(ng_1n^{-1}, c(ng_2n^{-1}],$$

where $[c] \in H^2(G, \mathbb{Z}^n), n \in N, g_1, g_2 \in G$. Finally let

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Diagram

From now, all crystallographic groups have a trivial center. It is easy to see, that in this case $(\mathbb{Z}^n)^G = 0$ and $Inn(\Gamma) = \Gamma$. We have a restriction homomorphism $F : Aut(\Gamma) \to Aut(\mathbb{Z}^n)$. In 1973 L. Charlap and A.T.Vasquez proved that the kernel F_0 of F is isomorphic to $Z^1(G, \mathbb{Z}^n)$ and the following diagram has exact rows and columns. This is our main technical tool.

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Diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}^n & \longrightarrow & \Gamma & \xrightarrow{p} & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow{\scriptstyle h_\Gamma} & & \\
0 & \longrightarrow & Z^1(G,\mathbb{Z}^n) & \longrightarrow & Aut(\Gamma) & \xrightarrow{F} & N_\alpha & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^1(G,\mathbb{Z}^n) & \longrightarrow & Out(\Gamma) & \longrightarrow & N_\alpha/G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Diagram

Lemma

Assume $Z(\Gamma) = \{e\}$, *then*

1. $H^1(G, \mathbb{Z}^n) \simeq (\mathbb{Q}^n/\mathbb{Z}^n)^G = H^0(G, \mathbb{Q}^n/\mathbb{Z}^n)$;
2. $Z^1(G, \mathbb{Z}^n) \simeq \{m \in \mathbb{Q}^n \mid \forall_{g \in G} \; gm - m \in \mathbb{Z}^n\} = A^0(\Gamma)$ *as* N_α *modules;*
3. $A(\Gamma) = N_{Aff(\mathbb{R}^n)}(\Gamma) = \{a \in Aff(\mathbb{R}^n) \mid \forall_{\gamma \in \Gamma} a\gamma a^{-1} \in \Gamma\} \simeq Aut(\Gamma)$.

Diagram

From the diagram and Bieberbach theorem we can see that for torsion-free crystallogarphic group Γ we have the following groups isomorphism

$$\mathrm{Out}(\Gamma) \simeq \mathrm{Aff}(M)/(S^1)^k,$$

where $M = \mathbb{R}^n/\Gamma$ and k is the first Betti number of M.

Theorem

For every $n \geqslant 2$ there exists a crystallographic group Γ of dimension n with $Z(\Gamma) = Out(\Gamma) = \{e\}$.

Example

Let $\Gamma_1 = G_1 \ltimes \mathbb{Z}^2$ be the crystallographic group of dimension 2 with holonomy group $G_1 = D_{12}$, where

$$D_{12} = gen\left\{\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$$

is the dihedral group of order 12.

Comments

Definition

A group G is said to be complete if its center $Z(G)$ and outer automorphism group are both trivial.

Complete groups were introduced by O. Hölder in 1895. He has proved already that the symmetric group S_n is complete provided $n \neq 2$ or 6. J. L. Dyer and E. Formanek proved in 1975 that if F is a non-cyclic free group then $Aut(F)$ is complete.

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In 2005 M. Belolipetsky and A. Lubotzky proved that for every $n \geqslant 2$ and every finite group G there exist infinitely many discrete, cocompact and torsion free subgroups $\Gamma \subset \mathrm{Isom}(\mathbb{H}^n)$ with $\mathrm{Isom}\mathbb{H}^n/\Gamma \simeq G$. Moreover, from Mostow rigidity, for $n \geqslant 3$, $Out(\Gamma) \simeq \mathrm{Isom}\mathbb{H}^n/\Gamma$. In particular for a trivial group G we obtain examples of complete groups.

In 2003 R. Waldmüller found an example of complete, torsion free crystallographic group Γ of dimension 141 with holonomy group M_{11} (Mathieu group)

$$0 \longrightarrow \mathbb{Z}^{141} \longrightarrow \Gamma \longrightarrow M_{11} \longrightarrow 0.$$

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Proof

We shall need a few lemmas.

Lemma

$Aut(\Gamma)$ is a crystallographic group if and only if $Out(\Gamma)$ is a finite group.

Proof: We start with an observation that $Z^1(G, \mathbb{Z}^n)$ is a free abelian group of rank n which is a faithful N_α module. First, assume that $Aut(\Gamma)$ is a crystallographic group with the maximal abelian subgroup M. From Bieberbach's theorems, M is the unique normal maximal abelian subgroup of $Aut(\Gamma)$. Hence, $M = Z^1(G, \mathbb{Z}^n)$, and $Out(\Gamma)$ is a finite group. The reverse implication is obvious.

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Lemma

Let G, H be finite groups and $H \subset G \subset GL(n,\mathbb{Z})$. If the group $N_{GL(n,\mathbb{Z})}(H)$ is finite, then $N_{GL(n,\mathbb{Z})}(G)$ is finite.

Proof: From the assumption, $Aut(H)$ and $Aut(G)$ are finite and we have monomorphisms:

$$N_{GL(n,\mathbb{Z})}(H)/C_{GL(n,\mathbb{Z})}(H) \xrightarrow{\bar{\phi}} Aut(H)$$

and

$$N_{GL(n,\mathbb{Z})}(G)/C_{GL(n,\mathbb{Z})}(G) \xrightarrow{\bar{\phi}} Aut(G),$$

where $\bar{\phi}$ is induced by $\phi(s)(g) = sgs^{-1}, g \in G, s \in GL(n,\mathbb{Z})$. Since $C_{GL(n,\mathbb{Z})}(G) \subset C_{GL(n,\mathbb{Z})}(H)$, our Lemma is proved.

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where $\bar{\phi}$ is induced by $\phi(s)(g) = sgs^{-1}, g \in G, s \in GL(n, \mathbb{Z})$. Since $C_{GL(n,\mathbb{Z})}(G) \subset C_{GL(n,\mathbb{Z})}(H)$, our Lemma is proved.

Proof

Lemma

Let G, H be finite groups and $H \subset G \subset GL(n,\mathbb{Z})$. If the group $N_{GL(n,\mathbb{Z})}(H)$ is finite, then $N_{GL(n,\mathbb{Z})}(G)$ is finite.

Proof: From the assumption, $Aut(H)$ and $Aut(G)$ are finite and we have monomorphisms:

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Proof

Collorary

If $|\ Out(\Gamma)\ | < \infty$, *then* $|\ Out(Aut(\Gamma))\ | < \infty$.

Proof

Let Γ be a crystallographic group with a holonomy group G of dimension n. Assume that the group $H^1(G, \mathbb{Z}^n) = \{e\}$, and the group $Out(\Gamma)$ is finite.

Inductively, put $\Gamma_0 = \Gamma$ and $\Gamma_{i+1} =$A(Γ_i), for $i \geqslant 0$.

Lemma

$\exists N$ such that $\Gamma_{N+1} = \Gamma_N$.

Proof: We start from observations that for $i > 0$, Γ_i is a crystallographic group, $Z(\Gamma_i) = \{e\}$ and $M_0 = M_i$, where $M_i = A^0(\Gamma_{i-1}) \subset \Gamma_i$ is the maximal abelian normal subgroup (a subgroup of translations). Let $G_i = \Gamma_i / M_i$. From definition we can consider (G_i) as a nondecreasing sequence of finite subgroups of $GL(n, \mathbb{Z})$. From Bieberbach theorems and from the diagram, there is only a finite number of possibilties for G_i. Hence $\exists N \in \mathbb{N}$ such that $\forall_{i > N} \; G_i = G_N$. This finishes the proof.

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Proof

Example

Let $\Gamma_2 = G_2 \ltimes \mathbb{Z}^3$ *be the crystallographic group of dimension 3, with holonomy group* $G_2 = S_4 \times \mathbb{Z}_2$ *generated by matrices*

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Here S_4 *denotes the symmetric group on four letters.*

For $i = 1, 2$ *we have*

$$N_{GL(n_i,\mathbb{Z})}(G_i) = G_i \text{ and } H^1(G_i, \mathbb{Z}^{n_i}) = 0,$$

where n_i *is the rank of* Γ_i*. Hence* $A(\Gamma_i) = \Gamma_i$*, and* $Out(\Gamma_i) = \{e\}$*, for* $i = 1, 2$*.*

Proof

Now we are ready to finish the proof of Theorem. The cases $n = 2, 3$ are done in the above examples. Assume $n \geqslant 4$.
Assume Γ' is a crystallographic group given by a short exact sequence

$$0 \to \mathbb{Z}^n \to \Gamma' \to S_{n+1} \to 0$$

with a holonomy representation s.t. $H^1(S_{n+1}, \mathbb{Z}^n) = 0$. (It is enough to consider S_{n+1} modul $\mathbb{Z}^{n+1}$ with permutation action on the standard basis $e_1, e_2, ..., e_{n+1}$ divided by a submodule of rank one generated by $\{e_1 + e_2 + ... + e_{n+1}\}$.)
Hence, Γ' satisfies the assumption of the last Lemma and the sequence $\Gamma_0 = \Gamma'$, $\Gamma_{i+1} = A(\Gamma_i)$ stabilizes, i.e., $\exists N$ such that $\forall_{i \geqslant N} \quad \Gamma_i = \Gamma_N$. Finally, $Out(\Gamma_N) = \{e\}$ and $Z(\Gamma_N) = \{e\}$.

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Problems and Question

1. Find a new example of a torsion free crystallographic group Γ with trivial center and trivial outer automorphism group.

2. For any finite group G find an example of (torsion free) crystallographic group Γ such that $\mathrm{Out}(\Gamma) = \mathrm{G}$.

3. Let G a finite group. Are there exist a natural number n and a subgroup $A \subset S_n$ of the permutation group S_n, such that

$$N_G(A)/A \simeq G?$$

Here $N_G(A)$ is a normalizer of the subgroup A in S_n.

Thank You