Spin and Spin$^{\mathbb{C}}$ structure on flat manifolds

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11 March 2014

Crystallographic group

Let us denote by $E(n)$ the isometry group $Isom(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$ of the n-dimensional Euclidean space.

Definition

A crystallographic group of dimension n is a cocompact and discrete subgroup of $E(n)$.

Example

1. Z^n

2. If $(B, \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}), (I, \begin{pmatrix} 0 \\ 1 \end{pmatrix}) \in E(2)$, where $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then the group $\Gamma \subset E(2)$ generated by the above elements is a crystallographic group of dimension 2.

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Bieberbach theorems

The first part of the eighteenth Hilbert problem was about the description of discrete and cocompact groups of isometries of $\mathbb{R}^n$.
The answer for the above Hilbert problem was given by the German mathematician L. Bieberbach in 1913.

Theorem

(Bieberbach) *1. If $\Gamma \subset E(n)$ is a crystallographic group then the set of translations $\Gamma \cap (I \times \mathbb{R}^n)$ is a torsion free and finitely generated abelian group of rank n, and is a maximal abelian and normal subgroup of finite index.*

2. For any natural number n, there are only a finite number of isomorphism classes of crystallographic groups of dimension n.

3. Two crystallographic groups of dimension n are isomorphic if and only if they are conjugate in the group $A(n) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$.

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Flat manifold I

Definition

A flat manifold M^n of dimension n is a compact connected Riemannian manifold without boundary with sectional curvature equal to zero.

Example

1. *torus* $\mathbb{R}^n/\mathbb{Z}^n \simeq \underbrace{S^1 \times S^1 \times \cdots \times S^1}_{n}$
2. $\mathbb{R}^n/\Gamma$, *where* $\Gamma \subset E(n)$ *is a torsion free crystallographic group*

Remark

Any flat manifold $M^n \simeq \mathbb{R}^n/\Gamma$, *where* $\Gamma = \pi_1(M^n)$. Γ *is a torsion free crystallographic group of rank n.*

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Flat manifold II

From the theorems of Bieberbach the fundamental group $\pi_1(M^n) = \Gamma$ (Bieberbach group) determines a short exact sequence

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} G \to 0,$$

where $\mathbb{Z}^n$ is a torsion free abelian group of rank n and G is a finite group with is isomorphic to the holonomy group of M^n.

Collorary

Any flat manifold $M^n \simeq \mathbb{R}^n/\Gamma \simeq \mathbb{R}^n/\mathbb{Z}^n/\Gamma/\mathbb{Z}^n \simeq T^n/G.$

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Clifford algebra

Definition

By a Clifford algebra over the real numbers we shall understand an associative algebra with unity, generated by elements

$$\{e_1, e_2, \ldots, e_n\}$$

and with relations

$$\forall i, e_i^2 = -1,$$

$$\forall i, j, e_i e_j = -e_j e_i,$$

where $1 \leqslant i, j \leqslant n$*. We define* $C_0 = \mathbb{R}$*.*

It is easy to see that $C_1 = \mathbb{C}$ and $C_2 = \mathbb{H}$, where $\mathbb{H}$ is the four-dimensional quaternion algebra. Moreover, $\mathbb{R}^n \subset C_n$ and $\dim_{\mathbb{R}} C_n = 2^n$, where $\mathbb{R}^n$ is n-dimensional $\mathbb{R}$-vector space with the basis $e_1, e_2, \ldots, e_n$.

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A group Spin I

We have the following homomorphisms (involutions) on C_n :

(i) $^* : e_{i_1} e_{i_2} \dots e_{i_k} \mapsto e_{i_k} e_{i_{k-1}} \dots e_{i_2} e_{i_1}$,

(ii) $' : e_i \mapsto -e_i$,

(iii) $^- : a \mapsto (a')^*, a \in C_n$.

Suppose $C_n^0 = \{x \in C_n \mid x' = x\}$. It is easy to observe that

$$\forall a, b \in C_n, (ab)^* = b^* a^*.$$

Definition

We define subgroups of C_n,

$$Pin(n) = \{x_1 x_2 \dots x_k \mid x_i \in S^{n-1} \subset \mathbb{R}^n \subset C_n, i = 1, 2, \dots k\},$$

$$Spin(n) = Pin(n) \cap C_n^0.$$

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Spin-structure I

Definition

A closed oriented manifold W^n has a Spin*-structure if and only if the second Stiefel-Whitney class $w_2(W^n) = 0$.*

Remark

A Spin*-structure on the manifold W^n is a lift of δ to* $\mathrm{BSpin}(n)$*, giving a commutative diagram:*

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\begin{array}{ccc}
 & & \mathrm{BSpin}(n) \\
 & \nearrow & \downarrow{\scriptstyle B(\lambda_n)} \\
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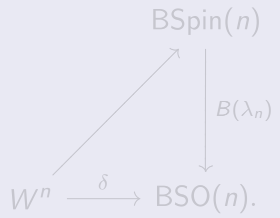

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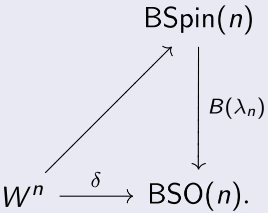

Spin-structure II

For flat manifolds we have.

Theorem

An oriented flat manifold M^n (a Biebarbach group $\pi_1(M^n) = \Gamma$) has a Spin-structure if and only if there exists a homomorphism

$$\epsilon : \Gamma \longrightarrow \mathrm{Spin}(n)$$

such that

$$\lambda_n \circ \epsilon = p,$$

where $\lambda_n : \mathrm{Spin}(n) \longrightarrow SO(n)$ is a covering map.

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Given a homomorphism ϵ with $p = \lambda_n \circ \epsilon$ one defines $B(\epsilon) : B\Gamma \to B\,\mathrm{Spin}(n)$ and one gets a spin structure as described above. For the proof of other direction we shall use the remark. Since $H^2(SO(n), \mathbb{Z}_2) = \mathbb{Z}_2$, the second Stiefel-Whitney class $w_2 \in H^2(M^n, \mathbb{Z}_2) = H^2(\Gamma, \mathbb{Z}_2)$ defines the upper row of the following diagram.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & \bar{\Gamma} & \longrightarrow & \Gamma & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow p & & \\
1 & \longrightarrow & \mathbb{Z}_2 & \longrightarrow & Spin(n) & \xrightarrow{\lambda_n} & SO(n) & \longrightarrow & 1
\end{array} .$$

Summing up, if M^n has a spin-structure then $w_2 = 0$ and the first row of the above diagram splits. Hence, there exists a homomorphism $\epsilon : \Gamma \to Spin(n)$, such that $\lambda_n \circ \epsilon = p$.

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Group $\mathrm{Spin}^{\mathbb{C}}$

The group $\mathrm{Spin}^{\mathbb{C}}(n)$ is given by

$$\mathrm{Spin}^{\mathbb{C}}(n) = (\mathrm{Spin}(n) \times S^1)/\{1, -1\}$$

where $\mathrm{Spin}(n) \cap S^1 = \{1, -1\}$. Moreover, there is a homomorphism of groups

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Now, we recall some facts about the group $\mathrm{Spin}^{\mathbb{C}}$. We start with homomorphisms

- $i : \mathrm{Spin}(n) \to \mathrm{Spin}^{\mathbb{C}}(n)$ is the natural inclusion $i(g) = [g, 1]$.
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$\mathrm{Spin}^{\mathbb{C}}$-structure

The manifold W^n has a $\mathrm{Spin}^{\mathbb{C}}$-structure if and only if there exists $\tilde{w}_2 \in H^2(W^n, \mathbb{Z})$ such that $red(\tilde{w}_2) = w_2$, where $w_2 \in H^2(W^n, \mathbb{Z}_2)$ and $red : H^2(W^n, \mathbb{Z}) \to H^2(W^n, \mathbb{Z}_2)$ is a homomorphism induced by the natural homomorphism $\mathbb{Z} \to \mathbb{Z}_2$.

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$\mathrm{Spin}^{\mathbb{C}}$ - structure

Definition

A $\mathrm{Spin}^{\mathbb{C}}$*-structure on the manifold* W^n *is a lift of* δ *to* $\mathrm{BSpin}^{\mathbb{C}}(n)$*, giving a commutative diagram*:

$$\begin{array}{ccc} & & \mathrm{BSpin}^{\mathbb{C}}(n) \\ & \nearrow & \downarrow B(\bar{\lambda}_n) \\ W^n & \xrightarrow{\delta} & \mathrm{BSO}(n). \end{array}$$

Theorem 1

Theorem

Let M^n be a flat oriented manifold with $H^2(M^n, \mathbb{R}) = 0$. M^n has a $\mathrm{Spin}^{\mathbb{C}}$*-structure if and only if there exists a homomorphism* $\epsilon : \Gamma \to \mathrm{Spin}^{\mathbb{C}}(n)$ *such that*

$$\bar{\lambda}_n \circ \epsilon = p.$$

Proof

Let us assume that there exists a homomorphism $\epsilon : \Gamma \to \mathrm{Spin}^{\mathbb{C}}(n)$ such that $\bar{\lambda}_n \epsilon = p$. Then, it defines a map $B(\epsilon) : B\Gamma = M^n \to \mathrm{BSpin}^{\mathbb{C}}(n)$ such that $B(\bar{\lambda}_n)B(\epsilon) = B(p)$ up to homotopy.

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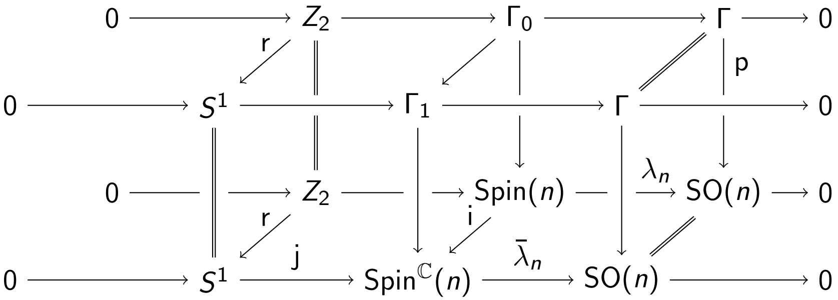
Proof

To go the other way, assume $M^n = B\Gamma$ admits a $\mathrm{Spin}^{\mathbb{C}}(n)$ structure. We have a commutative diagram



where Γ_0 is defined by the second Stiefel-Whitney class $w_2 \in H^2(\Gamma, \mathbb{Z}_2)$ and Γ_1 is defined by the element $r_*(w_2) \in H^2(\Gamma, S^1)$. Here $r : \mathbb{Z}_2 \longrightarrow S^1$ is a group monomorphism.

Proof

Let $p^2 : H^2(SO(n), K) \to H^2(\Gamma, K)$ be a homomorphism induced by the holonomy homomorphism p, for $K = \mathbb{Z}_2, S^1$. From definition there exists an element $x_2 \in H^2(SO(n), \mathbb{Z}_2) = \mathbb{Z}_2$ such that $p^2(x_2) = w_2$ and $p^2(r_*(x_2)) = r_*(p^2(x_2)) = r_*(w_2)$. Moreover we have two infinite sequences of cohomology which are induced by the following commutative diagram of groups

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$$0 \to S^1 \to \Gamma_1 \to \Gamma \to 0$$

of the above "big" diagram splits. Hence there exists a homomorphism $\epsilon : \Gamma \to \mathrm{Spin}^{\mathbb{C}}(n)$ which satisfies $\bar{\lambda}_n \circ \epsilon = p$. This proves theorem.

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Corollary

As an immediate corollary we have

Collorary

Let M^n be an oriented flat manifold with fundamental group Γ. If there exists a homomorphism $\epsilon : \Gamma \to \mathrm{Spin}^{\mathbb{C}}(n)$ such that

$$\bar{\lambda}_n \circ \epsilon = p.$$

then M^n has a $\mathrm{Spin}^{\mathbb{C}}$-structure.

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Examples

1. Because of the inclusion $i : \mathrm{Spin}(n) \to \mathrm{Spin}^{\mathbb{C}}(n)$ each spin structure on M^n induces a $\mathrm{Spin}^{\mathbb{C}}$ structure.
2. Any oriented compact manifold of dimension up to four has a $\mathrm{Spin}^{\mathbb{C}}$ structure (see R. E. Gompf, "$\mathrm{Spin}^{\mathbb{C}}$-structures and homotopy equivalences")
3. From the Example 2 and B.Putrycz, A. Szczepański "Existence of spin structure on flat four-manifolds" we have that there exist three four dimensional flat manifolds without Spin structure but with $\mathrm{Spin}^{\mathbb{C}}$ structure.
4. There exists a compact 5-dimensional manifold Q without $\mathrm{Spin}^{\mathbb{C}}$-structure with the fundamental group $\pi_1(Q) = 1$. (see T. Friedrich "Dirac operators in Riemannian geometry")

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HW-manifold

Definition

By Hantzsche-Wendt manifold (HW-manifold) M^n we shall understand any oriented flat manifold of dimension n with a holonomy group $(\mathbb{Z}_2)^{n-1}$

Examples in five-dimension

Proposition

Two HW*-manifolds of dimension five have not the* $\mathrm{Spin}^{\mathbb{C}}$*-structure.*

Theorem2

Definition

The HW*-manifold* M^n *of dimension* n*, is cyclic if and only if* $\pi_1(M^n)$ *is generated by the following elements*

$$\beta_i = (B_i, (0, 0, 0, \dots, 0, \underbrace{1/2}_{i}, 1/2, 0, \dots, 0)), 1 \leqslant i \leqslant n-1,$$

$$\beta_n = (\beta_1 \beta_2 \dots \beta_{n-1})^{-1} = (B_n, (1/2, 0, \dots, 0, -1/2)).$$

We have

Theorem

Cyclic HW*-manifolds have not the* $\mathrm{Spin}^{\mathbb{C}}$*-structure.*

Theorem2

Definition

The HW-manifold M^n of dimension n, is cyclic if and only if $\pi_1(M^n)$ is generated by the following elements

$$\beta_i = (B_i, (0,0,0,\ldots,0,\underbrace{1/2}_{i},1/2,0,\ldots,0)), 1 \leqslant i \leqslant n-1,$$

$$\beta_n = (\beta_1\beta_2\ldots\beta_{n-1})^{-1} = (B_n, (1/2,0,\ldots,0,-1/2)).$$

We have

Theorem

Cyclic HW-manifolds have not the $\mathrm{Spin}^{\mathbb{C}}$*-structure.*