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Crystallographic groups with trivial center and outer automorphism group

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Abstract

Let Γ be a crystallographic group of dimension n, i.e. a discrete, cocompact subgroup of $\mathrm{Isom}\,(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$. For any $n \geqslant 2$, we construct a crystallographic group with a trivial center and trivial outer automorphism group.

1. *Introduction*

Let Γ be a discrete, cocompact subgroup of $O(n) \ltimes \mathbb{R}^n = \mathrm{Isom}\,(\mathbb{R}^n)$ i.e. a crystallographic group. If Γ is a torsion free group, then $M = \mathbb{R}^n/\Gamma$ is a flat manifold (that is a compact Riemannian manifold without boundary with the sectional curvature $K_x = 0$ for any $x \in M$). Moreover $\pi_1(M) = \Gamma$.

In 2003 R. Waldmüller found a torsion free crystallographic group $\Gamma \subset O(141) \ltimes \mathbb{R}^{141}$ (a flat manifold $M = \mathbb{R}^{141}/\Gamma$) with the following properties:

$$Z(\Gamma) = \{e\} \tag{1·1 a}$$

and

$$\mathrm{Out}\,(\Gamma) = \{e\}, \tag{1·1 b}$$

where $Z(\Gamma)$ is the center of the group Γ, and $\mathrm{Out}\,(\Gamma) = \mathrm{Aut}\,(\Gamma)/\,\mathrm{Inn}\,(\Gamma)$ denotes the group of outer automorphisms of Γ (see [**7**, appendix C] and [**8**]). Equivalently, (1·1 a) means that the abelianization of Γ is finite (the first Betti number of M is equal to zero). Moreover, if both conditions (1·1 a) and (1·1 b) are satisfied, then the group of affine diffeomorphisms $\mathrm{Aff}\,(M)$ of the manifold M is trivial (see [**2**] and [**7**]).

We do not know if there exist such flat manifolds in dimensions less than 141. For example (see [**4**]), in dimensions up to six such Bieberbach groups do not exist. In this paper we are interested in the existence of not necessarily torsion free crystallographic groups with the above properties. We shall prove that for any $n \geqslant 2$ there exists a crystallographic group of dimension n which satisfies conditions (1·1 a–b).

The main motivation for us is the paper by M. Belolipetsky and A. Lubotzky, [**1**]. For any $n \geqslant 3$ they found an infinite family of hyperbolic compact manifolds of dimension n with the following property: for every manifold M from this family, $\mathrm{Out}\,(\pi_1(M)) = \{e\}$. Since the center of the fundamental group of a compact hyperbolic manifold is trivial, the above result gives us an infinite family of groups which satisfy conditions (1·1 a–b). The construction of

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the above hyperbolic examples uses the properties of simple Lie groups of $\mathbb{R}$-rank one and, in particular, follows from the existence of non arithmetic lattices.

In our construction the most important are Bieberbach theorems and specific properties of crystallographic groups.

2. *Crystallographic groups with trivial center and outer automorphism group*

In this part we shall prove our main result. Let Γ be a crystallographic group. From Bieberbach's theorems (see [**7**, chapter 2]) we have a short exact sequence of groups

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \Gamma \xrightarrow{p} G \longrightarrow 0,$$

where $\mathbb{Z}^n$ is the maximal abelian normal subgroup of Γ and G is a finite group. Moreover, let $h_\Gamma : G \to \mathrm{GL}(n, \mathbb{Z})$ be the integral holonomy representation, given by the formula

$$\forall_{g\in G}\ h_\Gamma(g)(e) = \bar{g}e\bar{g}^{-1},$$

where $\bar{g} \in \Gamma$, $p(\bar{g}) = g$ and $e \in \mathbb{Z}^n$. Let

$$N := N_{\mathrm{GL}(n,\mathbb{Z})}(h_\Gamma(G)) = \{X \in \mathrm{GL}(n, \mathbb{Z}) \mid \forall_{f\in h_\Gamma(G)}\ \ XfX^{-1} \in h_\Gamma(G)\}$$

be the normaliser of $h_\Gamma(G)$ in $\mathrm{GL}(n, \mathbb{Z})$. In the case when $Z(\Gamma) = \{e\}$, we have the following commutative diagram [**7**, pp. 65–69] with exact rows and columns:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{Z}^n \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "h_\Gamma"] & 0 \\
0 \arrow[r] & Z^1(G, \mathbb{Z}^n) \arrow[r] \arrow[d] & \mathrm{Aut}(\Gamma) \arrow[r, "F"] \arrow[d] & N_\alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^1(G, \mathbb{Z}^n) \arrow[r] \arrow[d] & \mathrm{Out}(\Gamma) \arrow[r] \arrow[d] & N_\alpha/G \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2·1}$$

where $Z^1(G, \mathbb{Z}^n)$ is the group of 1-cocycles. Moreover

$$N_\alpha = \{n \in N \mid n * \alpha = \alpha\}$$

and $\alpha \in H^2(G, \mathbb{Z}^n)$ is the cohomology class of the first row of the diagram. The action $* : N \times H^2(G, \mathbb{Z}^n) \to H^2(G, \mathbb{Z}^n)$ is defined by the formula

$$n * [a] = [n * a],$$

where $n \in N$, $a \in Z^2(G, \mathbb{Z}^n)$, $[a]$ is the cohomology class of a and

$$\forall_{g_1,g_2\in G}\ n * a(g_1, g_2) = na(n^{-1}g_1n, n^{-1}g_2n). \tag{2·2}$$

We have the following proposition:

PROPOSITION 2·1. Aut(Γ) *is a crystallographic group if and only if* Out(Γ) *is a finite group.*

Proof. We start with an observation that $Z^1(G, \mathbb{Z}^n)$ is a free abelian group of rank n which is a faithful N_α module. First, assume that Aut(Γ) is a crystallographic group with the maximal abelian subgroup M. From [**2**, proposition I·4·1], M is the unique normal maximal abelian subgroup of Aut(Γ). Hence, $M = Z^1(G, \mathbb{Z}^n)$, and Out(Γ) is a finite group. The reverse implication is obvious. This completes the proof of the proposition.

Let us formulate our main result.

THEOREM 2·2. *For every* $n \geqslant 2$ *there exists a crystallographic group* Γ *of dimension* n *with* $Z(\Gamma) = \mathrm{Out}(\Gamma) = \{e\}$.

We shall need few lemmas and examples.

LEMMA 2·3. *Let* G, H *be finite groups and* $H \subset G \subset \mathrm{GL}(n, \mathbb{Z})$. *If the group* $N_{\mathrm{GL}(n,\mathbb{Z})}(H)$ *is finite, then* $N_{\mathrm{GL}(n,\mathbb{Z})}(G)$ *is finite.*

Proof. From the assumption, Aut(H) and Aut(G) are finite. Moreover, we have monomorphisms:

$$N_{\mathrm{GL}(n,\mathbb{Z})}(H)/C_{\mathrm{GL}(n,\mathbb{Z})}(H) \xrightarrow{\bar{\phi}} \mathrm{Aut}(H)$$

and

$$N_{\mathrm{GL}(n,\mathbb{Z})}(G)/C_{\mathrm{GL}(n,\mathbb{Z})}(G) \xrightarrow{\bar{\phi}} \mathrm{Aut}(G),$$

where $\bar{\phi}$ is induced by $\phi(s)(g) = sgs^{-1}, g \in G, s \in \mathrm{GL}(n, \mathbb{Z})$. Since $C_{\mathrm{GL}(n,\mathbb{Z})}(G) \subset C_{\mathrm{GL}(n,\mathbb{Z})}(H)$, our lemma is proved.

Using the above lemma for the groups $h_\Gamma(G) \subset N_\alpha$, Proposition 2·1 and [**6**, theorem 1] we get

COROLLARY 2·4. *If* $|\,\mathrm{Out}(\Gamma)\,| < \infty$, *then* $|\,\mathrm{Out}(\mathrm{Aut}(\Gamma))\,| < \infty$.

LEMMA 2·5. *Assume* $Z(\Gamma) = \{e\}$, *then*:

(i) $H^1(G, \mathbb{Z}^n) \simeq (\mathbb{Q}^n/\mathbb{Z}^n)^G = H^0(G, \mathbb{Q}^n/\mathbb{Z}^n)$;

(ii) $A^0(\Gamma) := \{m \in \mathbb{Q}^n \mid \forall_{g\in G}\ \ gm - m \in \mathbb{Z}^n\} \simeq Z^1(G, \mathbb{Z}^n)$ *as* N_α *modules*;

(iii) $A(\Gamma) := N_{\mathrm{Aff}(\mathbb{R}^n)}(\Gamma) = \{a \in \mathrm{Aff}(\mathbb{R}^n) \mid \forall_{\gamma\in\Gamma}\ a\gamma a^{-1} \in \Gamma\} \simeq \mathrm{Aut}(\Gamma)$.

Proof. To the short exact sequence of G-modules

$$0 \longrightarrow \mathbb{Z}^n \longrightarrow \mathbb{Q}^n \longrightarrow \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow 0$$

we have the following long exact sequence of cohomology groups attached:

$$0 \longrightarrow H^0(G, \mathbb{Z}^n) \longrightarrow H^0(G, \mathbb{Q}^n) \longrightarrow H^0(G, \mathbb{Q}^n/\mathbb{Z}^n) \longrightarrow H^1(G, \mathbb{Z}^n) \longrightarrow H^1(G, \mathbb{Q}^n) \longrightarrow \cdots$$

Since $H^1(G, \mathbb{Q}^n) = 0$ and by assumption $Z(\Gamma) \simeq (\mathbb{Z}^n)^G = H^0(G, \mathbb{Z}^n) = 0$ we also get $H^0(G, \mathbb{Q}^n) = (\mathbb{Q}^n)^G = 0$ and part (i) follows.

Now consider a homomorphism $\Phi\colon A^0(\Gamma) \to Z^1(G, \mathbb{Z}^n)$ of N_α modules given by the formula

$$\forall_{m\in A^0(\Gamma)} \forall_{g\in G}\, \Phi(m)(g) = gm - m.$$

Note that the action '$*$' defined by equation (2·2) can be extended to any cocycle group (see

[**2**, page 168]) and this is the N_α module structure on $Z^1(G, \mathbb{Z}^n)$ that we use. Recall that $H^1(G, \mathbb{Q}^n) = 0$, hence every cocycle from the group $Z^1(G, \mathbb{Q}^n)$ is a coboundary and Φ is onto. Easy calculation shows that

$$\ker \Phi = (\mathbb{Q}^n)^G = 0$$

and by the isomorphism theorem we prove part (ii) of the lemma.

By [**7**, theorem 5·2] we have the following short exact sequence

$$0 \longrightarrow (\mathbb{R}^n)^G \longrightarrow A(\Gamma) \longrightarrow \mathrm{Aut}\,(\Gamma) \longrightarrow 1.$$

Using again the triviality of the center of Γ we get that $(\mathbb{R}^n)^G = 0$ and the groups $A(\Gamma)$ and $\mathrm{Aut}\,(\Gamma)$ are isomorphic.

We get the following modification of the diagram (2·1).

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mathbb{Z}^n \arrow[r] \arrow[d] & \Gamma \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "h_\Gamma"] & 0 \\
0 \arrow[r] & A^0(\Gamma) \arrow[r] \arrow[d] & A(\Gamma) \arrow[r, "F"] \arrow[d] & N_\alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathbb{Q}^n/\mathbb{Z}^n)^G \arrow[r] \arrow[d] & \mathrm{Out}\,(\Gamma) \arrow[r] \arrow[d] & N_\alpha/G \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{2·3}$$

Let Γ be a crystallographic group of rank n with trivial center and holonomy group G. Moreover, assume that the group $H^1(G, \mathbb{Z}^n) = 0$, and the group $\mathrm{Out}\,(\Gamma)$ is finite. Inductively, put $\Gamma_0 = \Gamma$ and $\Gamma_{i+1} = A(\Gamma_i)$, for $i \geqslant 0$.

LEMMA 2·6. *$\exists N$ such that $\Gamma_{N+1} = \Gamma_N$.*

Proof. We start from observations that for $i > 0$, Γ_i is a crystallographic group, $Z(\Gamma_i) = \{e\}$ and $M_0 = M_i$, where $M_i = A^0(\Gamma_{i-1}) \subset \Gamma_i$ is the maximal abelian normal subgroup (a subgroup of translations). Let $G_i = \Gamma_i/M_i$. From definition we can consider (G_i) as a nondecreasing sequence of finite subgroups of $\mathrm{GL}\,(n, \mathbb{Z})$. From Bieberbach theorems [**7**, chapter 2] and from diagrams (2·1) and (2·3), there is only a finite number of possibilities for G_i. Hence $\exists N \in \mathbb{N}$ such that $\forall_{i>N}\ \ G_i = G_N$. This finishes the proof.

Example 2·1. Let $\Gamma_1 = G_1 \ltimes \mathbb{Z}^2$ be the crystallographic group of dimension 2 with holonomy group $G_1 = D_{12}$, where

$$D_{12} = \mathrm{gen}\left\{\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right\}$$

is the dihedral group of order 12. Moreover, let $\Gamma_2 = G_2 \ltimes \mathbb{Z}^3$ be the crystallographic group

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of dimension 3, with holonomy group $G_2 = S_4 \times \mathbb{Z}_2$ generated by matrices

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Here S_4 denotes the symmetric group on four letters.

LEMMA 2·7. *For $i = 1, 2$ Γ_i is centerless and we have*

$$N_{\mathrm{GL}\,(n_i,\mathbb{Z})}(G_i) = G_i$$

and

$$H^1(G_i, \mathbb{Z}^{n_i}) = 0,$$

where n_i is the rank of Γ_i.

Proof. First of all note that the representations of both groups defined by the identity maps are absolutely irreducible (and non-trivial). Hence the center

$$Z(\Gamma_i) \simeq (\mathbb{Z}^{n_i})^{G_i} = 0$$

is trivial and using Schur's Lemma [**5**, proposition 4, page 13] one gets

$$C_{\mathrm{GL}\,(n_i,\mathbb{Z})}(G_i) = \{\pm I_{n_i}\} \subset G_i,$$

where I_{n_i} id the identity matrix of degree n_i, for $i = 1, 2$.

Now $\mathrm{Out}\,(G_i)$ is a cyclic group of order two for $i = 1, 2$. Consider non-inner automorphism of G_1 defined as follows

$$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \longmapsto \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \longmapsto -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

An easy calculation shows that this automorphism cannot be realized as a conjugation by an element of $\mathrm{GL}\,(2, \mathbb{Z})$.

As for the group G_2, if you identify it with the group generated by the cycles $(1\ 2)$, $(1\ 2\ 3\ 4)$ and $(5, 6)$ you'll get

$$B \longleftrightarrow (1, 2) \text{ and } C \longleftrightarrow (2, 3, 4)(5, 6).$$

Consider an automorphism of G_2 which corresponds to the automorphism of the permutation group defined by

$$(1\ 2) \longmapsto (1\ 2)(5\ 6), \quad (1\ 2\ 3\ 4) \longmapsto (1\ 2\ 3\ 4)(5, 6), \quad (5, 6) \longmapsto (5, 6).$$

In that case B is mapped to $-B$ and hence traces of those matrices differ, the automorphism cannot be realized by a conjugation inside $\mathrm{GL}\,(3, \mathbb{Z})$.

The cohomology groups can be calculated using Lemma 2·5.

COROLLARY 2·8. $A(\Gamma_i) = \Gamma_i$ *and* $\mathrm{Out}\,(\Gamma_i) = \{e\}$ *for* $i = 1, 2$.

Now we are ready to finish the proof of the main theorem.

Proof of Theorem 2·2. The cases $n = 2, 3$ are done by Corollary 2·8. Assume $n \geqslant 4$. Let $n = 2k + 3i$, where $i \in \{0, 1\}$. Put $\Gamma' = \Gamma_1^k \times \Gamma_2^i$. Then Γ' is centerless and since in

[**3**, theorem 3·4] the torsion-free assumption is not really necessary, the bottom exact sequence of the diagram (2·3) looks as follows

$$0 \longrightarrow 0 \longrightarrow \mathrm{Out}\,(\Gamma') \longrightarrow S_k \longrightarrow 0.$$

Hence, Γ' satisfies the assumption of Lemma 2·6 and the sequence $\Gamma_0 = \Gamma'$, $\Gamma_{i+1} = A(\Gamma_i)$ stabilizes, i.e., $\exists N$ such that $\forall_{i \geqslant N}\ \Gamma_i = \Gamma_N$. Moreover, $\mathrm{Out}\,(\Gamma_N) = \{e\}$ and $Z(\Gamma_N) = \{e\}$.

REFERENCES

[**1**] M. BELOLIPETSKY and A. LUBOTZKY. Finite groups and hyperbolic manifolds. *Invent. Math.* **162** (2005), no. 3, 459–472.

[**2**] L.S. CHARLAP. *Bieberbach Groups and Flat Manifolds.* Universitext (Springer-Verlag. New York, 1986).

[**3**] R. LUTOWSKI. On symmetry of flat manifolds. *Exp. Math.* **18** (2009), no. 3, 201–204.

[**4**] R. LUTOWSKI. Finite outer automorphism groups of crystallographic groups. *Exp. Math.* **22** (2013), no. 4, 456–464.

[**5**] J.P. SERRE Linear Representations of Finite Groups (Springer-Verlag, New York-Heidelberg, 1977).

[**6**] A. SZCZEPAŃSKI. Problems on Bieberbach groups and flat manifolds. *Geom. Dedicata* **120** (2006), 111–118.

[**7**] A. SZCZEPAŃSKI. *Geometry of the crystallographic groups*. *Algebra and Discrete Mathematics*, vol. 4 (World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012).

[**8**] R. WALDMÜLLER. A flat manifold with no symmetries. *Exp. Math.* **12** (2003), no. 1, 71–77.