Minimal non-solvable Bieberbach groups

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February 23, 2023

Abstract

It has been shown by several authors that there exists a non-solvable Bieberbach group of dimension 15. In this note we show that this is in fact a minimal dimension for such kind of groups.

1 Introduction

Let Γ be an n-dimensional crystallographic group, i.e. a discrete and cocompact subgroup of $\mathrm{Iso}(\mathbb{R}^n) = O(n) \ltimes \mathbb{R}^n$, the group of isometries of an n-dimensional euclidean space. By the first Bieberbach theorem (see [20, Theorem 2.1]), the structure of Γ is described by the following short exact sequence

$$0 \longrightarrow L \longrightarrow \Gamma \xrightarrow{\pi} G \longrightarrow 1, \tag{1}$$

where G is a finite group (a holonomy group of Γ) and L is a free abelian group of rank n and it is the unique maximal abelian normal subgroup of Γ. The conjugation of elements of L in Γ gives L the structure of a G-module and so the map $\varphi\colon G \to \mathrm{GL}(L)$, defined by the formula

$$\varphi_g(l) = \gamma l \gamma^{-1},$$

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2020 *Mathematics Subject Classification*: Primary 20H15, Secondary 20F16, 20D10.
Key words and phrases: Bieberbach group, non-solvable, torsion-free, virtually abelian.

where $g \in G, l \in L$ and $\gamma \in \Gamma$ is such that $\pi(\gamma) = g$, is a homomorphism called *integral holonomy representation of* Γ.

In the case when Γ is in addition torsion-free, it is called a *Bieberbach group*, the orbit space $X = \mathbb{R}^n/\Gamma$ is a Riemannian manifold with zero sectional curvature (a *flat manifold*) and $\pi_1(X) = \Gamma$.

In this short note we deal with a structure of fundamental groups of flat manifolds – with their solvability to be precise. It is known that there exists a non-solvable Bieberbach group in dimension 15, see [6, proof of Theorem 2.1]. Recently J.A. Hillman asked, whether 15 is in fact the precise lower bound for the dimension of such groups, see [8]. We show that in fact it is:

Theorem 1.1. *The minimal dimension of a non-solvable Bieberbach group is* 15*.*

Since the solvability of a Bieberbach group is fully described by the solvability of its holonomy group, in Section 2 we will focus on finite non-solvable groups, with additional assumption that they are minimal, i.e. they do not have non-solvable proper subgroups. This allows us to reduce the number of groups to consider. The problem of finding a non-solvable Bieberbach group of minimal dimension can be in theory solved only by computer calculations. In practice, our attempts in larger dimensions failed because of taking too long time or not enough computer resources. We deal with this problem in Section 3 by determining minimal non-solvable groups of order up to one million. Having them at hand, the rest of calculations is quite efficient. Finally, in Section 4, we show that there is no non-solvable Bieberbach group in dimension less than 15.

Remark 1.2. In the paper we will use an abbreviation *MNS*, which will mean *minimal non-solvable*. It will be used in the context of finite and infinite groups as well.

2 Finite minimal non-solvable groups

In the poset of non-solvable subgroups of a given *finite* non-solvable group there is a minimal element. Hence the following definition is very natural.

Definition 2.1. A finite non-solvable group G is called *minimal non-solvable* if every proper subgroup of G is solvable.

Remark 2.2. In [21] Thompson defined a notion of a minimal simple group as a non-abelian finite simple group all of whose proper subgroups are solvable. As we will see, minimal non-solvable groups do not have to be simple, hence we consider strictly larger class of groups.

Lemma 2.3. *Any finite MNS group is perfect.*

Proof. If the commutator subgroup of G is proper, then G is solvable-by-abelian and hence – solvable. □

Since all subgroups of a solvable group are solvable, we get:

Lemma 2.4. *A finite group is MNS if and only if it is non-solvable and its maximal subgroups are solvable.*

Lemma 2.5. *Let G be a finite MNS group and let $\varphi\colon G \to H$ be a group epimorphism onto a non-trivial group H. Then H is a MNS group.*

Proof. Every proper subgroup of H is an epimorphic image of a proper, and hence solvable, subgroup of G. □

Lemma 2.6. *Let G be a finite MNS group. Then G has a unique maximal normal subgroup.*

Proof. Let H and K be maximal normal subgroups of G, such that $K \neq H$. By assumption, G/H is a simple non-abelian group. Since $H \subsetneq HK \lhd G$, we have $G = HK$. By the second isomorphism theorem

$$G/H = HK/H \cong K/(K \cap H)$$

is simple and hence K is non-solvable. Since every proper subgroup of G is solvable, we get $K = G$, a contradiction. □

3 Minimal non-solvable subgroups of $\mathrm{GL}_n(\mathbb{Z})$

For the purposes of this article we need to determine conjugacy classes of irreducible MNS subgroups of $\mathrm{GL}_n(\mathbb{Z})$, for $n \leq 10$. Maximal irreducible subgroups of $\mathrm{GL}_n(\mathbb{Z})$ for $n \leq 3$ have been calculated already in the nineteenth century, for $n = 4$ in 1965 by Dade [2], for $5 \leq n \leq 9$ by Plesken in papers [14, 15, 16, 17, 18] and for $n = 10$ by Souvignier in [19]. A library of those groups is available in GAP [4]. By Lemma 2.4 this should be enough for calculation of all MNS subgroups of $\mathrm{GL}_n(\mathbb{Z})$, however higher dimensions may be quite involved in terms of computational resources and hence we take another approach. Nevertheless, our calculations show that

Lemma 3.1. *Let $n \leq 10$ be a natural number. There is no MNS subgroup of $\mathrm{GL}_n(\mathbb{Z})$ of order greater than 10^6.*

By Lemma 2.3 every finite MNS group is perfect. Hence we can use the beforementioned criterion from Lemma 2.4 to the library of finite perfect groups from GAP to find all MNS groups of order up to 10^6 (*small MNS groups*). We get

Proposition 3.2. *There are 159 MNS groups of order less than or equal to 10^6. They are listed in Tables 1 and 2.*

Knowing small MNS groups, we will gather information about their rational representations. Let G be a group and $\mathrm{Irr}_K(G)$ be the set of characters of irreducible representations of G over the field K. By [11, Corollary 10.2(b)]

$$\mathrm{Irr}_{\mathbb{Q}}(G) = \{\chi_{\mathbb{Q}} : \chi \in \mathrm{Irr}(G)\}.$$

In the above formula, for every $\chi \in \mathrm{Irr}(G)$, we have

$$\chi_{\mathbb{Q}} = m_{\mathbb{Q}}(\chi) \sum \chi^{\sigma},$$

where $m_{\mathbb{Q}}(\chi)$ denotes the Schur index of χ and the sum is taken over $\sigma \in \mathrm{Gal}(\mathbb{Q}(\chi)/\mathbb{Q})$.

Remark 3.3. The crucial part in the determination $\chi_{\mathbb{Q}}$ for $\chi \in \mathrm{Irr}(G) = \mathrm{Irr}_{\mathbb{C}}(G)$ is the Schur index $m_{\mathbb{Q}}(\chi)$. The calculations are available with the GAP package WEDDERGA [1]. In some cases a more efficient approach based on calculating lower and upper bound for $m_{\mathbb{Q}}(\chi)$ is taken. The lower bound depends on the Frobenius-Schur index $\nu_2(\chi)$ of χ. It is set to 2 if $\nu_2(\chi) = -1$ and to 1 in other cases. The upper bound calculation is based on [11, Lemma 10.4].

Using the above remark we were able to compute $\mathrm{Irr}_{\mathbb{Q}}(G)$ for every small MNS group G. Looking at the faithful characters, we get

Proposition 3.4. *If G is a finite MNS irreducible subgroup of $\mathrm{GL}_n(\mathbb{Z})$ for $n \leq 10$, then G is one of the following groups:*

a) $n = 4$: A_5;

b) $n = 5$: A_5;

c) $n = 6$: $A_5, L_3(2)$;

d) $n = 7$: $L_3(2), L_2(8), L_3(2)N2^3$;

e) $n = 8$: $A_5 2^1 = \mathrm{SL}_2(5), L_3(2), L_3(2)2^1 = \mathrm{SL}_2(7), L_2(8)$.

In particular:

f) for $n \in \{1, 2, 3\}$ all finite subgroups of $\mathrm{GL}_n(\mathbb{Z})$ are solvable;

g) for $n \in \{9, 10\}$ there is no minimal non-solvable irreducible subgroup of $\mathrm{GL}_n(\mathbb{Z})$.

4 Minimal non-solvable Bieberbach groups

Assume that an n-dimensional Bieberbach group Γ is given by the short exact sequence (1).

Definition 4.1. Let Γ be a Bieberbach group as above. We will call Γ *minimal non-solvable* (*MNS*), if every subgroup Γ' of Γ such that

a) Γ' is of smaller dimension than Γ or

b) $\Gamma' = \pi^{-1}(H)$ for some proper subgroup H of G

is solvable.

Since we will often use specific cases of [6, Theorem 2.1], [13, Theorem V.1] and [13, Theorem VI.1], we put them here in the following form

Corollary 4.2. *Let G be a finite group. Let $m(G)$ be the smallest dimension of a Bieberbach group with holonomy G. Then $m(A_5) = m(L_3(2)) = 15$ and $m(A_5 2^1) \geq 15$.*

Proof. It is enough to use formulas in the before-mentioned theorems, noting that $A_5 \cong L_2(4) \cong L_2(5)$ and $L_3(2) \cong L_2(7)$. □

Theorem 1.1 is a direct result of the above corollary and the following proposition.

Proposition 4.3. *If Γ is a minimal non-solvable Bieberbach group, then $n \geq 15$.*

Directly from the definitions of finite and Bieberbach MNS groups we have

Corollary 4.4. *Holonomy group of a MNS Bieberbach group is MNS.*

Lemma 4.5. *Holonomy representation of a MNS Bieberbach group Γ does not have any trivial constituent.*

Proof. If the holonomy representation of Γ has a trivial constituent, then by [7, Proposition 1.4] we have a short exact sequence

$$1 \longrightarrow \Gamma' \longrightarrow \Gamma \longrightarrow \mathbb{Z} \longrightarrow 0.$$

By construction Γ' is of dimension less than the dimension of Γ and by assumption – solvable. Hence Γ is solvable, a contradiction. □

Lemma 4.6. *Let Γ, given by* (1), *be a MNS Bieberbach group. Let*

$$\mathbb{Q} \otimes_{\mathbb{Z}} L = L_1 \oplus \ldots, \oplus L_k \tag{2}$$

be a decomposition of $\mathbb{Q}G$-module into irreducible components. Let $\rho_i \colon G \to \mathrm{GL}(L_i)$ be a representation associated with the module L_i. Then:

a) $\dim_{\mathbb{Q}}(L_i) \geq 4$ *for every* $1 \leq i \leq k$,

b) if $\rho_i(G)$ is a simple group for some i, then $\bigoplus_{j \neq i} L_j$ is a faithful $\mathbb{Q}G$-module.

Proof. Let $i \in \{1, \dots, k\}$. By Lemma 4.5 ρ_i is non-trivial. Using Proposition 3.4 we get that if $\dim(L_i) < 4$ then $\rho_i(G)$ is solvable and hence – by Lemma 2.5 – G is not minimal non-solvable. Using Corollary 4.4 we get that Γ is not MNS.

By Lemma 2.6, the kernel of ρ_i is the maximal normal subgroup of G and hence $\ker \rho_j \subset \ker \rho_i$ for $1 \leq j \leq k$. Since the holonomy representation is faithful, we get

$$\bigcap_{j \neq i} \ker \rho_j \subset \bigcap \ker \rho_j = 1.$$

□

Corollary 4.7. *Let Γ be a minimal non-solvable Bieberbach group of dimension n. Then $n > 12$.*

Proof. By [12, Theorem 1], the holonomy representation of a non-abelian Bieberbach group contains, over the rationals, at least two non-isomorphic constituents. In particular, it is reducible (see [9]). Hence, by Lemma 4.6, $n \geq 8$. Let Γ be defined by (1) with rational holonomy representation decomposition given by (2). Using Proposition 3.4 and Lemma 2.6 we get that if $8 \leq n \leq 12$ and $k = 2$, then the possibilities for $\{\dim L_1, \dim L_2\}$ and G are as follows:

a) $\{4,4\}, \{4,5\}, \{4,6\}, \{5,5\}, \{5,6\}, \{6,6\}$ and $G = A_5$ or $G = L_3(2)$,

b) $\{4,8\}$ and $G = A_5 2^1$.

By Corollary 4.2 we get that $n \geq 15$, a contradiction.

If $k = 3$ then $\dim L_i = 4$ and $\ker \rho_i$ is the maximal subgroup of G, for $i = 1, 2, 3$. Hence all the kernels are equal and – because the holonomy representation $\rho_1 \oplus \rho_2 \oplus \rho_3$ is faithful – they are trivial. In that case $G = A_5$ and – as above – $n \geq 15$. □

From [9, Lemmas 2.1 and 2.2(a)] one gets the following lemma. For the sake of completeness, we give the proof here, following the before-mentioned lemmas.

Lemma 4.8. *Let L be a G-lattice, U be a subgroup of G of prime order p and $\alpha \in H^2(G, L)$. If $\operatorname{res}^G_U(\alpha) \neq 0$ then at least one constituent of $\mathbb{C} \otimes_{\mathbb{Z}} L$ lies in the principal p-block of G.*

Proof. Let $\mathbb{Z}_p$ denote the ring of p-adic integers and $i \colon L \to \mathbb{Z}_p \otimes_{\mathbb{Z}} L$ be the inclusion. By [13, Remark II.1(ii)], the restriction $\operatorname{res}^G_U i_*(\alpha) \neq 0$ and hence $H^2(G, \mathbb{Z}_p \otimes_{\mathbb{Z}} L) \neq 0$. By [10, Lemma 2.2.25] some direct summand U of $\mathbb{Z}_p \otimes_{\mathbb{Z}} L$ lies in the principal $\mathbb{Z}_p G$-block.

By the Brauer theorem $\mathbb{Q}(\zeta)$ is a splitting field for G, where $\zeta = \exp 2\pi i/|G|$ (see [11, (10.3)]). Let $\varphi \colon \mathbb{Q}(\zeta) \to \overline{\mathbb{Q}_p}$ be an embedding over $\mathbb{Q}$, where $\overline{\mathbb{Q}_p}$ is the algebraic closure of the field $\mathbb{Q}_p$ of p-adic numbers. Let $K = \varphi(\mathbb{Q}(\zeta))\mathbb{Q}_p$ and R be its ring of integers. Then φ induces a bijection between $\operatorname{Irr}(G)$ and $\operatorname{Irr}_K(G)$, which preserves p-blocks (see [5, (7.10)]). We finish by noting that there is a direct summand of $R \otimes_{\mathbb{Z}_p} U$ which lies in the principal p-block (see [3, Section VI.1]). □

We are ready to prove Proposition 4.3 and hence – the main theorem.

Proof of Proposition 4.3. Let Γ be as in (1), with the rational holonomy module decomposition into k irreducible summands as in (2). Denote by N the maximal normal subgroup of the holonomy group G. Note that we are left with the cases $n = 13$ or $n = 14$.

Let $k = 2$ and $\dim L_1 \leq \dim L_2$. If $\dim L_1 \leq 5$ then by Proposition 3.4 we get $\dim L_1 = 5, \dim L_2 = 8$ and $G = A_5 2^1$, but then, by Corollary 4.2, the dimension of Γ is at least 15. If $\dim L_1 = 6$, then $G/N \cong L_3(2)$ and either $N = 1$, which is excluded – again by Corollary 4.2 – or $N = C_2^3$ and $G = L_3(2)N2^3$, see Proposition 3.4. In order to get a faithful G-lattice of rank 13 or 14 one has to use one of two irreducible characters of degree 7, but in that case none of the possible characters of G lies in the principal 3-block, see Table 3. If $\dim L_1 = 7$ then two of the three possibilities for G are the same as before – namely $L_3(2), L_3(2)N2^3$ – and excluded for the same reasons. In the third case $G = L_2(8)$ and we get only one rational representation of dimension 7, hence it is excluded by [12, Theorem 1].

If $k = 3$, then the possibilities for $(\dim L_1, \dim L_2, \dim L_3)$ – assuming non-decreasing order – are as follows: $(4, 4, 5), (4, 4, 6), (4, 5, 5)$. By Lemma 2.6 and Proposition 3.4 $G = A_5$, for which $n \geq 15$ by Corollary 4.2. □

5 Two non-solvable Bieberbach groups

In this section we give explicit constructions of Bieberbach groups with holonomy groups A_5 and $L_3(2)$. Let Γ be one of those groups, with holonomy G. The integral holonomy representation of Γ is a direct sum of irreducible *left* G-lattices $L_1, \ldots, L_k$, where $L_i = \mathbb{Z}^{n_i}$ for $i = 1, \ldots, k$. We give matrices of actions of generators as well as the images of representatives $\delta_1, \ldots, \delta_k$ of cohomology classes under the isomorphisms $H^2(G, \mathbb{Z}^{n_i}) \cong H^1(G, \mathbb{Q}^{n_i}/\mathbb{Z}^{n_i})$. In addition, for each pair (L_i, δ_i) we give prime numbers p such that restriction of cohomology class of δ_i to a subgroup of G of order p is non-zero. All the data show that in fact we define a torsion-free crystallographic group.

Remark 5.1. All representations, except one, have been taken from [22]. The construction of the exception is explicitly presented.

5.1 Bieberbach group with holonomy A_5

The presentation of A_5 is as follows:

$$A_5 = \langle a, b \mid a^2 = b^3 = (ab)^5 = 1 \rangle$$

1. Lattice L_1, $p = 3$.

$$a \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & -1 & -1 & -1 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Cohomology class:

$$a \mapsto \begin{pmatrix} 0 & \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}^T \qquad b \mapsto \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}^T$$

2. Lattice L_2, $p = 2$.

$$a \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & -1 & -1 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 \end{pmatrix}$$

Cohomology class:

$$a \mapsto \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}^T \qquad b \mapsto \begin{pmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \end{pmatrix}^T$$

3. Lattice L_3, $p = 5$.

$$a \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Cohomology class:

$$a \mapsto \begin{pmatrix} \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{2}{5} & \frac{3}{5} & \frac{2}{5} \end{pmatrix}^T \qquad b \mapsto \begin{pmatrix} \frac{3}{5} & \frac{3}{5} & \frac{2}{5} & \frac{4}{5} & \frac{1}{5} & \frac{2}{5} \end{pmatrix}^T$$

5.2 Bieberbach group with holonomy $L_3(2)$

The presentation of $L_3(2)$ is as follows:

$$L_3(2) = \langle a, b \mid a^2 = b^3 = (ab)^7 = [a, b]^4 = 1 \rangle$$

1. Lattice L_1, $p = 2, 3$. This is a sublattice of the 7-dimensional one from [22] with basis

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}.$$

$$a \mapsto \begin{pmatrix} -1 & 0 & -1 & -1 & -1 & 3 & -2 \\ 0 & -1 & -1 & -1 & -1 & 3 & -2 \\ -1 & -1 & -1 & 0 & -1 & 3 & -2 \\ -1 & -1 & 0 & -1 & -1 & 3 & -2 \\ -1 & -1 & -1 & -1 & -1 & 4 & -2 \\ -1 & -1 & -1 & -1 & 0 & 3 & -2 \\ 0 & 0 & 0 & 0 & 2 & -2 & 1 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & -2 & 1 \\ 1 & 0 & 0 & 1 & 0 & -2 & 1 \\ 0 & 1 & 0 & 1 & 0 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

   Cohomology class:

$$a \mapsto \begin{pmatrix} \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \frac{2}{3} & \frac{2}{3} & \frac{1}{6} & 0 \end{pmatrix}^T \qquad b \mapsto \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & \frac{1}{2} & \frac{2}{3} & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \end{pmatrix}^T$$

2. Lattice L_2, $p = 7$.

$$a \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad b \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

   Cohomology class:

$$a \mapsto \begin{pmatrix} \frac{4}{7} & \frac{3}{7} & \frac{2}{7} & \frac{4}{7} & \frac{4}{7} & \frac{3}{7} & 0 & 0 \end{pmatrix}^T \qquad b \mapsto \begin{pmatrix} \frac{2}{7} & \frac{6}{7} & \frac{4}{7} & 0 & \frac{1}{7} & \frac{4}{7} & 0 & \frac{4}{7} \end{pmatrix}^T$$

Acknowledgments

The authors would like to thank J.A. Hillman for pointing our attention to the problem of finding non-solvable Bieberbach group of minimal dimension.

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Order	Id	Description
60	1	A_5
120	1	$A_5\, 2^1$
168	1	$L_3(2)$
336	1	$L_3(2)\, 2^1 = SL_2(7)$
504	1	$L_2(8)$
1092	1	$L_2(13)$
1344	2	$L_3(2)\,\mathrm{N}\,2^3$
1920	4	$A_5\, 2^1\,\mathrm{E}\,2^4$
2184	1	$L_2(13)\, 2^1 = SL_2(13)$
2448	1	$L_2(17)$
2688	3	$L_3(2)\, 2^1 \times \mathrm{N}\,2^3$
3840	6	$A_5\, 2^1\,\mathrm{E}\,2^4\,\mathrm{E}\,2^1$
4860	2	$A_5\,\mathrm{N}\,3^{4'}$
4896	1	$L_2(17)\, 2^1 = SL_2(17)$
5616	1	$L_3(3)$
6072	1	$L_2(23)$
7500	2	$A_5\,\mathrm{N}\,5^3$
9720	2	$A_5\, 2^1 \times \mathrm{N}\,3^{4'}$
9828	1	$L_2(27)$
10752	4	$L_3(2)\,\mathrm{N}\,2^3\,\mathrm{A}\,2^3$
10752	7	$L_3(2)\,\mathrm{N}\,2^3 \times \mathrm{N}\,2^{3'}$
10752	9	$L_3(2)\,\mathrm{N}\,2^3\,\mathrm{E}\,2^{3'}$
12144	1	$L_2(23)\, 2^1 = SL_2(23)$
15000	2	$A_5\, 2^1 \times \mathrm{N}\,5^3$
19656	1	$L_2(27)\, 2^1 = SL_2(27)$
21504	8	$L_3(2)\, 2^7$
21504	16	$L_3(2)\, 2^7$
21504	22	$L_3(2)\, 2^1 \times (\mathrm{N}\,2^3\,\mathrm{E}\,2^{3'})$
25308	1	$L_2(37)$
29120	1	Sz(8)
30720	11	$A_5\, 2^1\,\mathrm{E}\,2^4\,\mathrm{A}\,2^4$
30720	22	$A_5\, 2^1\,\mathrm{E}\,2^4\;\mathrm{C}\;2^{4'}$
32256	2	$L_2(8)\,\mathrm{N}\,2^6$
32736	1	$L_2(32)$
39732	1	$L_2(43)$
43008	19	$L_3(2)\, 2^1\,(\mathrm{N}\,2^3 \times \mathrm{N}\,2^{3'})\,\mathrm{E}\,2^1$
50616	1	$L_2(37)\, 2^1 = SL_2(37)$
51888	1	$L_2(47)$
57624	2	$L_3(2)\,\mathrm{N}\,7^3$
58240	1	Sz(8) 2^1
64512	2	$L_2(8)\,\mathrm{N}\,2^6\,\mathrm{E}\,2^1$ I
64512	3	$L_2(8)\,\mathrm{N}\,2^6\,\mathrm{E}\,2^1$ II
64512	4	$L_2(8)\,\mathrm{N}\,2^6\,\mathrm{E}\,2^1$ III
74412	1	$L_2(53)$
79464	1	$L_2(43)\, 2^1 = SL_2(43)$
103776	1	$L_2(47)\, 2^1 = SL_2(47)$
115248	2	$L_3(2)\, 2^1 \times \mathrm{N}\,7^3$
116480	1	Sz(8) $2^1 \times 2^1$
129024	2	$L_2(8)\mathrm{N}(2^6\,\mathrm{E}\,2^1\;\mathrm{A}\;)\;\mathrm{C}\;2^1$
129024	3	$L_2(8)\,\mathrm{N}\,2^6\mathrm{N}(2^1 \times 2^1)$ I
129024	4	$L_2(8)\,\mathrm{N}\,2^6\mathrm{N}(2^1 \times 2^1)$ II
129024	5	$L_2(8)\,\mathrm{N}\,2^6\mathrm{N}(2^1 \times 2^1)$ III
148824	1	$L_2(53)\, 2^1 = SL_2(53)$
150348	1	$L_2(67)$
155520	12	$A_5 \,\#\, 2^5\, 3^4$ [11]
194472	1	$L_2(73)$
240000	11	$A_5 \,\#\, 2^5\, 5^3$ [11]
258048	2	$L_2(8)\,\mathrm{N}\,(\,2^6\,\mathrm{N}\,(2^1 \times 2^1\;\mathrm{A}\,)\,)\mathrm{C}\,2^1$
258048	3	$L_2(8)\,\mathrm{N}\,2^6\,\mathrm{N}\,(2^1 \times 2^1 \times 2^1)$
285852	1	$L_2(83)$
300696	1	$L_2(67)\, 2^1 = SL_2(67)$
311040	14	$A_5 \,\#\, 2^6\, 3^4$ [13]
367416	3	$L_3(2)\,\mathrm{N}\,3^7$
388944	1	$L_2(73)\, 2^1 = SL_2(73)$
393660	4	$A_5\,\mathrm{N}\,3^{4'}\,\mathrm{A}\,3^{4'}$
456288	1	$L_2(97)$
460992	4	$L_3(2) \,\#\, 2^3\, 7^3$ [4]
480000	13	$A_5 \,\#\, 2^6\, 5^3$ [13]
516096	1	$L_2(8)\,\mathrm{N}\,(\,2^6\,\mathrm{N}\,(\,2^1 \times 2^1 \times 2^1\;\mathrm{A}\,)\,)\;\mathrm{C}\,2^1$
546312	1	$L_2(103)$
571704	1	$L_2(83)\, 2^1 = SL_2(83)$
607500	4	$A_5 \,\#\, 3^4\, 5^3$ [4]
612468	1	$L_2(107)$
721392	1	$L_2(113)$
734832	3	$L_3(2)\, 2^1 \times \mathrm{N}\,3^7$
787320	4	$A_5\, 2^1 \times \mathrm{N}\,3^{4'}\,\mathrm{A}\,3^{4'}$
912576	1	$L_2(97)\, 2^1 = SL_2(97)$
921984	6	$L_3(2) \,\#\, 2^4\, 7^3$ [6]
937500	7	$A_5\,\mathrm{N}\,5^3\,\mathrm{E}\,5^3$
937500	8	$A_5\,\mathrm{N}\,5^3\,\mathrm{C}\,5^3$

Table 1: MNS groups of order up to 10^6 listed in [10]. Order and id give identification of perfect group in GAP.

Order	Id	Order	Id	Order	Id	Order	Id	Order	Id	Order	Id
61440	13	86016	40	172032	128	688128	178	688128	191	983040	64
61440	14	86016	41	172032	129	688128	179	688128	192	983040	66
61440	15	122880	88	172032	151	688128	180	688128	226	983040	67
61440	52	122880	89	172032	152	688128	181	688128	227	983040	69
61440	53	122880	90	245760	566	688128	182	688128	228	983040	72
61440	54	122880	218	344064	191	688128	183	688128	229	983040	104
61440	76	122880	219	344064	268	688128	184	688128	230	983040	105
86016	24	172032	1	344064	269	688128	185	688128	231	983040	361
86016	25	172032	91	344064	291	688128	186	688128	232	983040	362
86016	26	172032	92	491520	19	688128	187	688128	251	983040	363
86016	27	172032	93	491520	21	688128	188	688128	252	983040	371
86016	28	172032	94	688128	176	688128	189	688128	253	983040	372
86016	35	172032	95	688128	177	688128	190	688128	254	983040	373
86016	36										

Table 2: MNS groups of order up to 10^6 not listed in [10]. Order and id give identification of perfect group in GAP.

	$1a$	$2a$	$2b$	$3a$	$4a$	$4b$	$6a$	$7a$	$7b$	$8a$	$8b$	m	B_2	B_3	B_7
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	3	3	−1	0	−1	−1	0	α	$\overline{\alpha}$	1	1	1	1	2	1
χ_3	3	3	−1	0	−1	−1	0	$\overline{\alpha}$	α	1	1	1	1	3	1
χ_4	6	6	2	0	2	2	0	−1	−1	0	0	1	1	4	1
χ_5	7	7	−1	1	−1	−1	1	0	0	−1	−1	1	1	1	2
χ_6	7	−1	−1	1	3	−1	−1	0	0	1	−1	1	1	5	3
χ_7	7	−1	−1	1	−1	3	−1	0	0	−1	1	1	1	5	4
χ_8	8	8	0	−1	0	0	−1	1	1	0	0	1	1	1	1
χ_9	14	−2	−2	−1	2	2	1	0	0	0	0	1	1	5	5
χ_{10}	21	−3	1	0	1	−3	0	0	0	−1	1	1	1	6	6
χ_{11}	21	−3	1	0	−3	1	0	0	0	1	−1	1	1	7	7

$\alpha = -(1 + i\sqrt{7})/2$

Table 3: Character table of $L_3(2)N2^3$. Conjugacy classes are named by the orders of their elements, suffixed by a letter, m denotes the Schur index and p-blocks in column B_p are labeled by natural numbers.