Hantzsche-Wendt manifolds

Andrzej Szczepański

University of Gdańsk, Poland

Moscow, September 16, 2016

Crystallographic groups

Let $\mathbb{R}^n$ be n-dimensional Euclidean space, with isometry group $E(n) = O(n) \ltimes \mathbb{R}^n$.

Definition

Γ is a crystallographic group of rank n iff it is a discrete and cocompact subgroup of $E(n)$.

A Bieberbach group is a torsion free crystallographic group.

Basic properties

Theorem

(Bieberbach, 1910)

- *If Γ is a crystallographic group of dimension n, then the set of all translations of Γ is a maximal abelian subgroup of a finite index.*
- *There is only a finite number of isomorphic classes of crystallographic groups of dimension n.*
- *Two crystallographic groups of dimension n are isomorphic if and only if there are conjugate in the group affine transformations $A(n) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$.*

Pure abstract point of view

Theorem

(Zassenhaus, 1947) A group Γ is a crystallographic group of dimension n if and only if, it has a normal maximal abelian subgroup $\mathbb{Z}^n$ of a finite index.

Holonomy representation

Definition

Let Γ be a crystallographic group of dimension n with translations subgroup $A \simeq \mathbb{Z}^n$. A finite group $\Gamma/A = G$ we shall call a holonomy group of Γ.

Let $(A, a) \in E(n)$ and $x \in \mathbb{R}^n$. Γ acts on $\mathbb{R}^n$ in the following way:

$$(A, a)(x) = Ax + a.$$

Definition

Let Γ be n-dimensional Bieberbach group. We have the following short exact sequence of groups.

$$0 \to \mathbb{Z}^n \to \Gamma \stackrel{p}{\to} \Gamma/\mathbb{Z}^n = H \to 0.$$

Let us define a homomorphism $h_\Gamma : H \to GL(n, \mathbb{Z})$. Put

$$\forall h \in H, h_\Gamma(h)(e_i) = \bar{h}^{-1} e_i \bar{h},$$

where $p(\bar{h}) = h$ and $e_i \in \mathbb{Z}^n$ is a standard basis. h_Γ is called a holonomy representation of a group Γ.

Flat manifold

Let $\Gamma \subset E(n)$ be a torsion free crystallographic group. Since Γ is cocompact and discrete subgroup, then the orbit space $\mathbb{R}^n/\Gamma$ is a manifold. If Γ is not torsion free then the orbit space $\mathbb{R}^n/\Gamma$ is an orbifold.

Definition

The above manifolds (orbifolds) we shall call "flat".

From elementary covering theory any compact Riemannian manifold (orbifold) with sectional curvature equal to zero is flat.

Example

Flat surfaces:

- torus $S^1 \times S^1$,
- Klein bottle $S^1 \times S^1/\mathbb{Z}_2$

We shall see that many properties of the Bieberbach Groups correspond to properties of flat manifolds.

Definition

Definition

Any flat manifold of dimension n with holonomy group $(\mathbb{Z}_2)^{n-1}$ we shall call Generalized Hantzsche-Wendt (GHW for short) manifold. An oriented GHW we shall call Hantzsche-Wendt manifold (HW for short).

A fundamental group of GHW (HW) flat manifolds we shall call GHW (HW) Bieberbach group.

Exercise: Let M^n be a flat Hantzsche-Wendt manifold of dimension n. Shaw that n is an odd natural number.

Example

Klein Bottle is GHW, 3-dimensional flat oriented manifold M^3 with non-cyclic holonomy $\mathbb{Z}_2 \times \mathbb{Z}_2$ is HW manifold. J.Conway called it "didicosm".

It can be proved that $\pi_1(M^3)$ is a Fibonacci group $F(2,6)$, where

$$F(2,6) = \{x_1, ..., x_6 \mid x_1 x_2 = x_3, x_2 x_3 = x_4, ..., x_6 x_1 = x_2\}.$$

It is clear, that any n-dimensional HW group is related to an element of the second cohomology group $H^2(\mathbb{Z}_2^{n-1}, \mathbb{Z}^n)$. Hence the number of non-isomorphic GHW groups of given dimension growth exponentialy.

Theorem

(J.P.Rossetti, A.S. 2005) Let Γ *be a* n*-dimensional GHW group. Then*

$$h_\Gamma((\mathbb{Z}_2)^{n-1}) \subset GL(n,\mathbb{Z})$$

is a set of the diagonal matrices with ± 1 *on diagonal.*

The proof is by induction and used the following lemmas.

Lemma 1 Let $\rho : \mathbb{Z}_2^{n-1} \to GL(n,\mathbb{Z})$ be a diagonal faithful integral representation with -Id $\notin Im(\rho)$. Then there are $g \in \mathbb{Z}_2^{n-1}$ and $1 \leq i \leq n$ such that $\rho(g) =$ diag$[-1,-1,...,-1,\underbrace{1}_{i},-1,...,-1]$.

Moreover, if $Im(\rho) \not\subset SL(n,\mathbb{Z})$, then there are $g \in \mathbb{Z}_2^{n-1}$ and $1 \leq i \leq n$ such that $\rho(g) =$ diag$[1,...,1,\underbrace{-1}_{i},1,...,1]$.

Lemma 2 Let Γ be a n-rank Bieberbach group with translation lattice Λ. Suppose that $(B, b) \in \Gamma$ and B has eigenvalues 1 and -1, with corresponding eigenspaces V^+ and V^- of dimension 1 and $n-1$ respectively.

Then $\Lambda = (\Lambda \cap V^+) \oplus (\Lambda \cap V^-)$, and the orthogonal projection of b onto V^+ lies in $1/2(\Lambda \cap V^+) \setminus (\Lambda \cap V^+)$.

Rational homology sphere

Theorem

If $M = \mathbb{R}^n/\Gamma$, where Γ is HW, then M is a rational homology sphere.

We calculate rational homology of M from the following formula

$$H_i(M, \mathbb{Q}) = (\Lambda^i(\mathbb{Z}^n))^{\mathbb{Z}_2^{n-1}}.$$

From definition of the holonomy representation they are zero for all $i \neq 0, n$. Since HW is connected and oriented, then M is a rational homology sphere.

Abelianization

Theorem

(B.Putrycz, 2006) Let M be any HW flat manifold of dimension $n > 3$, then

$$H_1(M, \mathbb{Z}) = (\mathbb{Z}_2)^{n-1}.$$

In the proof are used methods proposed by R. Miatello and J. P. Rossetti.

Homology over $\mathbb{Z}$

In 2008 K. Dekimpe and N. Petrosyan calculate, by constructing a free resolution, homology of low dimensional oriented GHW groups/manifolds. In dimension five there are this same (for 2 groups). But in dimension 7 there are four classes on 62 groups. They find an algorithm.

Let us present the table for dimension five (2 groups):

$H_k(\Gamma)$	k = 0	k = 1	k = 2	k = 3	k = 4	k = 5
	$\mathbb{Z}$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	0	$\mathbb{Z}$

,

for dimension seven (62 groups):

$H_k(\Gamma)$	k=0	k=1	k=2	k=3	k=4	k=5	k=6	k=7
I	$\mathbb{Z}$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^8$	$\mathbb{Z}_4^6$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^6$	0	$\mathbb{Z}$
II	$\mathbb{Z}$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^9$	$\mathbb{Z}_2^{10} \oplus \mathbb{Z}_4^2$	$\mathbb{Z}_2^9$	$\mathbb{Z}_2^6$	0	$\mathbb{Z}$
III	$\mathbb{Z}$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^4 \oplus \mathbb{Z}_4^4$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^6$	0	$\mathbb{Z}$
IV	$\mathbb{Z}$	$\mathbb{Z}_2^6$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^8 \oplus \mathbb{Z}_4^2$	$\mathbb{Z}_2^8$	$\mathbb{Z}_2^6$	0	$\mathbb{Z}$

.

Cohomological rigidity

Definition

Two flat manifolds M_1 and M_2 are cohomological rigid iff a homeomorphism between M_1 and M_2 is equivalent to an isomorphism of graded rings $H^*(M_1.\mathbb{F}_2)$ and $H^*(M_2, \mathbb{F}_2)$.

Theorem

(J. Popko, A.S. 2013) Hantzsche-Wendt manifolds are cohomological rigid.

Idea of proof:

We apply the Lyndon-Hochschild-Serre spectral sequence $\{E_r^{p,q}, d_r\}$ of the covering $T^n \to T^n/(\mathbb{Z}_2)^{n-1}$ with $\mathbb{F}_2$ coefficients. Since a holonomy representation h_Γ is diagonal $E_2^{p,q} = H^p((\mathbb{Z}_2)^{n-1}) \otimes H^q(\mathbb{Z}^n)$. We only use the multiplicative structure of the first and second cohomology group. We have an exact sequence

$$H^1(\mathbb{Z}^n, \mathbb{F}_2) \xrightarrow{d_2} H^2((\mathbb{Z}_2)^{n-1}, \mathbb{F}_2) \xrightarrow{p^*} H^2(\Gamma, \mathbb{F}_2),$$

where d_2 is a transgression, p^* is induced by the homomorphism $p : \Gamma \to \mathbb{Z}_2^{n-1}$ and Γ is a HW-group.

Idea of proof:

A $\mathbb{F}_2$ - rank of

$$\text{Im}(d_2) \subset H^2((\mathbb{Z}_2)^{n-1}, \mathbb{F}_2) \subset H^*((\mathbb{Z}_2)^{n-1}, \mathbb{F}_2) \simeq \mathbb{F}_2[x_1, x_2, ..., x_{n-1}]$$

is equal to n.
Let $T_i = d_2(t_i)$, where $t_1, t_2, ..., t_n$ is the standard $\mathbb{F}_2$ - basis of $H^1(\mathbb{Z}^n, \mathbb{F}_2)$ for $i = 1, 2, ..., n$.
Moreover, let

$$D = \{y \in \text{Im}(d_2) \mid y \in \text{is a product of two polynomials of degree 1}\}.$$

Idea of proof:

Main Lemma. The set D define a HW-manifold in an unique way up to affine equivalence.

Remark. 1. $D = \{T_1, T_2, ..., T_n\}$ or
2. we can rediscover the set of generators $T_1, T_2, ..., T_n$ from the set D.

(The set D has less than $n+2$ elements.)

Definition

A Bieberbach group $\Gamma \subset SO(n) \ltimes \mathbb{R}^n$ has a spin structure if and only if there exists a homomorphism $\epsilon : \Gamma \to Spin(n)$ such that $\mathrm{pr}_1 = \lambda_n \circ \epsilon$. Here pr_1 is a projection on the first component, and $\lambda_n : Spin(n) \to SO(n)$ is a universal covering.

It is well known that $H^2(SO(n), \mathbb{Z}_2) = \mathbb{Z}_2$. We define a group $Spin(n)$ as a middle group in a non-trivial short exact sequence

$$0 \to \mathbb{Z}_2 \to Spin(n) \xrightarrow{\lambda_n} SO(n) \to 0.$$

More about Spin

Let C_n be Clifford's algebra over the real numbers. By definition it is an associative algebra with unity, generated by elements

$$\{e_1, e_2, \dots, e_n\}$$

and with relations

$$\forall i, e_i^2 = -1,$$

$$\forall i, j, e_i e_j = -e_j e_i,$$

where $1 \leq i, j \leq n$. We define $C_0 = \mathbb{R}$. It is easy to see that $C_1 = \mathbb{C}$ and $C_2 = \mathbb{H}$, where $\mathbb{H}$ is the four-dimensional quaternion algebra. Moreover, $\mathbb{R}^n \subset C_n$ and $\dim_{\mathbb{R}} C_n = 2^n$, where $\mathbb{R}^n$ is n-dimensional $\mathbb{R}$-vector space with the basis $e_1, e_2, \dots, e_n$.

We have the following homomorphisms (involutions) on C_n :

(i) $^* : e_{i_1} e_{i_2} \ldots e_{i_k} \mapsto e_{i_k} e_{i_{k-1}} \ldots e_{i_2} e_{i_1}$,

(ii) $' : e_i \mapsto -e_i$,

(iii) $^- : a \mapsto (a')^*, a \in C_n$.

Suppose $C_n^0 = \{x \in C_n \mid x' = x\}$. It is easy to observe that

$$\forall a, b \in C_n, (ab)^* = b^* a^*.$$

We define subgroups of C_n,

$$Pin(n) = \{x_1 x_2 \ldots x_k \mid x_i \in S^{n-1} \subset \mathbb{R}^n \subset C_n, i = 1, 2, \ldots k\},$$

$$Spin(n) = Pin(n) \cap C_n^0.$$

Theorem

(B. Putrycz, J. P. Rossetti, 2009) Let Γ be an oriented Hanztsche-Wendt group of dimension $2n+1 \geq 5$, then Γ has not a spin structure.

A few words about the proof.

Let $\beta_i = (B_i, b_i) \in \Gamma$ be generators of Γ, for $i = 1, 2, ..., 2n$, where $B_i = \text{diag}[-1, -1, ... -1, \underbrace{1}_{i}, -1, ..., -1]$. It is easy to see that $\lambda_n(\pm e_1 e_2, ..., e_{i-1} e_{i+1} ... e_{2n+1}) = B_i$ and $\lambda_n(\pm e_i e_j) = \text{diag}[1, ..., 1, \underbrace{-1}_{i}, 1, ..., 1, \underbrace{-1}_{j}, 1, ...1]$. Moreover $\forall i, j \ (e_i e_j)^2 = -1$ and $(e_{i_1} e_{i_2} ... e_{i_{2m}})^2 = (-1)^{m \ mod \ 2}$.

Let $n = 2m + 1$ and let $\epsilon : \Gamma \to Spin(n)$ be a homomorphism s.t. $\lambda_n \circ \epsilon = \mathrm{pr}_1$. From above $\forall i \;\; \epsilon(\beta_i) = \pm e_1 e_2 ... e_{i-1} e_{i+1} ... e_{2m+1}$. Hence $\epsilon(t_i) = \epsilon((\beta_i)^2) = (-1)^{m \;\; mod \; 2}$.
We consider two cases. For m even $\epsilon(Z^n) = 1$. We have $\epsilon(\beta_1 \beta_2) = \pm e_1 e_2$. Finally $\epsilon((\beta_1 \beta_2)^2) = 1 = (\pm e_1 e_2)^2 = -1$, and we have a contradiction.

For m odd a proof is rather more difficult.

Let $M = \mathbb{R}^n/\Gamma$ be a flat manifold.

Definition

A holonomy representation $\Psi_\Gamma : H \to GL(n, \mathbb{Z})$ is essentially complex if there exists a matrix $A \in GL(n, \mathbb{R})$, such that,

$$\forall h \in H, A\Psi_\Gamma(h)A^{-1} \in GL(\frac{1}{2}n, \mathbb{C}).$$

Theorem

(F.E.A. Johnson, E. Rees, 1991) The following conditions are equivalent:

- M *is a flat Kähler manifold,*
- Ψ_Γ *is essentially complex,*
- Γ *is a discrete cocompact torsion-free subgroup of* $U(\frac{1}{2}n) \ltimes \mathbb{C}^{\frac{1}{2}n}$.

In the same paper is given the following characterization of an essentially complex representation.

$$\Psi_\Gamma : H \to GL(n, \mathbb{Z})$$

is essentially complex if and only if n is an even number and each $\mathbb{R}$-irreducible summand of Ψ_Γ which is also $\mathbb{C}$-irreducible occurs with even multiplicity.

Definition

A flat manifold has a $\mathbb{C}$omplex structure if and only if their holonomy representation is essentially complex.

In Algebraic geometry the flat Kähler manifolds are called hyperelliptic varieties.

2-dimensional hyperelliptic varieties :

holonomy	CARAT notations
1	15.1.1
$\mathbb{Z}_2$	18.1.1; 18.1.2
$\mathbb{Z}_3$	35.1.1; 35.1.2
$\mathbb{Z}_4$	25.1.2; 27.1.1
$\mathbb{Z}_6$	70.1.1

There is also a list of all 3-dimensional complex flat manifolds. There are 174 such objects. If a holonomy group of a flat manifold M is a subgroup of $SU(n)$, then M is called Calabi-Yau manifold.

Theorem

(Hodge, 1941) Let M be n-dimensional complex Kähler manifold. We have:

- $H^r(M,\mathbb{C}) = \Sigma_{p+q=r} H^{p,q}(M)$,
- *if $h^{p,q}(M)$ = $\dim_{\mathbb{C}} H^{p,q}(M)$, then $h^{p,q}(M) = h^{q,p}(M)$,*
- *number $b_r(M) = \Sigma_{p+q=r} h^{p,q}(M)$ is even if r is odd.*

The table of numbers $\{h^{p,q}(M), 0 \leq p, q \leq n\}$ is called the Hodge diamond of M.

Definition

A flat Kähler manifold of complex - dimension n with $\mathbb{Z}_2^{n-1}$ holonomy group is called a complex Hantzsche-Wendt manifold.

They are Calabi-Yau manifolds. (Holonomy group is a subgroup of $SU(n)$.) Here we present their Hodge diamond:

$$
\begin{array}{c}
\binom{n}{0} \\
0 \qquad\qquad 0 \\
0 \qquad\qquad \binom{n}{1} \qquad\qquad 0 \\
0 \qquad\qquad 0 \qquad\qquad 0 \qquad\qquad 0 \\
0 \qquad\qquad 0 \qquad\qquad \binom{n}{2} \qquad\qquad 0 \qquad\qquad 0 \\
\cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \\
\binom{n}{0} \qquad \binom{n}{1} \qquad \binom{n}{2} \qquad \cdots \qquad \cdots \qquad \binom{n}{n-1} \qquad \binom{n}{n} \\
\cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \qquad\qquad \cdots \\
0 \qquad\qquad 0 \qquad\qquad \binom{n}{n-2} \qquad\qquad 0 \qquad\qquad 0 \\
0 \qquad\qquad 0 \qquad\qquad 0 \qquad\qquad 0 \\
0 \qquad\qquad \binom{n}{n-1} \qquad\qquad 0 \\
0 \qquad\qquad 0 \\
\binom{n}{0}
\end{array}
$$

Thank You.