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HIGH DIMENSIONAL KNOT GROUPS AND HNN EXTENSIONS OF THE FIBONACCI GROUPS

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Abstract

We shall present a new class of examples of high dimensional knot groups. All of them are HNN extensions of the Fibonacci groups. We give also some characterization of these groups.

1 Introduction

Let $K : S^n \to S^{n+2}$ be an n-knot with group $\pi_1(S^{n+2} \setminus K(S^n))$. Kervaire characterized (see for instance [6] and [9]) the groups of n-knots (for each $n \geq 3$) as the finitely presentable groups G with $G/G' \simeq Z$, $H_2(G,Z) = 0$ and which are normal closure of some single element (the groups satisfying the last condition are said to have weight 1). Such groups are called high dimensional knot groups. The above conditions are also necessary when $n = 1$ or 2, but are then no longer sufficient. In this paper we shall present a new class of examples of high dimensional knot groups. It is not clear which of them are 2-knot groups. We have a proof of it only in special cases. See Corollary 1. Generally, it seems to be an open problem (cf.Corollary

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2). Our results arise from the geometric interpretations of Fibonacci groups. Fibonacci groups are abstract groups defined by the presentations:

$$F(r,n) = \{a_1, a_2, ..., a_n \mid a_1a_2...a_r = a_{r+1},$$

$$a_2a_3...a_{r+1} = a_{r+2}, ..., a_{n-1}a_na_1...a_{r-2} = a_{r-1},$$

$$a_na_1a_2...a_{r-1} = a_r\}$$

where $r > 0, n > 0$, and all subscripts are assumed to be reduced modulo n (see [2], [10]).

In [3] and [4] an interesting geometric interpretation of the group $F(2,2n)$ is shown. Let L_1 be the figure eight knot. Let K'_n be the n-fold cyclic branched covering of L_1. Then $\pi_1(K'_n) = F(2,2n)$.
Moreover in [11] the following construction is considered: Let L_2 be the n-twist spin of L_1. Then L_2 is a 2-knot, whose complement L is a bundle over S^1, with group Z_n and fibre a space homeomorphic to K'_n minus a point (punc K'_n). Hence $\pi_1(L)$ is a 2-knot group which maps onto Z with $F(2,2n)$ as a kernel.

The second motivation for us was the following example (cf. [5] page 129): Let $\Phi' : F(2,6) \to F(2,6)$ be an automorphism given by formula

$$\Phi'(a_i) = a_{i+1}, \text{ for } (1 \leq i \leq 6).$$

Moreover let $F(2,6)^*$ be a HNN extension of $F(2,n)$ presented by

$$\{F(2,6), t \mid tft^{-1} = \Phi'(f), f \in F(2,6)\}.$$

Then $F(2,6)^*$ is a 2-knot group (cf.[5] and Corollary 1).

Our main result generalizes this example and has a connection with the previous one.

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2 Main Result

Let $\Phi : F(r,n) \to F(r,n)$ be an automorphism given by formula:

$$\Phi(a_i) = a_{i+1}, \text{ for } 1 \leq i \leq n.$$

Let us define the HNN extension

$$F(r,n)^* = \{F(r,n), t \mid tft^{-1} = \Phi(f), f \in F(r,n)\}.$$

We have

Theorem 1 *$F(r,n)^*$ is a high dimensional knot group if and only if $r = 2$.*

Proof: First we prove that Φ is a *meridional* automorphism if and only if $r = 2$. That is the normal closure $K(r,n)$ in $F(r,n)$ of $\{f^{-1}\Phi(f) \mid f \in F(r,n)\}$ is $F(r,n)$ only for $r = 2$ (see [5] page 123). In fact, from the properties of the Fibonacci group, see [8] Theorem 1 and 6, we have an epimorphism $h : F(r,n) \to F(r,1) \simeq Z_{r-1}$. It is immediate from the definition of Φ that $K(r,n) \subset kerh$ and $K(r,n) = kerh$ if and only if $r = 2$. Hence only for $r = 2$ the abelianization of $F(r,n)^*$ is Z. And from now on we can assume that $r = 2$.
Next we show that *weight* of $F(2,n)^*$ is 1. We claim that $F(2,n)^*$ is the normal closure of $t^{-1}a_1$. Let B be the normal closure of this element. From definition we have $t^{-1}a_1, t^{-1}a_2, ..., t^{-1}a_n \in B$. Hence $\bar{a}_i = \bar{t}$ $(0 \leq i \leq n)$ in the quotient group $F(2,n)^*/B$. But the elements $\bar{a}_i$ $(0 \leq i \leq n)$ satisfy the relations of the Fibonacci group, therefore $\bar{a_1} = \bar{a_2} = ... = \bar{a_n} = \bar{t} = \bar{e}$. Here e denotes the neutral element and $\bar{a} = \nu(a)$, where $a \in F(2,n)^*$ and $\nu : F(2,n)^* \to F(2,n)^*/B$ is a natural map. This proves equality $F(2,n)^* = B$. To prove the last condition from the definition of the higher dimensional knot group we shall use the Hochschild - Serre spectral sequence of the short exact sequence of groups ([1], Theorem 6.3)

$$0 \to F(2,n) \to F(2,n)^* \to Z \to 0.$$

From the properties of the above spectral sequence it is enough to prove that

$$E^2_{2,0} = H_2(Z, H_0(F(2,n))) = 0,$$

$$E^2_{1,1} = H_1(Z, H_1(F(2,n))) = H_1(F(2,n))_Z = 0$$

and

$$E^2_{0,2} = H_0(Z, H_2(F(2,n))) = 0.$$

The first equality is obvious. For the proof of the next one we shall use the long exact sequence (see [1], page 47):

$$H_2(F(2,n)^*) \to H_2(Z) \to H_1(F(2,n))_Z \to H_1(F(2,n)^*) \to H_1(Z) \to 0.$$

We have $H_2(Z) = 0, H_1(Z) = Z$. Moreover we have already proved that $H_1(F(2,n)^*) = Z$. Hence $E^2_{1,1} = H_1(Z, H_1(F(2,n))) = 0$.
Since $F(2,n)$ has a finite presentation with equal numbers of generators and relations it is the fundamental group of a finite 2-complex X with $\chi(X) = 0$. As $H_1(X,Z)$ is the abelianization of $F(2,n)$, which is finite (see [7]), $H_2(X,Z) = 0$ and therefore $H_2(F(2,n),Z) = 0$, by Hopf's formula (see [1], page 46 exercise 5). Hence $E^2_{0,2} = H_2(F(2,n)^*) = 0$.

□

As we remark at the introduction the conditions of the high dimensional knot groups are also necessary when $n = 1$ or 2, but are then no longer sufficient. For example, it is proved in [6] (Th. 9 page 34) that if a group G satisfying the above conditions is a torsion free group such that G' is the fundamental group of closed orientable hyperbolic or Seifert fibred 3-manifold N, then G is the group of a fibred 2-knot with closed fibre N. Thus the following corollary should come as no surprise.

Corollary 1 *For $n \geq 3$ the group $F(2,2n)^*$ is a fibred 2-knot group with closed fibred K_n.*

Proof: From [3] it follows that the group $F(2,2n)$ is torsion free. From this and from the above we must only prove that the group $F(2,2n)^*$ is torsion free. But this follows from the definition of the automorphism Φ and HNN extension.

□

We should mention that the knot groups $F(2,2n)^*$ ($n \geq 3$)are not the groups of the n-twist spins of the figure eight knot. In fact, for $n = 3$, from [5] and from the definition of the group $F(2,6)^*$ (automorphism Φ' has order 6) it is not difficult to see that $F(2,6)^*$ is the group called $G(-1)$ in [5].

Moreover, for $n \geq 4$ we use at the definition of the group $F(2,2n)^*$ the automorphism Φ of order $2n$. But an automorphism in the corresponding n-twist spins group has order n (see [11] page 484).

In the case when $F(2,n)$ is a finite group we have:

Corollary 2 *If $F(2,n)$ is a finite group, then $F(2,n)^*$ is a 2-knot group only for $n = 3$.*

Proof: $F(2,3)^*$ is the group of the 3-twist spin of the trefoil knot. On the other hand for $n = 4,5,7$, $F(2,n)$ is not a 2-knot group as the meridional automorphism does not induce an isometry of the Farber-Levine pairing. (See [6] Th. II.3). For all larger values of n the group $F(2,n)$ is infinite. (See [10]).

□

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