Introduction to flat manifolds

Andrzej Szczepański

University of Gdańsk, Poland

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Crystallographic groups

Let $\mathbb{R}^n$ be n-dimensional Euclidean space, with isometry group $E(n) = O(n) \ltimes \mathbb{R}^n$.

Definition

Γ is a crystallographic group of rank n iff it is a discrete and cocompact subgroup of $E(n)$.

A Bieberbach group is a torsion free crystallographic group.

Basic properties

Theorem

(Bieberbach, 1910)

- *If Γ is a crystallographic group of dimension n, then the set of all translations of Γ is a maximal abelian subgroup of a finite index.*
- *There is only a finite number of isomorphic classes of crystallographic groups of dimension n.*
- *Two crystallographic groups of dimension n are isomorphic if and only if there are conjugate in the group affine transformations $A(n) = GL(n, \mathbb{R}) \ltimes \mathbb{R}^n$.*

Pure abstract point of view

Theorem

(Zassenhaus, 1947) A group Γ is a crystallographic group of dimension n if and only if, it has a normal maximal abelian subgroup $\mathbb{Z}^n$ of a finite index.

Holonomy representation

Definition

Let Γ be a crystallographic group of dimension n with translations subgroup $A \simeq \mathbb{Z}^n$. A finite group $\Gamma/A = G$ we shall call a holonomy group of Γ.

Let $(A, a) \in E(n)$ and $x \in \mathbb{R}^n$. Γ acts on $\mathbb{R}^n$ in the following way:

$$(A, a)(x) = Ax + a.$$

Definition

Let Γ be n-dimensional Bieberbach group. We have the following short exact sequence of groups.

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} \Gamma/\mathbb{Z}^n = H \to 0.$$

Let us define a homomorphism $h_\Gamma : H \to GL(n, \mathbb{Z})$. Put

$$\forall h \in H, h_\Gamma(h)(e_i) = \bar{h}^{-1} e_i \bar{h},$$

where $p(\bar{h}) = h$ and $e_i \in \mathbb{Z}^n$ is a standard basis. h_Γ is called a holonomy representation of a group Γ.

Flat manifold

Let $\Gamma \subset E(n)$ be a torsion free crystallographic group. Since Γ is cocompact and discrete subgroup, then the orbit space $\mathbb{R}^n/\Gamma$ is a manifold. If Γ is not torsion free then the orbit space $\mathbb{R}^n/\Gamma$ is an orbifold.

Definition

The above manifolds (orbifolds) we shall call "flat".

From elementary covering theory any compact Riemannian manifold (orbifold) with sectional curvature equal to zero is flat.

Example

Flat surfaces:

- torus $S^1 \times S^1$,
- Klein bottle $S^1 \times S^1/\mathbb{Z}_2$

We shall see that many properties of the Bieberbach Groups correspond to properties of flat manifolds.

Classification

From the second Bieberbach theorem there is only a finite number of flat manifolds of given dimension.
For example in dimension 3 there are 10 flat manifolds. Here classifiction was made in 1936. Then we have a computer program CARAT see

`https://www.mathb.rwth-aachen.de/carat/index.html`.
We have:
in dimension 4 - 74,
in dimension 5 - 1060,
in dimension 6 - 38746.

Remark: There exists $G \subset GL(6, \mathbb{Z})$ s.t. $\mid H^2(G, \mathbb{Z}^6) \mid = 2^{30} = 1073741824$.

Classification I

Theorem

(Calabi - 1957) Let Γ *be a torsion free crystallographic group of dimension* n *with an epimorphism* $f:\Gamma\to\mathbb{Z}$*. Then* $kerf$ *is a torsion free crystallographic group of dimension* $n-1$.

"the induction method of Calabi"

1. classify all torsion free crystallographic groups of rank $< n$;

2. classify all torsion free crystallographic groups of dimension n with finite abelianization;

3. classify all torsion free c.g. Γ of dimension n defined by the short exact sequence

$$0\to\Gamma_{n-1}\to\Gamma\to\mathbb{Z}\to 0,$$

where Γ_{n-1} is from point 1.

Classification II

Definition

Let M be a flat manifold with the fundamental group Γ, which acts by isometries on a flat torus T^k. Then Γ also acts by isometris on the space $\tilde{M} \times T^k$. We shall call the space $(\tilde{M} \times T^k)/\Gamma$ a flat toral extension of the manifold M.

Theorem

(A.T.Vasquez - 1970) For any finite group G there exists a natural number $n(G)$ with the following property: if M is any flat manifold with holonomy group G, then M is a flat toral extension of some flat manifold of dimension $\leq n(G)$.

Classification II

The third way of the classification is called the Auslander-Vasquez method. It is related only to flat manifolds with given holonomy group G and has the following steps:
1. calculate the Vasquez invariant $n(G)$;
2. describe all flat toral extensions of the manifolds of dimension $\leq n(G)$;
3. classify all flat manifolds of dimension $\leq n(G)$.
For example for p-group, $n(G) = \Sigma_{C \in \mathcal{X}} \mid G : C \mid$, where $\mathcal{X}$ is a set of representatives of conjugacy classes of subgroups of G of prime order; G. Cliff, A. Weiss 1989. $n(A_5) = 16$; G. Cliff, Hongliu Zheng 1996.

Boundary problem

Theorem

(G. Hamrick, D.Royster, Inv. Math.1982) Every compact Riemannian flat manifold bounds a compact manifold.

We can ask: Does every compact Riemannian flat manifold bound a compcat hyperbolic manifold ?

Example

Let V be a hyperbolic (complete Riemannian with constant sectional curvature -1) manifold with one cusp. After cut a cusp we have a compact hyperbolic monifold V' with boundary $\partial V'$, where $\partial V'$ is flat.

We can ask: Let M^n be a flat manifold of dimension n. Is there some $V^{(n+1)}$ such that $(\partial((V')^{(n+1))} \simeq M^n$?

η invariant

In 2000 D. D. Long and A. Reid proved:

Theorem

Let V be a hyperbolic manifold with one cusp of dimension $4n$. If a flat manifold $M^{(4n-1)}$ of dimension $(4n-1)$ has geometric realization as $\partial V'$, then $\eta(M^{(4n-1)}) \in \mathbb{Z}$.

The proof is consequence of the Atiyah, Patodi and Singer theorem. Here we consider η-invariant of signature operator. Already in dimension 3 there exists a flat manifold M^3 such that $\eta(M^3) = 3/4 \notin \mathbb{Z}$.

Definition

A Bieberbach group $\Gamma \subset SO(n) \ltimes \mathbb{R}^n$ has a spin structure if and only if there exists a homomorphism $\epsilon : \Gamma \to Spin(n)$ such that $\mathrm{pr}_1 = \lambda_n \circ \epsilon$. Here pr_1 is a projection on the first component, and $\lambda_n : Spin(n) \to SO(n)$ is a universal covering.

It is well known that $H^2(SO(n), \mathbb{Z}_2) = \mathbb{Z}_2$. We define a group $Spin(n)$ as a middle group in a non-trivial short exact sequence

$$0 \to \mathbb{Z}_2 \to Spin(n) \xrightarrow{\lambda_n} SO(n) \to 0.$$

More about Spin

Let C_n be Clifford's algebra over the real numbers. By definition it is an associative algebra with unity, generated by elements

$$\{e_1, e_2, \ldots, e_n\}$$

and with relations

$$\forall i, e_i^2 = -1,$$

$$\forall i, j, e_i e_j = -e_j e_i,$$

where $1 \leq i, j \leq n$. We define $C_0 = \mathbb{R}$. It is easy to see that $C_1 = \mathbb{C}$ and $C_2 = \mathbb{H}$, where $\mathbb{H}$ is the four-dimensional quaternion algebra. Moreover, $\mathbb{R}^n \subset C_n$ and $\dim_{\mathbb{R}} C_n = 2^n$, where $\mathbb{R}^n$ is n-dimensional $\mathbb{R}$-vector space with the basis $e_1, e_2, \ldots, e_n$.

We have the following homomorphisms (involutions) on C_n :

(i) $^* : e_{i_1} e_{i_2} \dots e_{i_k} \mapsto e_{i_k} e_{i_{k-1}} \dots e_{i_2} e_{i_1}$,

(ii) $' : e_i \mapsto -e_i$,

(iii) $^- : a \mapsto (a')^*, a \in C_n$.

Suppose $C_n^0 = \{x \in C_n \mid x' = x\}$. It is easy to observe that

$$\forall a, b \in C_n, (ab)^* = b^* a^*.$$

We define subgroups of C_n,

$$Pin(n) = \{x_1 x_2 \dots x_k \mid x_i \in S^{n-1} \subset \mathbb{R}^n \subset C_n, i = 1, 2, \dots k\},$$

$$Spin(n) = Pin(n) \cap C_n^0.$$

Theorem

(B. Putrycz, J. P. Rossetti, 2009) Let Γ *be an oriented Hanztsche-Wendt group of dimension* $2n+1 \geq 5$*, then* Γ *has not a spin structure.*

A few words about the proof.

Let $\beta_i = (B_i, b_i) \in \Gamma$ be generators of Γ, for $i = 1, 2, ..., 2n$, where $B_i =$ diag$[-1, -1, ... -1, \underbrace{1}_{i}, -1, ..., -1]$. It is easy to see that $\lambda_n(\pm e_1 e_2, ..., e_{i-1} e_{i+1} ... e_{2n+1}) = B_i$ and $\lambda_n(\pm e_i e_j) =$ diag$[1, ..., 1, \underbrace{-1}_{i}, 1, ..., 1, \underbrace{-1}_{j}, 1, ...1]$. Moreover $\forall i, j \ \ (e_i e_j)^2 = -1$ and $(e_{i_1} e_{i_2} ... e_{i_{2m}})^2 = (-1)^{m \ \ mod \ 2}$.

Let $n = 2m + 1$ and let $\epsilon : \Gamma \to Spin(n)$ be a homomorphism s.t. $\lambda_n \circ \epsilon = \mathrm{pr}_1$. From above $\forall i \ \ \epsilon(\beta_i) = \pm e_1 e_2 ... e_{i-1} e_{i+1} ... e_{2m+1}$. Hence $\epsilon(t_i) = \epsilon((\beta_i)^2) = (-1)^{m \ \ mod \ 2}$.
We consider two cases. For m even $\epsilon(Z^n) = 1$. We have $\epsilon(\beta_1 \beta_2) = \pm e_1 e_2$. Finally $\epsilon((\beta_1 \beta_2)^2) = 1 = (\pm e_1 e_2)^2 = -1$, and we have a contradiction.

For m odd a proof is rather more difficult.

Spin structures in dimensions 4, 5 and 6

dim	# flat manifolds	# orientable flat manifolds	# spin flat manifolds
4	74	27	24
5	1060	174	88
6	38746	3314	760

Thank You.