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Cohomological rigidity of oriented Hantzsche–Wendt manifolds

J. Popko, A. Szczepański *

Institute of Mathematics, University of Gdańsk, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland

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ABSTRACT

By Hantzsche–Wendt manifold (for short *HW-manifold*) we understand any oriented closed Riemannian manifold of dimension n with a holonomy group $(\mathbb{Z}_2)^{n-1}$. Two *HW-manifolds* M_1 and M_2 are cohomological rigid if and only if a homeomorphism between M_1 and M_2 is equivalent to an isomorphism of graded rings $H^*(M_1, \mathbb{F}_2)$ and $H^*(M_2, \mathbb{F}_2)$. We prove that *HW-manifolds* are cohomological rigid.

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1. Introduction

Let M^n be a flat manifold of dimension n. By definition, this is a compact connected, Riemannian manifold without boundary with sectional curvature equal to zero. From

* Corresponding author.
E-mail addresses: jpopko@mat.ug.edu.pl (J. Popko), matas@univ.gda.pl (A. Szczepański).

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the theorems of Bieberbach ([1,8]) the fundamental group $\pi_1(M^n) = \Gamma$ determines a short exact sequence:

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} G \to 0, \tag{1}$$

where $\mathbb{Z}^n$ is a torsion free abelian group of rank n and G is a finite group which is isomorphic to the holonomy group of M^n. The universal covering of M^n is the Euclidean space $\mathbb{R}^n$ and hence Γ is isomorphic to a discrete cocompact subgroup of the isometry group $\mathrm{Isom}(\mathbb{R}^n) = \mathrm{O}(n) \ltimes \mathbb{R}^n = \mathrm{E}(n)$. In the above short exact sequence $\mathbb{Z}^n \cong (\Gamma \cap \mathbb{R}^n)$ and p can be considered as the projection $p : \Gamma \to G \subset \mathrm{O}(n) \subset \mathrm{E}(n)$ on the first component. An orthogonal representation p is equivalent (see [8]) to a holonomy representation. That is a homomorphism $\phi_\Gamma : G \to \mathrm{GL}(n, \mathbb{Z})$, given by a formula $\phi_\Gamma(g)(z) = \bar{g}z\bar{g}^{-1}$, where $\bar{g} \in \Gamma, g \in G, z \in \mathbb{Z}^n$ and $p(\bar{g}) = g$. Conversely, given a short sequence of the form (1), it is known that the group Γ is (isomorphic to) a Bieberbach group if and only if Γ is torsion free.

By Hantzsche–Wendt manifold (for short *HW-manifold*) M^n we understand any oriented flat manifold of dimension n with a holonomy group $(\mathbb{Z}_2)^{n-1}$. It is easy to see that n is always an odd number. Moreover, any HW-manifold has a diagonal holonomy representation, see [7]. It means $\pi_1(M^n)$ is generated by $\beta_i = (B_i, b_i) \in \mathrm{SO}(n) \ltimes \mathbb{R}^n, 1 \leq i \leq n$, where

$$B_i = \mathrm{diag}(-1, -1, ..., -1, \underbrace{1}_{i}, -1, -1, ..., -1) \tag{2}$$

and $b_i \in \{0, 1/2\}^n$. For other properties of M^n we send a reader to [8] and to next sections. We shall need

Definition 1. (See [4].) Two flat manifolds M_1 and M_2 are cohomological rigid if and only if a homeomorphism between M_1 and M_2 is equivalent to an isomorphism of graded rings $H^*(M_1, \mathbb{F}_2)$ and $H^*(M_2, \mathbb{F}_2)$.

Our main result is the following theorem.

Theorem. *Hantzsche–Wendt manifolds are cohomological rigid.*

The Theorem answers the question from [2, problem 4.3].

For the proof we introduce a new presentation of *HW-manifolds*. We consider these manifolds rather as a finite quotient of the torus than a quotient of the $\mathbb{R}^n$. Here, we use an obvious equivalence $\mathbb{R}^n/\Gamma = (\mathbb{R}^n/\mathbb{Z}^n)/G = T^n/G$, where Γ is a Bieberbach group from (1). According to the definition of n-dimensional *HW-manifold* we shall define a $(n \times n)$-*HW-matrix* A. The analysis of properties of the matrix A is used in the proof. Moreover, we apply the Lyndon–Hochschild–Serre spectral sequence $\{E_r^{p,q}, d_r\}$ of the covering $T^n \to T^n/G$ with $\mathbb{F}_2$ coefficients. Since a holonomy representation Φ_Γ is diagonal

$E_2^{p,q} = H^p((\mathbb{Z}_2)^{n-1}) \otimes H^q(\mathbb{Z}^n)$. We shall only use the multiplicative structure of the first and second cohomology group. In particular, we shall consider the properties of the transgression homomorphism $d_2 : H^1(\mathbb{Z}^n) \to H^2((\mathbb{Z}_2)^{n-1})$. Finally, another important point of the proof is an isomorphism of cohomology groups $H^1((\mathbb{Z}_2)^{n-1})$ and $H^1(\Gamma)$, which was proved in [6, Theorem 3.1]. Hence, we can consider elements of the image of the transgression homomorphism d_2 as homogeneous polynomials of degree two which are equivalent to polynomial functions.

Let us present a structure of the paper. In the next section, we give a "new-old" definition of *HW-manifold* and we outline the proof of the theorem. In section 3 we define *HW-matrix* and prove some of its properties.

At the last section, we present the proof of the **Main Lemma**.

2. Proof of the Main Theorem

Let $\mathcal{D} = \{g_i \mid i = 0, 1, 2, 3\}$, where $g_i : S^1 \to S^1$, and $\forall z \in S^1 \subset \mathbb{C}$,

$$g_0(z) = z, g_1(z) = -z, g_2(z) = \bar{z}, g_3(z) = -\bar{z}. \tag{3}$$

Equivalently, if $S^1 = \mathbb{R}/\mathbb{Z}, \forall [t] \in \mathbb{R}/\mathbb{Z}$,

$$g_0([t]) = [t], g_1([t]) = [t + \frac{1}{2}], g_2([t]) = [-t], g_3([t]) = [-t + \frac{1}{2}]. \tag{4}$$

Let $(t_1, t_2, ..., t_n) \in \mathcal{D}^n$ and $(z_1, z_2, ..., z_n) \in T^n = \underbrace{S^1 \times S^1 \times ... \times S^1}_{n}$. It is easy to see that $\mathcal{D} = \mathbb{Z}_2 \times \mathbb{Z}_2$, and $g_3 = g_1 g_2$. For $k = 1, 2, 3$ we have different projections

$$p^{(k)} : \mathcal{D} \to \mathbb{F}_2 = \{0, 1\} \tag{5}$$

such that $p^{(k)}(g_k) = 1$ and for $i = 1, 2, .., n$ we have homomorphisms

$$p^{(k)} \circ pr_i : \mathcal{D}^n \to \mathcal{D} \overset{p^{(k)}}{\to} \mathbb{F}_2 \tag{6}$$

given by the formula $p^{(k)} \circ pr_i(t_1, t_2, ..., t_i, ..., t_n) = p^{(k)}(t_i)$.

We summing up values of the projections $p^{(2)}$ and $p^{(3)}$ in Table 1.

Table 1
Values of the projections from $\mathcal{D} \to \mathbb{F}_2$.

	g_0	g_1	g_2	g_3
$p^{(2)}$	0	1	1	0
$p^{(3)}$	0	1	0	1

The next, obvious formula

$$\forall x \in \mathcal{D} \;\; x = p^{(2)}(x)2 + p^{(3)}(x)3 \tag{7}$$

will be useful later. We can define an action $\mathcal{D}^n$ on T^n as follows:

$$(t_1, t_2, ..., t_n)(z_1, z_2, ..., z_n) = (t_1 z_1, t_2 z_2, ..., t_n z_n). \tag{8}$$

We have

Proposition 1. *Let M^n be a HW-manifold of dimension n. Then there exists a subgroup $(\mathbb{Z}_2)^{n-1} \subset \mathcal{D}^n$ such that $M^n = T^n/(\mathbb{Z}_2)^{n-1}$, where the action $(\mathbb{Z}_2)^{n-1}$ on T^n is defined by (2) and (8).*

Proof. Let $\pi_1(M^n) = \Gamma$ and $(B_l, b_l) \in \Gamma$ be the generators (2), $l = 1, 2, .., n$. On each coordinate, (4) defines $g_j \in \mathcal{D}, j = 0, 1, 2, 3$ which are determinated by projections $p^{(1)} \circ pr_i, p^{(2)} \circ pr_i, p^{(3)} \circ pr_i$. □

Let us start to prove that the graded ring $H^*(M^n, \mathbb{F}_2)$ defines a manifold M^n. We have an exact sequence

$$0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} (\mathbb{Z}_2)^{n-1} \to 0, \tag{9}$$

where $\Gamma = \pi_1(M^n)$. As we mentioned already in the introduction the image of a holonomy representation $\Phi_\Gamma((\mathbb{Z}_2)^{n-1})$, is a subgroup of the group of all diagonal matrices of $GL(n, \mathbb{Z})$. Moreover (see [6]) $H^1(\Gamma, \mathbb{F}_2) = (\mathbb{F}_2)^{n-1}$ for any Hantzsche–Wendt group Γ of dimension n. That is an observation which we shall use during the proof.

Since $(\mathbb{Z}_2)^{n-1} \subset \mathcal{D}^n$ the above maps $p^{(k)} \circ pr_i, k = 1, 2, 3$ define homomorphisms from $(\mathbb{Z}_2)^{n-1} \to \mathbb{F}_2 \in Hom((\mathbb{Z}_2)^{n-1}, \mathbb{F}_2) = H^1((\mathbb{Z}_2)^{n-1}, \mathbb{F}_2) \overset{[6]}{=} H^1(M^n, \mathbb{F}_2)$. Hence we can define elements

$$T_i = (p^{(2)} \circ pr_i) \cup (p^{(3)} \circ pr_i) \in H^2((\mathbb{Z}_2)^{n-1}, \mathbb{F}_2),$$

where $\cup$ is a cup product. It is well known that $H^*((\mathbb{Z}_2)^{n-1}, \mathbb{F}_2)$ is isomorphic to $\mathbb{F}_2[x_1, x_2, ..., x_{n-1}]$. Hence the elements $p^{(k)} \circ pr_i = p_i^{(k)}$ correspond to

$$\sum_{j=1}^{n-1} p^{(k)}(pr_i(b_j))x_j = \sum_{j=1}^{n-1} p^{(k)}(A_{ji})x_j \in \mathbb{F}_2[x_1, x_2, ..., x_{n-1}], \tag{10}$$

where $b_1, b_2, ..., b_{n-1}$ is the basis of $(\mathbb{Z}_2)^{n-1}$ and $k = 2, 3; i = 1, 2, ..., n$. Here the matrix $A_{ij}, i = 1, 2, \dots, n-1; j = 1, 2, \dots, n$ is related to *HW-matrix* (Definition 2) from the next section.

We shall apply the Lyndon–Hochschild–Serre spectral sequence $\{E_r^{p,q}, d_r\}$ of (9). Since a holonomy representation Φ_Γ is diagonal $E_2^{p,q} = H^p((\mathbb{Z}_2)^{n-1}) \otimes H^q(\mathbb{Z}^n)$. Hence (see [3, Corollary 7.2.3 on p. 77]) we have an exact sequence (see [2, p. 770])

$$H^1(\mathbb{Z}^n, \mathbb{F}_2) \overset{d_2}{\to} H^2((\mathbb{Z}_2)^{n-1}, \mathbb{F}_2) \overset{p^*}{\to} H^2(\Gamma, \mathbb{F}_2), \tag{11}$$

where d_2 is a transgression and p^* is induced by the above homomorphism $p : \Gamma \to (\mathbb{Z}_2)^{n-1}$. In what follows we shall prove (see also [2, Theorem 2.7]) that a rank of

$$\mathrm{Im}(d_2) \subset H^2((\mathbb{Z}_2)^{n-1}, \mathbb{F}_2) \subset H^*((\mathbb{Z}_2)^{n-1}, \mathbb{F}_2) \simeq \mathbb{F}_2[x_1, x_2, ..., x_{n-1}]$$

is equal to n.

Let us define a basis $\hat{t}_i, i = 1, 2, \ldots, n$ of $H^1(\mathbb{Z}^n, \mathbb{F}_2) = \mathrm{Hom}(\mathbb{Z}^n, \mathbb{F}_2)$. For $k \in \mathbb{Z}$, we shall write $\bar{k} = 0$ if k is even and $\bar{k} = 1$ if k is odd. Let $(k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$ and let

$$\hat{t}_i(k_1, k_2, \ldots, k_n) = \bar{k}_i, i = 1, 2, \ldots, n.$$

We have

Proposition 2. $d_2(\hat{t}_i) = T_i = (p^{(2)} \circ pr_i) \cup (p^{(3)} \circ pr_i)$. *Moreover elements* $T_i, i = 1, 2, \ldots, n$ *are a basis of* $Im(d_2)$.

Proof. By Theorem 2.5 (ii) and Proposition 1.3 of [2] and using (10) it follows that

$$d_2(\hat{t}_i) = \sum_{A_{il}=1} x_i^2 + \sum_{i \neq j} x_i x_j,$$

where the second sum is taken for such i, j that

$$(A_{il}, A_{jl}) \in \{(1,2), (2,1), (1,3), (3,1), (3,1), (3,2), (2,3)\}.$$

On the other hand

$$T_l = p_l^{(2)} p_l^{(3)} = \sum_{i=1}^{n-1} p^{(2)}(A_{il}) p^{(3)}(A_{il}) x_i^2 + $$
$$+ \sum_{1 \leq i < j \leq n-1} (p^{(2)}(A_{il}) p^{(3)}(A_{jl}) + p^{(2)}(A_{jl}) p^{(3)}(A_{il}) x_i x_j. \tag{12}$$

Comparing coefficients of the above two polynomials finishes the proof. □

The main idea of the proof of rigidity is an application of the above Proposition 2. It means, we show that any *HW-manifold* M, of dimension greater than three, define elements in the cohomology ring $H^*(M, \mathbb{F}_2)$ which determines M up to affine equivalence. In the **Main Lemma**, we shall prove an existence of n linear independence elements $T_1, T_2 \dots, T_n \in \mathrm{Im}(d_2)$ such that for any $i = 1, 2, \dots, n$ $T_i = p_i q_i$. At the end of this section we give a method of a reconstruction of *HW-group* from the set $\{T_i\}_{i=1,2,\dots,n}$.

Let us define

$$D = \{y \in \mathrm{Im}(d_2) \mid y \text{ is a product of two polynomials of degree } 1\}. \tag{13}$$

We shall prove that D has less than $n+2$ elements from which we can reconstruct the basis $T_1, T_2, \dots, T_n$ of $\mathrm{Im}(d_2)$.

Main Lemma. *Let $n > 3$, then there are the following possibilities for the structure of the set D:*

1. $D = \{T_1, T_2, \dots, T_n\}$;

2. $D = \{T_1, T_2, \dots, T_n, T_i + T_j\}$, and we can find a polynomial p *of degree one such that* p $\mid T_i$ *and* p $\mid T_j$ *for some $1 \le i, j \le n$. In the second case we can rediscover the set of generators $T_1, T_2, \dots, T_n$.* □

Let M be *HW-manifold* of dimension n. From the **Main Lemma**, we know that there is a set $D = \{T_1, T_2, ..., T_n\} \subset \mathrm{Im}(d_2)$ such that any T_i is a product of two polynomials p_i and $q_i, i = 1, 2, \dots, n$ of a degree one. Let V be $(n-1)$-dimensional $\mathbb{F}_2$ vector space. We define a linear map $h : V^* \to \mathcal{D}^n$, which simple version is (7) such that

$$h_i(x) = p_i(x)2 + q_i(x)3, \text{ for } i = 1, 2, \dots, n, \tag{14}$$

where $p_i, q_i \in V \simeq V^{**}$. Hence, through formulas (10), (12) and the Table 1, $\mathrm{Im}(h)$, defines a Hantzsche–Wendt group.

Example 1. 1. Let $V = \mathrm{gen}\{x_1, x_2, x_3\}$ and $D = \{x_1^2 + x_1x_2, x_1x_2 + x_1x_3 + x_2^2 + x_2x_3\}$. Put $p_1 = x_1, q_1 = x_1 + x_2, p_2 = x_1 + x_2, q_2 = x_2 + x_3$. Hence a homomorphism $h(x_1^*) = (1, 2), h(x_2^*) = (3, 1)$ and $h(x_3^*) = (0, 3)$. Here x_1^*, x_2^*, x_3^* is a dual basis of V^*. Finally we define a subgroup of $\mathcal{D}^2$ which generators are rows of the matrix

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 3 \end{bmatrix}.$$

2. Let $\mathbb{Z}_2^{n-1} \subset \mathcal{D}^n$ be a *HW-group*, and D a set from the Proposition 2. Assume that $D = \{p_1q_1, p_2q_2, \dots, p_nq_n\}$. Then

$$h_i(x) = p_i(x)2 + q_i(x)3 = p(x_i)2 + q(x_i)3 = x_i.$$

Hence for $x \in \mathbb{Z}_2^{n-1}, h(x) = x$ and $Im(h) = \mathbb{Z}_2^{n-1}$.

Let $\phi : H^*(M_1, \mathbb{F}_2) \to H^*(M_2, \mathbb{F}_2)$ be an isomorphism of cohomology rings of *HW-manifolds* M_1 and M_2. From the **Main Lemma** for the both manifolds we have the sets of elements D_1 and D_2 such that $\phi(D_1) = D_2$. Hence we obtain the affine equivalence manifolds M_1 and M_2. □

3. Properties of Hantzsche–Wendt matrices

Let us illustrate the Proposition 1 on two *HW-manifolds* of dimension 5, (see [8]). We shall denote by Γ_1 and Γ_2 its fundamental groups.

Example 2. A group $\Gamma_1 \subset E(5)$ is generated by

$$(B_1, (1/2, 1/2, 0, 0, 0)), (B_2, (0, 1/2, 1/2, 0, 0)),$$
$$(B_3, (0, 0, 1/2, 1/2, 0)), (B_4, (0, 0, 0, 1/2, 1/2)).$$

From above $\mathbb{R}^5/\Gamma_1 \simeq T^5/(\mathbb{Z}_2)^4$, where $(\mathbb{Z}_2)^4 \subset \mathcal{D}^5$ is defined by

$$(g_1, g_3, g_2, g_2, g_2), (g_2, g_1, g_3, g_2, g_2),$$
$$(g_2, g_2, g_1, g_3, g_2), (g_2, g_2, g_2, g_1, g_3).$$

Moreover a group $\Gamma_2 \subset E(5)$ is generated by

$$(B_1, (1/2, 0, 1/2, 1/2, 0)), (B_2, (0, 1/2, 1/2, 1/2, 1/2)),$$
$$(B_3, (1/2, 1/2, 1/2, 1/2, 1/2)), (B_4, (1/2, 0, 1/2, 1/2, 1/2)).$$

Hence, $\mathbb{R}^5/\Gamma_2 \simeq T^5/(\mathbb{Z}_2)^4$ where generators of a group $(\mathbb{Z}_2)^4 \subset \mathcal{D}^5$ are following

$$(g_1, g_2, g_3, g_3, g_2), (g_2, g_1, g_3, g_3, g_3),$$
$$(g_3, g_3, g_1, g_3, g_3), (g_3, g_2, g_3, g_1, g_3).$$

In what follows we shall write i for g_i, $i = 0, 1, 2, 3$. Let A be a $(n \times m)$ matrix with coefficients $A_{ij} \in \mathcal{D}$. For short $A \in \mathcal{D}^{n\times m}$. Let A_i (A^j) denote i-row (j-column) of a matrix A.

Definition 2. By *HW-matrix* we shall understand a matrix $A \in \mathcal{D}^{n\times n}$ such that $A_{ii} = 1$, $A_{ij} \in \{2, 3\}$ for $i \neq j, 1 \leq i, j \leq n$ and if $X \subset \{1, 2, ..., n\}$ and $1 \leq \#X \leq n-1$ then the row $\sum_{i\in X} A_i$ has 1 on a some position.

Lemma 1. *Any HW-manifold of dimension n defines a $(n \times n)$ HW*-matrix.

Proof. Let (β_i, b_i), $1 \leq i \leq n-1$ be generators of the fundamental group of some n-dimensional HW-manifold M. Then i-generator defines i-row of some $(n \times n)$

HW-matrix, cf. (2), (4). See also Example 2 and Proposition 1. The last row is defined by the product $\beta_1\beta_2\dots\beta_{n-1}$ or equivalently is a sum of the first $(n-1)$ rows. It is easy to see that the first property of the above matrix follows from a definition, see [5, p. 4]. Since a holonomy group $(\mathbb{Z}_2)^{n-1}$ acts free on T^n (or equivalently $\pi_1(M)$ is a torsion free group) the last part of lemma follows. □

We shall present some properties of *HW-matrices*.

Remark 1. Let $\sigma \in S_n$ and let P_σ be the corresponding permutation matrix. It is not difficult to see that if A is *HW-matrix* then $P_\sigma A P_\sigma^{-1}$ also satisfies conditions of the Definition 2. Moreover, if A' is a conjugation matrix of A, where conjugation means exchange at some column numbers 2 for 3, then A' is again a *HW-matrix*. The *HW-matrix* is related to the matrix defined on page 6 of [5].

Remark 2. Let A be a $(n \times n)$ *HW-matrix*. Then

$$(p^{(2)} + p^{(3)})(A) = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 1 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1 & 0 & 1 \\ 1 & 1 & \dots & 1 & 1 & 0 \end{bmatrix}. \tag{15}$$

Let $A \in \mathcal{D}^{m\times n}$ be a $(m\times n)$ matrix with coefficients in $\mathcal{D}$ and $(\alpha_1, \alpha_2, ..., \alpha_n) \in \{2,3\}^n$. By $p^{(\alpha)}(A)$ we shall understand a $(m\times n)$-matrix with coefficients in $\mathbb{F}_2$ which a i-column is equal to $p^{(\alpha_i)}(A^i)$.

Let M be a matrix. By defect of M we shall understand a number

$$d(M) = \{\text{number of columns of M}\} - \text{rk}(M).$$

Lemma 2. *1. Let M_1 be a matrix M from which we remove some columns. Then*

$$d(M_1) \le d(M).$$

*2. If A is a HW-*matrix *of dimension n and $\alpha \in \{2,3\}$, then*

$$d(p^{(\alpha)}(A)) \le 1.$$

Proof. The first statement is clear. For the proof of a second one, let us assume that $d(p^{(\alpha)}(A)) > 1$. Hence $rk(p^{(\alpha)}(A)) < n-1$. By definition there exists a non-trivial $X \subset \{1,2,...,n-1\}$, such that $\sum_{i\in X} p^{(\alpha)}(A_i) = 0$. Finally $p^{(\alpha)}(\sum_{i\in X} A_i) = \sum_{i\in X} p^{(\alpha)}(A_i) = 0$. This contradicts the Definition 2. □

Lemma 3. *Let $m < n$ and $W \in \mathcal{D}^{m\times n}$ is a sub-matrix of some $(n \times n)$ HW-*matrix. *Then $rk(p^{(\alpha)}(W)) = m$.*

Proof. Similar to the proof of the last Lemma. □

A symmetric $(m \times m)$ matrix $A \in (\mathbb{F}_2)^{m\times m}$ defines a nonoriented graph, graph(A) with set of vertices $\{1, 2, ..., m\}$ and two different vertices i and j are connected if and only if $A_{ij} = 1$. We say that a matrix A is connected if a graph(A) is connected. Let $A \in \mathcal{D}^{m\times m}$ be a symmetric matrix, then $p^{(i)}(A)$ are symmetric with coefficients in $\mathbb{F}_2, i = 2, 3$. We shall write $i \sim_2 j$ if i, j are at the same connected component of a matrix $p^{(2)}(A)$. Similar definition is for a relation $i \sim_3 j$.

Lemma 4. *Let a HW-*matrix *M have the following decomposition on the blocks:*

$$M = \begin{bmatrix} * & 2 & * \\ C & A & D \\ * & 3 & * \end{bmatrix}, \tag{16}$$

where A is a symmetric matrix and $2, 3$ are block matrices with all rows and columns equal 2 and 3 correspondingly. Then

(I) *if $i \sim_2 j \implies D_i = D_j$;*
(II) *if $i \sim_3 j \implies C_i = C_j$.*

Proof. For the proof of (I) let us assume that i, j (where $i < j$) are connected by a 2-edge; i.e. $A_{i,j} = 2$. Let r be some column of a matrix D. Let us consider a diagonal submatrix of the matrix M related to (i, j, r). It looks like

$$\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 3 & 3 & 1 \end{bmatrix}. \tag{17}$$

The sums of the first two columns are zero. Since Lemma 3 a sum of elements of the last one is not zero. Hence $a = b$. We have just proved that if $A_{i,j} = 2$ then $D_i = D_j$. It also means that if $i \sim_2 j$ then $D_i = D_j$. The proof of the second point of the lemma is similar. □

The next lemmas are about possibilities of complement of some matrices to a *HW-matrix*. We shall first consider an odd case.

Lemma 5. *Let $A \in \mathcal{D}^{m\times m}$ be a symmetric matrix with 1 on the diagonal and $\{2, 3\}$ off the diagonal with a column sums equal to 1. Assume that $m > 1$. Then a matrix*

$$K_A = \begin{bmatrix} 2 \\ A \\ 3 \end{bmatrix}, \tag{18}$$

*cannot be complement to HW-*matrix.

Proof. Let us assume that there exists a *HW-matrix*

$$\begin{bmatrix} * & 2 & * \\ C & A & D \\ * & 3 & * \end{bmatrix}. \tag{19}$$

From assumption m is an odd number and heights of the blocks 2 and 3 are also odd. We shall use induction. For $m = 3$

$$A = \begin{bmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{bmatrix}. \tag{20}$$

Here $a = 2$ or 3. If $a = 3$ then $\mathrm{rk}(p^{(2)}(A)) = 1$ and $d(p^{(2)}(A)) = 3 - 1 = 2 > 1$. From Lemma 2 it is impossible. For $a = 3$ the proof is the same. Let us assume that $m > 3$.

1. We shall consider a matrix $p^{(2)}(A)$. We claim that there is no such decomposition as

$$p^{(2)}(A) = B \oplus E,$$

such that a dimension of a matrix B is odd and > 1. In fact, in that case

$$A = \begin{bmatrix} \tilde{B} & 3 \\ 3 & \tilde{E} \end{bmatrix}. \tag{21}$$

Since a column sums of A are equal to 1 and height of a block 3 under $\tilde{B}$ is even, a column sums of $\tilde{B}$ are 1. If K_A has complement then $K_{\tilde{B}}$ has a complement (where a dimension of a block 3 is greater on a dimension of E). But by induction it is impossible, since $1 < \text{dimension}(\tilde{B}) < m$.

2. We claim that there is no such a nontrivial decomposition as

$$p^{(2)}(A) = B \oplus E \oplus F.$$

In fact since m is odd we have two possibilities:

(a) dimension of one component is odd and other components have dimension even
(b) dimension of all components are odd.

In the case (a) $\dim(B \oplus E) > 1$ and odd. Hence we consider decomposition $p^{(2)}(A) = (B \oplus E) \oplus F$. But it is a previous case 1.

In case (b), since $m > 3$ there exists a component (for example B) which dimension is > 1. In that case we have a decomposition $p^{(2)}(A) = B \oplus (E \oplus F)$ which was already considered in the point 1.

3. By definition we have a decomposition

$$p^{(2)}(A) = B_1 \oplus \ldots \oplus B_s,$$

where all components are connected matrices. From the above we can assume that $s = 2$ and odd component has a graph equal to one point or $s = 1$. Equivalently,

(a) $A = \left[\begin{smallmatrix} 1 & 3 \\ 3 & B \end{smallmatrix}\right]$ and $p^{(2)}(B)$ is connected or

(b) $p^{(2)}(A)$ is connected.

In the first case

$$p^{(3)}(A) = \left[\begin{smallmatrix} 1 & 1 \\ 1 & p^{(3)}(B) \end{smallmatrix}\right]. \tag{22}$$

Hence $p^{(3)}(A)$ is connected. Summing up, we have

(a) $A = \left[\begin{smallmatrix} 1 & 3 \\ 3 & B \end{smallmatrix}\right]$ and both $p^{(2)}(B)$ and $p^{(3)}(A)$ are connected or

(b) $p^{(2)}(A)$ is connected.

If we exchange $p^{(2)}$ for $p^{(3)}$ in the above points 1., 2. and 3. with the similar arguments, we obtain finally two cases:

(a) $A = \left[\begin{smallmatrix} 1 & 3 \\ 3 & B \end{smallmatrix}\right]$ and both $p^{(2)}(B)$ and $p^{(3)}(A)$ are connected or

(b) both $p^{(2)}(A)$ and $p^{(3)}(A)$ are connected.

We come back to the beginning of the proof. We shall try to figure out matrices C and D. From definition of $\sim_3$ and because $p^{(3)}(A)$ is connected we conclude that all rows of the matrix C are identical. By conjugation we can assume that $C = 2$. Using the same arguments and definition of $\sim_2$ together with a connectedness of $p^{(2)}(B)$ we conclude that with exception of the first row, all rows of the matrix D are the same. By conjugation and permutation we can assume that the first row of the matrix D is equal to $[2, ...2, 3, ..., 3]$. All other rows of a matrix D consist only 3. Summing up a matrix

$$W = [C \;\; A \;\; D]$$

is following

$$\begin{bmatrix} 2 & \begin{bmatrix} 1 & 3 \\ 3 & B \end{bmatrix} & \begin{matrix} 2 & 3 \\ 3 & 3 \end{matrix} \\ 2 & & \end{bmatrix}. \tag{23}$$

Apply homomorphisms: $p^{(3)}, [p^{(2)}, p^{(3)}], p^{(2)}, p^{(2)}$ to the corresponding columns we get a matrix

$$W' = \begin{bmatrix} \begin{matrix} 0 \\ 0 \end{matrix} & \begin{bmatrix} 1 & 1 \\ 0 & p^{(3)}(B) \end{bmatrix} & \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \end{bmatrix}. \tag{24}$$

We have $\mathrm{rk} W' = 1 + \mathrm{rk}(p^{(3)}(B))$. From assumption sums of columns of a matrix A are equal to 1. Hence sums of columns of a matrix

$$(p^{(2)}, p^{(3)})A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & p^{(3)}(B) \end{smallmatrix}\right] \tag{25}$$

are also equal to 1 and sums of columns of a matrix $p^{(3)}(B)$ are equal to 0. It means $\mathrm{rk}(p^{(3)}(B)) < m - 1$ and also $\mathrm{rk}(W') < m$. From Lemma 3

$$\mathrm{rk}(W') = \mathrm{rk}(W) = \text{number of rows } (W) = m.$$

Hence a matrix W cannot be a matrix of some rows of HW-*matrix*.

We have to still consider a case when matrices $p^{(2)}(A)$ and $p^{(3)}(A)$ are connected. Similar to the above consideration, using relation $\sim_2$ and $\sim_3$ plus conjugation we can assume that

$$[C \;\; A \;\; D] = [2 \;\; A \;\; 3].$$

Hence all nonempty sums of rows of a matrix A include 1. For $m > 1$ it is impossible. □

The next lemma is an even version of the Lemma 5.

Lemma 6. *Let $A \in \mathcal{D}^{m\times m}$ be a symmetric matrix with* 1 *on the diagonal and* $\{2, 3\}$ *off the diagonal with a column sums equal to* 3*. Assume that $m > 1$. Then a matrix*

$$K_A = \begin{bmatrix} 2 \\ A \\ 3 \end{bmatrix}, \tag{26}$$

cannot be a complement to some HW-matrix.

Proof. As in the proof of the previous lemma let us assume that there exists a HW-*matrix*

$$\begin{bmatrix} * & 2 & * \\ C & A & D \\ * & 3 & * \end{bmatrix}. \tag{27}$$

From assumption and Definition 2 m is an even number and a hight of the block 2 is even and 3 is odd. We shall use induction. For $m = 4$.

1. On the beginning let us consider the case, where $p^{(2)}(A)$ is not connected. We have two cases of matrices of dimension 4:

(a) $A = \left[\begin{smallmatrix} 1 & 3 \\ 3 & B \end{smallmatrix}\right]$, where B has a dimension 3 and

(b) $A = \left[\begin{smallmatrix} B & 3 \\ 3 & E \end{smallmatrix}\right]$, where matrices A, B have rank two.

The case (a) is impossible since $1+3 \neq 3$. In the case (b) matrices A and B are symmetric with columns sums equal to 3. Hence $B = E = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right]$, and

$$p^{(2)}(A) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{28}$$

From the other side a matrix $p^{(2)}(K_A)$ has rows of 1 $(p^{(2)}(2) = 1)$ and rows of 0 $(p^{(2)}(3) = 0)$. These rows are linear combination of rows of $p^{(2)}(A)$ and

$$\mathrm{rk} p^{(2)}(K_A) = \mathrm{rk} p^{(2)}(A) = 2.$$

Finally $\mathrm{d}(K_A) = 4 - 2 = 2 > 1$ and from Lemma 2 we are done.

2. As the second step let us consider the case where $p^{(3)}(A)$ is not connected. We have to consider two cases of matrices of dimension 4:

(a) $A = \left[\begin{smallmatrix} 1 & 2 \\ 2 & B \end{smallmatrix}\right]$, and

(b) $A = \left[\begin{smallmatrix} B & 2 \\ 2 & E \end{smallmatrix}\right]$, and B and E have dimension 2.

In the case (a) a matrix B is symmetric of dimension 3 with sums of columns 1. If K_A has complement to *HW-matrix* then also a matrix K_B has this possibility. But it is impossible by Lemma 5. In case (b) matrices B, E are symmetric with sums of columns 3. Hence $B = E = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right]$ and

$$p^{(2)}(A) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}. \tag{29}$$

In the matrix $p^{(2)}(K_A)$ we have rows of 1 and 0. They are linearly dependent from the rows of $p^{(2)}(A)$. Hence

$$\mathrm{rk} p^{(2)}(K_A) = \mathrm{rk} p^{(2)}(A) = 1$$

and

$$\mathrm{d}(K_A) = 4 - 1 = 3 > 1.$$

From Lemma 2 the matrix K_A has not complement to the *HW-matrix*.

3. By the above points 1. and 2. we have that $p^{(2)}(A)$ and $p^{(3)}(A)$ are connected matrices. As in the proof of Lemma 5 using relations $\sim_2, \sim_3$ and conjugations of matrices we can assume that

$$[C \;\; A \;\; D] = [2 \;\; A \;\; 3].$$

By assumption a sum of all rows of the above matrix has 1 on a some position. We can see easily that it is impossible at the first and the third block. For a matrix A it is also impossible since m is even. This contradicts our assumption that $m < n$.

Let us assume that $m > 4$. We shall consider three steps.

1. Assume that $p^{(2)}(A)$ is not connected. We have to consider two cases:

(a) $p^{(2)}(A)$ is a direct sum of two odd blocks,

(b) $p^{(2)}(A)$ is a direct sum of two even blocks.

Hence $A = \left[\begin{smallmatrix} B & 3 \\ 3 & E \end{smallmatrix}\right]$. In the case (a) since dimensions of B, E are odd and sums of column of A are 3 we obtain that sums of column of B and E are 0. Moreover, if B is an odd diagonal submatrix of *HW-matrix* then by Definition 2 a sum of rows of B should enclose 1. But this is impossible and also case (a) is impossible.

In case (b) since dimensions of B, E are even and sums of column of A are 3 we obtain that sums of column of B and E are 3. Moreover either the matrix B or the matrix E has rank > 2. Assume the matrix B has such a property. If a matrix K_A has complement, then a matrix K_B has complement to *HW-matrix*. But by induction it is impossible.

2. Assume that $p^{(3)}(A)$ is not connected. We have to consider two cases. The same as in the step 1.

(a) $p^{(3)}(A)$ is a direct sum of two odd blocks,

(b) $p^{(3)}(A)$ is a direct sum of two even blocks.

Hence $A = \left[\begin{smallmatrix} B & 2 \\ 2 & E \end{smallmatrix}\right]$. In the first case since dimensions of B, E are odd and sums of column of A are 3 we obtain that sums of column of B and E are 1. Moreover, either the matrix B or the matrix E has rank > 2. Assume the matrix B has such a property. If a matrix K_A has complement then (after permutation of indexes) a matrix K_B has complement to *HW-matrix*. But by Lemma 5 it is impossible. In the second case, since dimensions of B, E are even and sums of column of A are 3 we obtain that sums of column of B and E are 3. Moreover, either the matrix B or E has rank > 2. Assume the matrix B has such a property: If a matrix K_A has complement then a matrix K_B has complement to *HW-matrix*. But by induction it is impossible.

We can assume that matrices $p^{(2)}(A)$ and $p^{(3)}(A)$ are connected. As in the previous cases we can assume that

$$[C \;\; A \;\; D] = [2 \;\; A \;\; 3].$$

By Definition 2 a sum of all rows should enclose 1. Since m is even and $m < n$ we have a contradiction. □

4. Proof of the Main Lemma

We keep the notation from previous sections, but we also need a new definitions. Denote by $\mathcal{P}_n$ an algebra of all subsets of the set $\{1, 2, \dots, n\}$. Let $\mid U \mid$ denote the number of elements of a set $U \in \mathcal{P}_n$ modulo two. We have an isomorphism of algebras $I : \mathbb{F}_2^n \to \mathcal{P}_n$, where

$$I(x) = \{i \mid x_i = 1\}, x \in \mathbb{F}_2^n \tag{30}$$

is an indicator.

Definition 3. Let A be a *HW-matrix*. The function $J : \mathcal{P}_n \to \mathcal{P}_n$ is defined by

$$J(U) = \{s \mid \sum_{i \in U} A_{is} = 1\}, \tag{31}$$

where $U \in \mathcal{P}_n$.

Remark 3. In what follows we shall use a formula (10) with a basis $b_i, 1 \leq i \leq n-1$. Let us consider a map $l : \mathcal{P}_n \to \mathbb{F}_2[x_1, \ldots, x_{n-1}]$ given by a formula

$$l_Z := \sum_{i \in Z} x_i. \tag{32}$$

In this language the formula (10) for $k = 2, 3$ we can write as

$$\sum_{j=1}^{n-1} p^{(k)} A_{ji} x_j = l_S$$

where $S = \{p^{(k)}(A_{1,i}), p^{(k)}(A_{2,i}), ..., p^{(k)}(A_{n-1,i})\}$.

Proposition 3. *The map J has the following properties:*

1. *$U \neq 0, 1$ then $J(U) \neq 0$, here $0, 1$ denote the trivial additive and multiplicative element of the algebra $\mathcal{P}_n$ respectively;*
2. *$J(U + 1) = J(U)$ where $U + 1 = U'$ denotes a complement of the subset U in the set $\{1, 2, ..., n\}$;*
3. *$J(\{i\}) = \{i\}, i = 1, 2, ..., n$;*
4. *if $\mid U \mid = 1$ then $J(U) \subset U$;*
5. *if $\mid U \mid = 0$ then $J(U) \subset U'$.*

Proof. Elementary calculations with support of the matrix (15). □

Any polynomial of $\mathbb{F}_2[x_1, x_2, \ldots, x_n]$ we shall identify with a polynomial map $\mathbb{F}_2^n \to \mathbb{F}_2$. Hence by indicator function (30) the formula (32) has the following presentation $l_Z(e_j) = \{j \in Z\}$, where $Z \in \mathcal{P}_n$. Since the transgressive elements $T_i \in \mathbb{F}_2[x_1, \ldots, x_{n-1}]$ we define a split monomorphism of rings $\mathbb{F}_2[x_1, \ldots, x_{n-1}] \xrightarrow{\phi} \mathbb{F}_2[x_1, \ldots, x_n]$ such that $\bar{T}_i = \phi(T_i) \in \mathbb{F}_2[x_1, \ldots, x_n], i = 1, \ldots, n$. Here, $\phi(x_i) = x_i + x_n, i = 1, 2, \ldots, n-1$. Obviously $\#D = \#\phi(D)$.

From definition, for polynomial functions $\bar{T}_i$ we have $\bar{T}_i(e_j) = \delta_{ij}$, where $1 \leq i, j \leq n$ and $e_i \in (\mathbb{F}_2)^n$ is the standard basis. Hence, by the isomorphism (30) a map J (see Definition 3) is equivalent to a function $T : \mathbb{F}_2^n \to \mathbb{F}_2^n, T(x) = (\bar{T}_1(x), \bar{T}_2(x), \ldots, \bar{T}_n(x))$, where $x \in \mathbb{F}_2^n$. Hence and from an equation (12) we have a commutative diagram

$$\begin{array}{ccc} \mathbb{F}_2^n & \xrightarrow{T} & \mathbb{F}_2^n \\ \downarrow I & & \downarrow I \\ \mathcal{P}_n & \xrightarrow{J} & \mathcal{P}_n \end{array} . \tag{33}$$

We shall use these observations in the proof of the **Main Lemma**. Moreover, we shall apply a remark that homogeneous polynomials of degree 2 are recognized by their polynomial functions. Let $S, Z_1, Z_2 \in \mathcal{P}_n$. From definition if

$$\sum_{i \in S} \bar{T}_i = l_{Z_1} \cdot l_{Z_2}$$

then $S = Z_1 \cap Z_2$.

Proposition 4. *The following conditions are equivalent.*

(i) $\sum_{i \in S} \bar{T}_i = l_{Z_1} \cdot l_{Z_2}$

(ii) $\forall U \in \mathcal{P}_n |J(U)S| = |UZ_1| \cdot |UZ_2|$.

Proof. We shall use (33) and an isomorphism I. Let $x \in \mathbb{F}_2^n, U = I(x)$. We have

$$\sum_{i \in S} \bar{T}_i(x) = \sum_{i \in S \cap I(\bar{T}(x))} 1 = | \; I(\bar{T}(x)) \cap S \; | = | \; J(I(x)) \cap S \; | = | \; J(U) \cap S \; | \, .$$

From the other side

$$l_{Z_1}(x) \cdot l_{Z_2}(x) = \sum_{i \in Z_1 \cap I(x))} 1 \cdot \sum_{i \in Z_2 \cap I(x))} 1 = |UZ_1| \cdot |UZ_2|.$$

This finishes a proof. □

Corollary 1. *Let us assume the condition* (ii) *of Proposition 4, then*

1. *$|Z_1|$ or $|Z_2|$ is even,*
2. *if $S \neq 0$ then $|Z_1|$ and $|Z_2|$ are even*
3. *if $S \neq Z_1$ and $S \neq Z_2$ then $Z_1 \cup Z_2 = 1$.*

Proof. 1. Since $J(1) = J(\{1, 2, \ldots, n\}) = 0$ the condition is true.

2. Since $J(U) = J(U') = J(U + 1)$ we have

$$|UZ_1||UZ_2| = |(1 + U)Z_1||(1 + U)Z_2|.$$

Hence

$$|Z_1||Z_2| + |Z_1||UZ_2| + |Z_2||UZ_1| = 0.$$

From a point 1. we can assume that $|Z_1| = 0$ (or $|Z_2| = 0$) and $|Z_2||UZ_1| = 0$. If $|Z_2| = 1$ then $\forall U \in \mathcal{P}_n, |UZ_1| = 0$ and $Z_1 = 0$. Since $S = Z_1 \cdot Z_2 \neq 0$ we have a contradiction.

3. Let $a \in Z_1 \setminus S, b \in Z_2 \setminus S$ and $c \notin Z_1 \cup Z_2$. Put $U = \{a, b, c\}$. We have $J(U)S \subset US = 0$ and $UZ_1 = \{a\}, UZ_2 = \{b\}$. Hence

$$0 = |J(U)S| = |UZ_1||UZ_2| = 1 \cdot 1 = 1.$$

This is a contradiction. □

Definition 4. Define

$$\sigma_a^S := \sum_{i \in S} A_{a,i},$$

where $a \in \{1, 2, \ldots, n\}, S \subset \{1, 2, \ldots, n\}$ and $A \in \mathcal{D}^{n \times n}$.

Let us present relations between the above definition and the function J.

Proposition 5. *Let A be $(n \times n)$ HW-matrix, $a, b \in \{1, 2, \ldots, n\}$ and $S \in \mathcal{P}_n$. Then*

1. $|J(\{a, b\})S| = \sigma_a^S + \sigma_b^S$, *where* $a, b \notin S$;
2. $|J(\{a, b\})S| = \sigma_a^S + \sigma_b^S + A_{a,b} + 1$, *where* $a \notin S$, $b \in S$.
3. $|J(\{a, b\})S| = \sigma_a^S + \sigma_b^S + A_{a,b} + A_{b,a}$, *where* $a, b \in S$.

Proof. 1. By a point 5. of Proposition 3 we know that $J(\{a, b\}) \subset \{a, b\}'$. If $J(\{a, b\})S = \emptyset$ we are done. On the contrary we shall consider the following cases.

(a) Assume $|S| = 1$ and $|J(\{a, b\})S| = 1$. We have two rows, which correspond to a and b,

$$\begin{matrix} 2 & 2 & \ldots & 2 & 2 \\ 2 & 3 & \ldots & 3 & 2 \end{matrix} \tag{34}$$

with a number of columns equal to $|S|$, and a number of columns with different coefficients equal to $J(\{a, b\})$. Hence a sum of the upper row is equal to 2 and a sum of the down row is equal to 3. This finishes a proof in this case.

(a') Assume $|S| = 1$ and $|J(\{a, b\})S| = 0$. We also have (34) and a sum of the upper row is equal to 2 and a sum of the down row is also equal to 2. This finishes a proof in this case.

(b) Assume $|S| = 0$. Then again we have two subcases $|J(\{a, b\})S| = 1$, then a sum of the upper row of (34) is equal to 0 and a sum of the down row is equal to 1. The proof of the case is complete. When $|J(\{a,b\})S| = 0$ a sum of the upper row of (34) is 0 and a sum of the down row is also 0. This finished a proof of point 1. The proofs of other cases are similar and we put it as an exercise. □

Using the above language we shall prove that for a *HW-manifold* there exists only a limited number of transgressive elements which are a product of degree one nontrivial polynomials.

Proposition 6. *Let A be a $(n \times n)$ HW*-matrix*, $(n > 3)$ then there does not exist not empty set $S \subset \{1, 2, \dots, n\}$ such that*

$$\forall U \in \mathcal{P}_n |J(U)S| = |US|. \tag{35}$$

Proof. It is the case $S = Z_1 = Z_2$. Let us assume (35). We are going to divide the proof into four steps.

Step 1. We claim that, if $a_1, a_2 \notin S$ and $b \in S$ then $A_{a_1,b} = A_{a_2,b}$. In fact, from (35) for $U = \{a_1, a_2\}$, $|J(\{a_1, a_2\})S| = |\{a_1, a_2\}S| = 0$. By Proposition 5 (1.), $\sigma^S_{a_1} = \sigma^S_{a_2} := \sigma$. If $a \notin S$ then from Proposition 5 (2.)

$$1 = |J(\{a,b\})S| = |\{a,b\}S| = \sigma^S_a + \sigma^S_b + A_{a,b} + 1 = \sigma + \sigma^S_b + A_{a,b} + 1.$$

Hence $\forall a \notin S, A_{a,b} = \sigma + \sigma^S_b$ and *Step 1.* is proved.

Step 2. We claim that, if $US = 0$ then $J(U)S = 0$. In fact from *Step 1.* all elements (numbers of columns) of $J(U)$ which are considered have not the first indexes from S and are equal each other. Then $J(U)S = 0$.

Step 3. We claim that, if $S \neq 0$ then $\#S = n - 1$. From *Step 2.* if $0 \neq U \subset S'$ then $J(U)S' \neq 0$. Let B be a diagonal submatrix of the matrix A related to the set S'. Then B is a quadratic matrix with 1 on the diagonal and $2, 3$ otherwise. Moreover all sums of rows of B have at some position an element 1. Hence, the only possible matrix B is (1×1) matrix.

Step 4. We claim that, if $S \neq 0$ then $n \leq 3$. For the proof, let us assume that $n > 3$. From the *Step 3.* we can assume that $S = \{2, 3, \dots, n\}$. Let l_2 denote a number of 2 at the first column of A. We shall prove that $|l_2| = 0$. In fact, we can assume that $0 < l_2 < n - 1$ and at the first column, from the top we have first 2 then going down we have 3. On the contrary, suppose that l_2 is odd and let v be a sum of the first $2l_2 + 1$ rows. Since $l_2 + 1$ is even v has not 1 on places $1, 2, \dots, l_2 + 1$. Then it has 1 on the position $> l_2 + 1$. Hence there exists $k \geq l_2 + 1$ such that $A_{1,r} \neq A_{k,r}$ or equivalently $A_{1,r} + A_{k,r} = 1$. Let us consider a diagonal submatrix

$$\begin{bmatrix} 1 & * & A_{1,r} \\ 2 & 1 & A_{k,r} \\ 3 & * & 1 \end{bmatrix}. \tag{36}$$

A sum of elements at the first column and at the third column is 0, then it at the second column has to be $\neq 0$. Let $U = \{1, k, r\}$. Since $j(U) \subset U$ and $n > 3$, $J(U) = \{k\}$. Finally

$$1 = |\{k\}S| = |J(U)S| = |US| = |\{k,r\}S| = 0.$$

That is a contradiction and l_2 is even. Moreover if l_3 is a number of 3 at the first column then $| \; l_3 = n - 1 - l_2 \; | = 0$ and a sum $1 + l_2 * 2 + l_3 * 2 = 1$. But a sum of all rows is zero and we have a contradiction. This finishes a proof. □

Corollary 2. *At the space $Im(d_2)$ we have not squares.*

Proof. If $l_Z \in \mathrm{Im}(d_2)$, then $S = Z = Z$. For $n > 3$ it is impossible. □

Proposition 7. *Let $S, Z \subset \{1, 2, \ldots, n\}$ such that $0 \neq S \neq Z$. Let A, J be as in Proposition 6. Assume that*

$$\forall U \in \mathcal{P}_n \mid J(U)S \mid = \mid US \mid \cdot \mid UZ \mid$$

then $\#S = 2, \mid Z \mid = 0$ and $S \subset Z$.

Proof. On the beginning we claim that up to permutation and conjugation,

$$A = \begin{bmatrix} * & 2 & * \\ * & B & * \\ * & 3 & * \end{bmatrix}, \tag{37}$$

where B is a symmetric matrix with a column sums 3. Moreover a block 2 has rows indexed by numbers from the set $Z \setminus S$ and a block 3 has rows indexed by numbers from the set $1 + Z = Z'$. In fact, from Proposition 4, $S \subset Z$ and Corollary 1, $S \subset Z$ and $|S| = |Z| = 0$. Let us change the indexes of A such that

$$A = \begin{bmatrix} * & E & * \\ * & B & * \\ * & F & * \end{bmatrix}, \tag{38}$$

and E has rows indexed by numbers from the set $Z \setminus S$, B has rows indexed by numbers from S and F is indexed by $1 + Z = Z'$. From the point 1 of Proposition 5, for $a, b \notin S$

$$\sigma_a^S + \sigma_b^S = |J(\{a,b\}S| = |\{a,b\}S| \cdot |\{a,b\}Z| = 0.$$

Hence $\sigma_a^S = \sigma_b^S$. Let $\sigma := \sigma_a^S$, for $a \notin S$.

By the point 2 of Proposition 5 for $b \in S$ and $a \notin Z$,

$$A_{a,b} = \sigma + \sigma_b^S. \tag{39}$$

From the above all columns of the matrix F are constant. Again from the point 2 of Proposition 5 for $b \in S, a \in Z \setminus S$,

$$A_{a,b} = \sigma + \sigma_b^S + 1. \tag{40}$$

It follows that also columns of the matrix E are constant. Let us conjugate columns of the matrix A such that $E = 2$. In that case $\sigma = 0$ since for $a \in Z \setminus S$ we have $\sigma = \sigma_a^S = |S| \cdot 2 = 0$.

From (40), for $b \in S, 2 = 0 + \sigma_b^S + 1$. Hence $\sigma_b^S = 3$ and $F = 3$, because from the formula (39) $A_{a,b} = 0 + 3$, for $a \in Z'$ and $b \in S$. Finally, from Proposition 5 for $a, b \in S$ we have

$$\begin{aligned} A_{a,b} + A_{b,a} = 3 + 3 + A_{a,b} + A_{b,a} = \sigma_a^S + \sigma_b^S + A_{a,b} + A_{b,a} = \\ = |J(\{a,b\})S| = |\{a,b\}S| \cdot |\{a,b\}Z| = 0. \end{aligned} \tag{41}$$

To finish a proof it suffices to apply Lemma 6. □

Proposition 8. *We keep the notation from the previous propositions. Let us assume* $S, Z_1, Z_2 \in \mathcal{P}_n$ *such that* $0 \neq S, S \neq Z_1, S \neq Z_2$ *and*

$$\forall U \in \mathcal{P}_n |J(U)S| = |UZ_1| \cdot |UZ_2|$$

then $\#S = 1, |Z_1| = |Z_2| = 0$ *and* $Z_1 + Z_2 = 1$.

Proof. A proof is similar to the proof of Proposition 7. On the beginning we show that (up to permutation and conjugation)

$$A = \begin{bmatrix} * & 2 & * \\ * & B & * \\ * & 3 & * \end{bmatrix}, \tag{42}$$

where B is a symmetric matrix of odd dimension with sums of columns 1, a block 2 is indexed by the set $Z_1 \setminus S$ and a block 3 is indexed by the set $Z_2 \setminus S$. In fact, from assumption and Corollary 1, $S = Z_1 Z_2, |Z_1| = |Z_2| = 0$ and $Z_1 + Z_2 = 1$. Hence $|S| = 1$. Let us change the order of rows in the matrix A such that

$$A = \begin{bmatrix} * & E & * \\ * & B & * \\ * & F & * \end{bmatrix} \tag{43}$$

and E is indexed by $Z_1 \setminus S$, B by S and F by $Z_2 \setminus S$. From Proposition 5 we have $\forall a, b \in Z_1 \setminus S, \sigma_a^S = \sigma_b^S := \sigma_E$. With similar consideration we have $\forall a, b \in Z_2 \setminus S, \sigma_a^S = \sigma_b^S := \sigma_F$. Moreover, by Proposition 5 (2) for $b \in S$ and $a \in Z_1 \setminus S$,

$$A_{a,b} = \sigma_E + \sigma_b^S + 1. \tag{44}$$

From the above, all columns of the matrix E are the same. By analogy for $b \in S$ and $a \in Z_2 \setminus S$,

$$A_{a,b} = \sigma_F + \sigma_b^S + 1 \tag{45}$$

and columns of the matrix F are also constant. Let us conjugate columns of A such that $E = 2$. Then $\sigma_E = 2$, because for $a \in Z_1 \setminus S, \sigma_E = \sigma_a^S = |S| \cdot 2 = 2$ and for $b \in S, \sigma_b^S = 1$. The last equality follows from (44) because $2 = 2 + \sigma_b^S + 1$. Similarly, by (45) for $b \in S$ and $a \in Z_2 \setminus S$, we have $A_{a,b} = \sigma_F + 1 + 1 = \sigma_F$ and the matrix F is constant and equal to σ_F. Finally, a matrix B is symmetric since from Proposition 5

$$\sigma_a^S + \sigma_b^S + A_{a,b} + A_{b,a} = |J(\{a,b\}S|$$

what means,

$$A_{a,b} + A_{b,a} = |\{a,b\}Z_1| \cdot |\{a,b\}Z_2| = 0.$$

We have still to show that $\sigma_F = 3$. In fact from assumption a column's sums of B are 1. Since B is symmetric the same is true for rows. Let us calculate a sum of some column of A:

$$(|Z_1| - |S|)2 + 1 + (|Z_2| - |S|)\sigma_F = 2 + 1 + \sigma_F = 3 + \sigma_F = 0.$$

To finish a proof of Proposition we have to apply Lemma 5. □

Summing up we have the following two possibilities:

I. $\#S = 1$ and $Z_1 + Z_2 = 1$;

II. $\#S = 2$ and $S = Z_1, S \neq Z_2$ or $S = Z_2.S \neq Z_1$.

Let us recall that $\mathrm{Im}(d_2)$ is a n-dimensional $\mathbb{Z}_2$-space generated by $T_i, i = 1, 2, 3, \dots, n$. We are interested in description of the set D of elements in $\mathrm{Im} d_2$ which are a product of two nontrivial linear polynomials, see (13). We claim that $D \leq n + 1$. In what follows, if it does not give a contradiction we shall write T_i for $\bar{T}_i, i = 1, \dots, n$.

Lemma 7. *Let $w \in D$, then $w = T_i$ or $w = T_j + T_k$ for some $1 \leq i, j, k \leq n$.*

Proof. On the beginning we shall prove that $T_i + T_j$ is a product of two nontrivial linear polynomials if and only if T_i, T_j have a common component. It means there exists $\mathrm{p} \neq 0$ s.t. $\mathrm{p}|T_i$ and $\mathrm{p}|T_j$. Let $T_i + T_j$ have a common component, then from the above case II we can assume that $j = i + 1$ and the matrix A enclose:

$$\begin{matrix} \dots \\ 2 \quad 2 \\ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \\ 3 \quad 3 \\ \dots . \end{matrix}$$

By definition

$$T_i = (x_1 + \cdots + x_i + x_{i+1})(x_i + x_{i+2} + \cdots + x_n)$$
$$T_{i+1} = (x_1 + \cdots + x_i + x_{i+1})(x_{i+1} + \cdots + x_n).$$

For the proof of the opposite conclusion we shall need

Definition 5. Let X be a subset of some monoid. By Γ_X we define a graph with the vertex set X and two vertices a, b are connected by an edge $a \underline{\;\;f\;\;} b$ if and only if $f|a$ and $f|b$. Put $\Gamma := \Gamma_{T_1,T_2,\ldots,T_n}$.

We claim that for $n > 3$ the graph

$$i \underline{\;\;f\;\;} j \underline{\;\;g\;\;} k \tag{46}$$

is not a subgraph of Γ, where $i := T_i, i = 1, 2, \ldots, n$. In fact we have two possibilities:

1. $f = g$. Let $i = 1, j = 2, k = 3$ and let $\mathfrak{I}$ be an ideal generated by $(f, T_4, \ldots, T_n)$ in the polynomial ring. Since there exists a nontrivial solution of system of $(n-2)$ linear equation in $(n-1)$ linear space an algebraic set $\mathrm{V}(\mathfrak{I})$ is not trivial. It means $0 \neq x \in \mathrm{V}(\mathfrak{I})$. From definition $x \in \mathrm{V}(\mathfrak{I}')$, where $\mathfrak{I}'$ is an ideal generated by $(T_1, T_2, \ldots, T_n)$. But it is impossible.

2. $f \neq g$. Using permutation of indexes and conjugation we can assume that in *HW-matrix* $A, j = i+1, k = i+2$. Recall that $S = \{i, i+1\}$ and A is as in Lemma 6. Hence it has a diagonal block related to rows (columns) $\{i, i+1, i+2\}$

$$\begin{bmatrix} 1 & 2 & b \\ 2 & 1 & a \\ 3 & 3 & 1 \end{bmatrix}, \tag{47}$$

and a matrix A has upper two first columns of (47) only elements 2, but lower only elements 3. Let us consider polynomials T_i, T_{i+1} and T_{i+2} for $x_s = 0, s \notin \{i, i+1, i+2\}$ and denote it by $\hat{T}_i$ respectively. We have

$$\hat{T}_i = (x_i + x_{i+1})(x_i + x_{i+2})$$

and

$$\hat{T}_{i+1} = (x_i + x_{i+1})(x_{i+1} + x_{i+2}).$$

The both polynomials are divided by $(x_i + x_{i+1})$. Hence $\hat{T}_{i+1}$ and $\hat{T}_{i+2}$ are divided by $(x_{i+1} + x_{i+2})$. From the above we can observe that

$$\hat{T}_{i+2} = (x_{i+1} + x_{i+2})(x_{i+2} + x_i). \tag{48}$$

By (48) and definition we get $a \neq b$. Hence a sum of all columns of the matrix (47) are equal to 0. But it is impossible, since $n > 3$. This finishes a proof of our claim and we have

Corollary 3. *For $n > 3$ all connected components of a graph Γ are points or edges $i \overset{f}{\text{———}} j$.* □

Corollary 4. *For $n > 3, D = \{T_1, T_2, ..., T_n\}$ or $\mathrm{D} = \{T_1, T_2, ..., T_n, T_i + T_j\}$ for some $1 \leq i, j \leq n$.*

Proof. Conversely, suppose that edges

$$1 \overset{f}{\text{———}} 2 \text{ and } 3 \overset{g}{\text{———}} 4$$

are components of the graph Γ. Let us consider an ideal $\mathfrak{J} = (f, g, T_5, \ldots, T_n)$ in polynomial ring. Since there exists a nontrivial solution of system of $(n-2)$ linear equation in $(n-1)$ linear space an algebraic set $\mathrm{V}(\mathfrak{J})$ is not trivial. It means $0 \neq x \in \mathrm{V}(\mathfrak{J})$. But from definition $x \in \mathrm{V}(\mathfrak{J}')$, where $\mathfrak{J}'$ is an ideal generated by $(T_1, T_2, \ldots, T_n)$. But it is impossible. This finishes a proof. □

Let us prove the **Main Lemma**.

Main Lemma. *Let $n > 3$, then there are the following possibilities for the structure of the set D:*

1. *$D = \{T_1, T_2, \ldots, T_n\}$;*
2. *$D = \{T_1, T_2, \ldots, T_n, T_i + T_j\}$, and we can find a polynomial p of degree one such that $\mathrm{p} \mid T_i$ and $\mathrm{p} \mid T_j$ for some $1 \leq i, j \leq n$. In the second case we can rediscover the set of generators $T_1, T_2, \ldots, T_n$.*

Proof. We start from the simple observation. If $i \neq j$ and $T_i = \mathrm{p} \cdot \mathrm{q}, T_j = \mathrm{p} \cdot \mathrm{r}$ then $\forall i = 1, 2, \ldots, n \quad \mathrm{q} + \mathrm{r}$ is not divided T_i. In fact $\mathrm{q} + \mathrm{r} \neq \mathrm{p}$ since in other case $T_i + T_j = \mathrm{p}(\mathrm{q} + \mathrm{r}) = \mathrm{p}^2$. By Corollary 2 it is impossible. Hence T_i and T_j are also not divided by $\mathrm{q} + \mathrm{r}$. Moreover, if $T_r = (\mathrm{q} + \mathrm{r})\mathrm{s}$ then $T_i + T_j + T_r = (\mathrm{q} + \mathrm{r})(\mathrm{p} + \mathrm{s})$. By Proposition 7, a decomposition for $\#S = 3$ is impossible. Let us prove the second point of the above lemma. From definition the graph $\Gamma_{T_1, \ldots, T_n}$ has connected components which are vertices for $r \notin \{i, j\}$, of the triangle with vertices $T_i, T_j, T_i + T_j$ and a constant label which is a component of T_i and T_j. Let $T_i = \mathrm{p} \cdot \mathrm{q}$ and $T_j = \mathrm{p} \cdot \mathrm{r}$ then $T_i + T_j = \mathrm{p}(\mathrm{q} + \mathrm{r})$. The triangle is a connected component of a graph because by (46) for $r \notin \{i, j\}$ elements $\mathrm{p}, \mathrm{q}, \mathrm{r}$ do not divide T_r. Also from the above simple observation, the element $(\mathrm{q} + \mathrm{r})$ does not divide T_r.

We continue the proof of the **Main Lemma**. Let $w = \xi\eta$ where ξ and η are linear polynomials. Let us define $s(w) := \xi + \eta$. Since *HW-manifolds* are oriented $\sum_i s(T_i) = 0$.

We claim that if $T_i + T_j \in D$, then $s(\xi) + s(\eta)$ recognizes subsets of order two of the set $\{T_i, T_j, T_i + T_j\}$. In fact, let $T_i = \mathrm{p} \cdot \mathrm{q}, T_j = \mathrm{p} \cdot \mathrm{r}$, then $T_i + T_j = \mathrm{p}(\mathrm{q} + \mathrm{r})$ and $s(T_i) + s(T_j) = \mathrm{q} + \mathrm{r}, s(T_i) + s(T_i + T_j) = \mathrm{r}, s(T_j) + s(T_i + T_j) = \mathrm{q}$.

Let $n > 3$, then there are the following possibilities for the structure of the set D:

1. $D = \{T_1, T_2, \ldots, T_n\}$;
2. $D = \{T_1, T_2, \ldots, T_n, T_i + T_j\}$, for some $1 \leq i, j \leq n$. Let $n > 3$ if D has n elements we are done. If it has $(n + 1)$ elements then the graph $\Gamma_{T_1, T_2, \ldots, T_n}$ has $(n - 2)$ discrete connected components D^c and a triangle. We proceed in two steps:

1. Put $s_{D^c} := \sum_{a \in D^c} s(a)$
2. From the triangle we take a unique pair ξ, η such that

$$s(\xi) + s(\eta) + s_{D^c} = 0.$$

Hence $\{T_1, T_2, \ldots, T_n\} = \{\xi, \eta\} \cup D$. This finishes a proof of the **Main Lemma**. $\square$

For illustration of possibilities of the structure of the set D we present two examples.

Example 3. Let $G \subset \mathcal{D}^5$ correspond to *HW-matrix* $\begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 2 & 1 & 3 & 2 & 2 \\ 3 & 2 & 1 & 3 & 2 \\ 3 & 2 & 3 & 1 & 3 \\ 3 & 3 & 3 & 2 & 1 \end{bmatrix}$.

The set

$$\begin{aligned} D = \{T_1 &= (x_1 + x_2)(x_1 + x_3 + x_4), T_2 = (x_1 + x_2 + x_3 + x_4)x_2, \\ T_3 &= (x_1 + x_3)(x_2 + x_3 + x_4)), T_4 = (x_1 + x_2 + x_4)(x_3 + x_4), \\ T_5 &= (x_1 + x_2 + x_3)x_4\}. \end{aligned} \tag{49}$$

From Remark 1 the above group is isomorphic to the group Γ_1 of the Example 2. The next example illustrates the second case of the **Main Lemma**.

Example 4. Let a matrix $\begin{bmatrix} 1 & 2 & 2 & 2 & 2 \\ 2 & 1 & 3 & 2 & 2 \\ 2 & 2 & 1 & 3 & 2 \\ 2 & 2 & 2 & 1 & 3 \\ 3 & 3 & 2 & 2 & 1 \end{bmatrix} \in \mathcal{D}^{5\times 5}$ be the second *HW-matrix* of dimension 5.

In this case we have

$$\begin{aligned} D = \{T_1 &= (x_1 + x_2 + x_3 + x_4)x_1, T_2 = (x_1 + x_2 + x_3 + x_4)x_2, \\ T_3 &= (x_1 + x_3 + x_4)(x_2 + x_3), T_4 = (x_1 + x_2 + x_4)(x_3 + x_4), \\ T_5 &= (x_1 + x_2 + x_3)x_4, T_1 + T_2\}. \end{aligned} \tag{50}$$

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