Properties of generalized Hantzsche-Wendt groups

A.Szczepański
Institute of Mathematics, University of Gdańsk
ul. Wita Stwosza 57, 80-952 Gdańsk, Poland
E-mail: matas @ paula.univ.gda.pl *

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1 Introduction

Let M be a closed Riemannian manifold of dimension n. We shall call M a Generalized Hantzsche-Wendt manifold (GHW manifold for short) (cf. [11]) if the holonomy group of M is isomorphic to $(\mathbb{Z}_2)^{n-1}$, where $\mathbb{Z}_2$ is the ring of integers modulo 2. Likewise, we shall call the fundamental group $\Gamma = \pi_1(M)$. By theorems of Cartan-Ambrose-Singer and Bieberbach [13, Cor. 3.4.7, p. 110], the fundamental group Γ is an extension of $\mathbb{Z}^n$ by $\mathbb{Z}_2^{n-1}$. Hence, we have the short exact sequnce of groups

$$(*) \quad 0 \to \mathbb{Z}^n \to \Gamma \xrightarrow{p} (\mathbb{Z}_2)^{n-1} \to 0.$$

Moreover, Γ is a discrete and torsion free subgroup of $E(n) = O(n) \ltimes \mathbb{R}^n$. Let us recall some known properties of the GHW manifolds (groups). In the oriented case, when Γ is contained in $SO(n) \ltimes \mathbb{R}^n$, which is only possible for n odd, they are rational homology spheres (cf.[11]), and there are pairs of such manifolds that are isospectral and but are not homeomorphic ([7]). It is also known that for all oriented GHW manifolds, the first homology group

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is exactly the holonomy group, (cf. [10]). We should also mention that there are relations between the oriented GHW groups and the Fibonacci groups $F(2n, 2(2n+1))$, (cf. [12]). More precisely, there exists an epimorphism $\Phi_{2n} : F(2n, 2(2n+1)) \to \Gamma_{2n+1}, n \geq 1$, where Γ_{2n+1} will be defined in section 3, Example 2 as a specific oriented GHW group of rank $(2n+1)$. Moreover, the Nielsen number is the absolute value of the Lefschetz number for any continuous map on a given orientable GHW; this is an analogue of a theorem of Anosov for continuous maps on nilmanifolds (cf. [2]). The number of GHW manifolds grows exponentially with the dimension ([7]). Recently, K.Dekimpe and N.Petrosyan [3] determined the homology of some oriented GHW in low dimension. They proved that in dimension 5 two oriented GHW groups have the same homology and in dimension 7 there are 4 classes of such manifolds with different homology.

In this note we establish a connection between the GHW groups and the theory of quadratic forms over $\mathbb{Z}_2$. Based on it, we introduce a new invariant that may be useful in the classification or recognition of Hantzsche-Wendt groups. We consider the quotient by subgroups of index two of the maximal abelian subgroup $\mathbb{Z}^n$. Let $A \subset \mathbb{Z}^n$ be a subgroup, such that $\mid \mathbb{Z}^n : A \mid = 2$. We shall prove (see Lemma 2) that $A \subset \Gamma$ is a normal subgroup and $F = \Gamma/A$ is a central extension of $\mathbb{Z}_2$ by $(\mathbb{Z}_2)^{n-1}$. Then, to each such extension we shall associate (Theorem 1) a quadratic form Q_A^Γ (over $\mathbb{Z}_2$) together with an alternating, bilinear, symmetric form B_A^Γ. It is not difficult to present the Gram-Matrix of B_A^Γ. In the last section, we shall give examples, and discuss some relations between the properties of B_A^Γ and the group F. For some special subgroups $D \subset \mathbb{Z}^n$, we shall present these relations with all details (see Theorem 4).

Roughly speaking, our method is to try to use this central extension to recognize the GHW groups. Computer calculations show that this method works at least in dimension 5 and 7.

In a special case this method was used in [4, Prop. 3.5,p.184] and [12, p.371-378].

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2 Definition of quadratic form

In this section Γ will denote a generalized Hantzsche-Wendt group. The above sequence $(*)$ defines, by conjugation, a holonomy representation

$$h_\Gamma : (\mathbb{Z}_2)^{n-1} \to GL(n, \mathbb{Z}).$$

From [11] it is known that the image $h_\Gamma((\mathbb{Z}_2)^{n-1})$ in $GL(n, \mathbb{Z})$ is a group of diagonal matrices with diagonal entries $\pm$.

We have,

Lemma 1 *The free abelian group $\mathbb{Z}^n$ has precisely $2^n - 1$ subgroups of index* 2.

Proof: We have a short exact sequence of abelian groups

$$0 \to 2(\mathbb{Z}^n) \to \mathbb{Z}^n \to (\mathbb{Z}_2)^n \to 0.$$

There are $2^n - 1$ different, nontrivial elements of $(\mathbb{Z}_2)^n$. Any such element defines a subgroup of index two.

$\square$

Lemma 2 *Let $A \subset \mathbb{Z}^n$ be a subgroup of index two of the maximal abelian subgroup of Γ. Then A is a normal subgroup of Γ, and F is the central extension of $\mathbb{Z}_2 = \mathbb{Z}^n/A$ by $\mathbb{Z}_2^{n-1}$.*

Proof: Assume $\Gamma \subset E(n) = O(n) \ltimes \mathbb{R}^n$. For any $a \in A$ and $(G, g) \in \Gamma$, where $G \in O(n)$ and $g \in \mathbb{R}^n$, we have

$$(G, g)(I, a)(G, g)^{-1} = (G, G(a) + g)(G^{-1}, -G^{-1}(g)) =$$

$$= (I, -g + G(a) + g) = (I, G(a)).$$

By definition and from Lemma 1, we have $2(\mathbb{Z}^n) \subset A$. Moreover, since G is the diagonal matrix, then $(G - I)x \in 2(\mathbb{Z}^n)$, for any $x \in \mathbb{Z}^n$. Hence $G(a) \in A$ and the Lemma is proved.

$\square$

Theorem 1 *To any generalized Hantzsche-Wendt group* Γ *of dimension* $n \geq 3$ *and any subgroup* A *of index two of the maximal abelian subgroup* $\mathbb{Z}^n$*, we can associate a quadratic function*

$$Q^\Gamma_A : (\mathbb{Z}_2)^{n-1} \to \mathbb{Z}_2$$

and the associated alternating, bilinear quadratic form

$$B^\Gamma_A : (\mathbb{Z}_2)^{n-1} \times (\mathbb{Z}_2)^{n-1} \to \mathbb{Z}_2.$$

Moreover, if $n > 3$*, and* Γ *is oriented, then* $B^\Gamma_A \not\equiv 0$ *for any* A*.*

Proof: We shall define the quadratic form Q^Γ_A and its associated alternating, bilinear quadratic form B^Γ_A with the help of our group $\Gamma \subset E(n) = O(n) \ltimes \mathbb{R}^n$. Let $(X, x), (Y, y) \in \Gamma$, mapped by p to $X, Y \in V := (\mathbb{Z}_2)^{n-1}$, where $X, Y \in O(n)$ and $x, y \in \mathbb{R}^n$. We have

$$B^\Gamma_A(X, Y) = (X, x)(Y, y)(X, x)^{-1}(Y, y)^{-1} = (X - I)y - (Y - I)x \in \mathbb{Z}_2 = \mathbb{Z}^n/A$$

and

$$Q^\Gamma_A(X) = (X, x)^2 = (X + I)x \in \mathbb{Z}_2 = \mathbb{Z}^n/A.$$

It is easy to see that B^Γ_A and Q^Γ_A are well defined. It means that they do not depend on the choice of an element $(X, x) \in \Gamma$. The bilinear form B^Γ_A is alternating. In fact,

$$B^\Gamma_A(X, X) = (X - I)(x + a) - (X - I)x = (X - I)a \in A,$$

where $(X, x), (X, x+a) \in p^{-1}(X)$ and $a \in \mathbb{Z}^n$. We still have to prove the last assertion. But this follows because the commutator subgroup of Γ is equal to the translation subgroup, cf. [10, Theorem 3.1].

$\square$

Remark 1 A. We shall see in the next section that for different translation subgroups A_1, A_2 it can happen that $B^\Gamma_{A_1} = B^\Gamma_{A_2}$. Moreover, for different Γ_1 and Γ_2, $B^{\Gamma_1}_A$ can be equal to $B^{\Gamma_2}_A$.
B. The function Q^Γ_A and the form B^Γ_A correspond to the central short exact sequence of finite groups (cf. Corollary 1)

$$0 \to \mathbb{Z}_2 \to F \to (\mathbb{Z}_2)^{n-1} \to 0.$$

Let us recall the basic facts about the quadratic forms over the field $\mathbb{Z}_2$. We shall follow [1] and [5]. Let V be a finite dimensional vector space over $\mathbb{Z}_2$ and $Q : V \to \mathbb{Z}_2$ be a function. We shall call Q a quadratic form if $Q(0) = 0$ and

$$(**) \quad B(x, y) = Q(x + y) + Q(x) + Q(y)$$

is bilinear, where $x, y \in V$.

Example 1 *Let V be the 2-dimensional vector space with basis a, b and $B(a, b) = 1, B(a, a) = B(b, b) = 0$. There are two quadratic forms $Q_1 : V \to \mathbb{Z}_2, Q_2 : V \to \mathbb{Z}_2$ compatible with B, with $Q_1(a) = Q_1(b) = 1$, and $Q_2(a) = Q_2(b) = 0$. Note that $Q_1(a + b) = Q_2(a + b)$.*

For a bilinear form B, satisfying $(**)$ we define the radical

$$R = \{x \in V \mid B(x, y) = 0 \ \forall y \in V\},$$

and its subspace $R_1 = \{x \in R \mid Q(x) = 0\}$. Observe that B induces a non-degenerate alternating bilinear form $\bar{B}$ on V/R, hence $\dim_{\mathbb{Z}_2} V/R = 2m$ (cf. [6, Theorem 8.1, p.586]). Let us assume that the radical $R = 0$. In that case we may find a basis $\{a_i, b_i \mid 1 \leq i \leq m\}$ for V such that $B(a_i, b_j) = \delta_{ij}, B(a_i, a_j) = B(b_i, b_j) = 0$. Then, with respect to the symplectic basis $\{a_i, b_i\}$, we define for Q the Arf invariant (see [1, p. 54]):

$$\mathbf{c}(Q) = \Sigma_{i=1}^m Q(a_i)Q(b_i) \in \mathbb{Z}_2.$$

It can be proved that $\mathbf{c}$ is independent of the choice of basis, [1]. If $R \neq 0$, let us assume that $R = R_1$. Then, it easy to see that Q defines Q' on V/R and the radical of Q' is zero. In that case we define $\mathbf{c}(Q) = \mathbf{c}(Q')$.

Let $p(Q) =$ number of elements of $x \in V$ such that $Q(x) = 1$ and let $n(Q) =$ number of $x \in V$ such that $Q(x) = 0$. Obviously $p(Q) + n(Q) = \mid V \mid$. Put $r(Q) = p(Q) - n(Q)$. We have the following equivalent results.

Theorem 2 ([1, Theorem III.1.14]) *Let $Q : V \to \mathbb{Z}_2$ be a quadratic form over $\mathbb{Z}_2$, and let R denote the radical of the associated bilinear form. Then the Arf invariant $\mathbf{c}(Q)$ is defined if and only if $R = R_1$. In general, if $R = R_1, Q$ is determined up to isomorphism by rank V, rank R and $\mathbf{c}(Q)$, while if $R \neq R_1$, then Q is determined by rank V and rank R. Note that in the latter case $r(Q) = 0$.*

$\square$

Theorem 3 ([5, Theorem 2.15]) *Any quadratic form $Q : V \to \mathbb{Z}_2$ is equivalent to $Q_1^r \oplus Q_2^s \oplus E^t \oplus G^h$, where $s = 0$ or 1, E is the one-dimensional form and G is the zero form. Moreover t can be chosen to be 0 or 1, and if $t = 1$ then s can be chosen to be 0.*

$\square$

Suppose that G_1 and G_2 are 2-groups and that G_i has unique normal subgroup $\langle z_i \rangle$ of order 2 for $i = 1, 2$. The central product $G_1 * G_2$ is defined by

$$G_1 * G_2 = (G_1 \times G_2)/\langle (z_1, z_2) \rangle.$$

It is not difficult to check that $D_8 * D_8 \simeq Q_8 * Q_8$ and that $D_8 * Z_4 \simeq Q_8 * \mathbb{Z}_4$. Here D_8 is the dihedral group of order 8 and Q_8 is the quaternion group of order 8. Finally, we have the following corollary of Theorems 1 and 3. Recall, that $F = \Gamma/A$.

Corollary 1 *Let Γ be a generalized Hantzsche-Wendt group of dimension $n \geq 3$ and A be a subgroup of index two of the maximal abelian subgroup $\mathbb{Z}^n$. Moreover, let $Q_A^\Gamma \sim Q_1^r \oplus Q_2^s \oplus E^t \oplus G^h$. Then F is isomorphic to one of the following: $\underbrace{D_8 * \ldots * D_8}_{r} \times \mathbb{Z}_2^h$, $\underbrace{D_8 * \ldots * D_8}_{r} * \mathbb{Z}_4 \times \mathbb{Z}_2^h$ or $\underbrace{D_8 * \ldots * D_8}_{r} * Q_8 \times \mathbb{Z}_2^h$.*

$\square$

3 Examples

Using the previous results, we shall describe the groups F in low dimensions and for the special family of the GHW groups.

Assume that B_A^Γ is non-degenerate or (equivalently) that the determinant of the Gram-Matrix of B_A^Γ is 1. Then we shall call F an extraspecial group, (cf. [12]) and its isomorphism class depends on the value of the Arf invariant.

Example 2 *Let $D \subset \mathbb{Z}^n$ be generated by the following elements:*

$$2e_1, e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n,$$

where $\{e_1,\dots,e_n\}$ *is the standard basis of* $\mathbb{Z}^n$*. It is easy to see that* D *is a subgroup of index two in* $\mathbb{Z}^n$*, see [12, p. 387]. For* $n \geq 2$*, let* Γ_n *be the subgroup of* $E(n)$ *generated by the set* $\{(B_i, s(B_i)) \mid 1 \leq i \leq n-1\}$*. Here* B_i *is the diagonal matrix*

$$B_i = \, diag(-1,\dots,-1,\underbrace{1}_{i},-1,\dots,-1)$$

and

$$s(B_i) = e_i/2 + e_{i+1}/2 \text{ for } 1 \leq i \leq n-1.$$

These groups are Hantzsche-Wendt for odd n*, [11]. From the definition (cf. the proof of Theorem 1), we have*

$$B_D^{\Gamma_n}(B_i, Bj) = \begin{cases} 0 & \text{if } i = j \\ 1 & \text{if } i = j+1 \\ 0 & \text{if } i \geq j+2 \end{cases}$$

and a matrix

$$X = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & 0\dots & 0 \\ \dots & \dots & \dots & \dots & \\ 0 & \dots 0 & 1 & 0 & 1 \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix}.$$

Moreover, for any $B_i \in (Z_2)^{n-1}$*, where* $1 \leq i \leq n-1$*,*

$$Q_D^{\Gamma_n}(B_i) = (B_i + I)(e_i/2 + e_{i+1}/2) = e_i \notin D.$$

From the above, we have $n = 2k+1$ *for some* $k \in \mathbb{N}$*. In order to calculate the Arf invariant* $\mathbf{c}$ *of* $Q_D^{\Gamma_n}$ *we have to transform* X *into a symplectic matrix. Let us introduce a new basis* $f_1, f_2, \dots, f_k, f_{k+1}, \dots, f_{2k}$ *:*

$$f_i = B_{2i-1}, 1 \leq i \leq k$$

and

$$f_{k+i} = B_{2i} + B_{2i+2} + B_{2i+4} + \dots + B_{2k}, 1 \leq i \leq k.$$

It is easy to see that the matrix X *with respect to the new basis is a symplectic one. From the previous section, we have*

$$\mathbf{c}(Q_D^{\Gamma_n}) = \Sigma_{i=1}^k Q_D^{\Gamma_n}(f_i) Q_D^{\Gamma_n}(f_{k+i}) \in \mathbb{Z}_2 = \mathbb{Z}^n/D.$$

Summing up, we obtain the following result:

Theorem 4 *For any odd number $n = 2k+1$, the group Γ_n/D is an extraspecial group of order 2^{2k+1}. In addition:*
If $k = 2l-1$ or $2l$ and if l is odd, then $\Gamma_n/D = \underbrace{D_8 * D_8 * \ldots D_8}_{k-1} * Q_8$.
If $k = 2l-1$ or $2l$ and if l is even, then $\Gamma_n/D = \underbrace{D_8 * D_8 * \ldots D_8}_{k}$.
If $k = 1$, then $\Gamma_3/D = Q_8$.

Proof: The bilinear form $B_A^{\Gamma_n}$ above is non-degenerate. From the above, $Q_D^{\Gamma_n}(f_{2k}) = Q_D^{\Gamma_n}(f_j) = 1$ for $j = 1, 2, \ldots k$ and $Q_D^{\Gamma_n}(f_{2k-1}) = 0$. We have

$$\mathbf{c}(Q_D^{\Gamma_n}) = \Sigma_{i=1}^k Q_D^{\Gamma_n}(f_i) Q_D^{\Gamma_n}(f_{k+i}) =$$

$$= \Sigma_{i=1}^k Q_D^{\Gamma_n}(f_{k+i}),$$

where

$$Q_D^{\Gamma_n}(f_{k+i}) = \begin{cases} 1 & \text{if } i \equiv k \mod 2 \\ 0 & \text{if } i \equiv (k-1) \mod 2. \end{cases}$$

Finally

$$\mathbf{c}(Q_D^{\Gamma_n}) = \begin{cases} 1 & \text{if } k = 2l-1, 2l \text{ for l odd} \\ 0 & \text{if } k = 2l-1, 2l \text{ for l even.} \end{cases}$$

□

Now we concentrate on generalized Hantzsche-Wendt groups Γ with degenerate bilinear forms B_A^Γ and a Gram-Matrix X. We have to find the radical R of B_A^Γ. In fact, $\dim_{\mathbb{Z}_2} R = \dim_{\mathbb{Z}_2} V - \mathrm{rk}(X)$. When $R = R_1$, we have $F = E \times \mathbb{Z}_2^k$, where $k = \dim_{\mathbb{Z}_2} R$ and E is an extraspecial group. From Theorems 2 and 3, we know that, if $R \neq R_1$, then $t = 1$ and $r(Q_A^\Gamma) = 0$.

Let us see what happens in dimension 3. From Theorem 4, we know that $\Gamma_3/D = Q_8$. It is easy to see, that there are seven possibilities for the subgroup $A \subset \mathbb{Z}^3$ of index two. For example, if

$$A_1 = \mathrm{gen}\{2e_1, e_2, e_3\}$$

we get the quadratic form Q_2 and hence $\Gamma_3/A_1 = D_8$. But for the subgroup

$$A_2 = \mathrm{gen}\{e_1 + e_2, 2e_2, e_3\}$$

the Gram-Matrix is zero. It is easy to see that $\Gamma_3/A_2 = \mathbb{Z}_4 \times \mathbb{Z}_2$ and $r(Q^{\Gamma_3}_{A_2}) = 0$. As a comment to the last part of Theorem 1, we mention here, that the abelianization of Γ_3 is equal to $\mathbb{Z}_4 \times \mathbb{Z}_4$, see [13, Corollary 3.5.10].

For the 12 GHW-groups of dimension 4 (cf. [9]), and our subgroup D of index two we have the following possibilities.

Γ/D	$\mathbb{Z}_4 \times (\mathbb{Z}_2)^2$	$Q_8 \times \mathbb{Z}_2$	$Q_8 * \mathbb{Z}_4$
$\mathrm{rk}(B^{\Gamma}_D)$	0	$2, R = R_1, \mathbf{c}(Q^{\Gamma}_D) = 0$	$2, R \neq R_1$
$\sharp$	1	7	4

.

The next example shows that our invariant distinguishes two oriented five-dimensional Hantzsche-Wendt groups.

Example 3 *There are only two oriented Hantzsche-Wendt groups of dimension five (cf. [11]), namely $G_1 = \Gamma_5$ and G_2 is the subgroup*

$$\{(B_1, (1/2, 0, 1/2, 0, 0)), (B_2, (0, 1/2, 0, 0, 0)),$$

$$(B_3, (0, 0, 1/2, 1/2, 0)), (B_4, (0, 0, 0, 1/2, 1/2))\},$$

of the isometry group $SO(5) \ltimes \mathbb{R}^5$. We know already the form $B^{\Gamma_5}_D = B^{G_1}_D$. In the second case, we have

$$B^{G_2}_D = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

$Q^{G_1}_D$ corresponds to the extraspecial 2-group $G_1/D = D_8 * Q_8$, see [12, p.388] and Theorem 4. Since the rank of the above matrix is 2, it is easy to see that $Q^{G_2}_D$ corresponds to the group $G_2/D = (\mathbb{Z}_2)^2 \times Q_8$.

There are 62 orientable, 7-dimensional GHW groups (cf. [8]). We have the following possibilities

Γ/D	E_1	E_2	$E_3 \times \mathbb{Z}_2$	$E_4 \times (\mathbb{Z}_2)^2$	$Q_8 \times (\mathbb{Z}_2)^4$
$\mathrm{rk}(B^{\Gamma}_D)$	$6, \mathbf{c}(Q^{\Gamma}_D) = 0$	$6, \mathbf{c}(Q^{\Gamma}_D) = 1$	$4, R \neq R_1$	$4, R = R_1$	$2, R = R_1$
$\sharp$	1	17	2	32	10

,

where E_1 and E_2 are extraspecial groups of order 128, $E_3 = Q_8 * Q_8 * \mathbb{Z}_4$ and E_4 is an extraspecial group of order 32.

Remark 2 *Only in two cases, and both for the group $Q_8 * Q_8 * \mathbb{Z}_4 \times \mathbb{Z}_2$, is the number t defined by Theorem 3 non - zero. In other words, in these cases $R \neq R_1$.*

Finally, computer calculation give another observation. Let Γ be any oriented GHW-group of dimension 7 and let $A_1, \ldots, A_{127}$ be the subgroups of index 2 in the maximal abelian subgroup. By the Theorem 2 it is easy to see, that there are only 8 possibilties for the groups Γ/A_i: $D_8 \times (\mathbb{Z}_2)^4, Q_8 \times (\mathbb{Z}_2)^4, D_8 * \mathbb{Z}_4 \times (\mathbb{Z}_2)^3, D_8 * D_8 \times (\mathbb{Z}_2)^2, D_8 * Q_8 \times (\mathbb{Z}_2)^2, Q_8 * Q_8 * \mathbb{Z}_4 \times \mathbb{Z}_2, D_8 * D_8 * D_8$ and $D_8 * D_8 * Q_8$.

Remark 3 *All 62 sequences $(\Gamma/A_i))_{i=1,\ldots,127}$ are different.*

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