Irreducible holomorphic symplectic manifolds

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K3 surface

Definition

A *K3 surface* X is a compact complex surface with trivial canonical bundle and $H^1(X, \mathcal{O}_X) = 0$.

Properties: 1. K3 surface is actually Kähler (Y. T. Siu 1983);

2. X is simply-connected;

3. Since canonical bundle is trivial then X has non-degenerate holomorphic symplectic structure;

4. Any two complex analytic K3 surfaces are diffeomorphic as smooth 4-manifolds (K. Kodaira 1963).

A. Beauville (1983) - F. Bogomolov (1974)

Theorem 2.1

Let X be a compact Kähler manifold with $c_1 = 0$ (equivalently X is a Ricci-flat) Then, there exists a finite covering $\pi : \widehat{X} \to X$ that splits as a product of complex tori, irreducible Calabi-Yau manifolds and irreducible symplectic manifolds.

Remark 2.2

1. (G. De Rham 1952) If a Riemannian manifold (X, g) is complete, simply connected and if its holonomy representation is reducible, then (X, g) is a Riemannian product.

3. (J. Cheeger, D. Gromoll 1971) X- compact connected with non-negative Ricci curvature. Then the universal covering space $\tilde{X}$ splits isometrically as $\tilde{X} = E \times N$, where E is a Euclidean space and N is closed and simply connected with Ricci $\geqslant 0$.

The Calabi conjecture 1957

Theorem 2.3

(S. T. Yau - 1977) Let M be a compact Kähler manifold with vanishing first Chern class c_1. Then there exists a Ricci-flat Kähler metric on M. Every such metric is uniquely determined by the cohomology class of its Kähler form.

On the definition of irreducible (holomorphic) symplectic manifolds

Definition

A complex manifold X is called irreducible (holomorphic) symplectic manifold (IHS) if and only if is a smooth compact Kählerian simply connected manifold with $H^0(X, \Omega_X^2) \simeq \mathbb{C}\sigma$, for an everywhere nondegenerate 2-form σ.

Remark 2.4

1. From the definition X *has even dimension* $2n$ *and its canonical bundle* ω_X *is trivial. Moreover* $h^{2,0}(X) = h^{0,2}(X) = 1$ *and* $h^{1,0}(X) = h^{0,1}(X) = 0$.

2. IHS = Irreducible compact hyper-Kähler manifolds

Main Result

Example

In complex dimension 2, the only examples are K^3 surfaces.

$(X$,g) is irreducible if its holonomy representation is irreducible $(X \neq X_1 \times X_2.)$

Theorem 2.5

(M. Schwald, 2022) Let X be a compact Kähler manifold such that $H^0(X, \Omega_X^2) \cong \mathbb{C}$ is generated by a holomorphic symplectic form. Then $h^1(X, \mathcal{O}_X) = 0$ if and only if X is simply connected and therefore an irreducible symplectic manifold.

An application of the decomposition theorem

Proposition 2.6

(M. Schwald 2018) Let (X, ω) *be a smooth symplectic manifold with* $h^1(X, \mathcal{O}_X) = 0$ *and* $h^0(X, \Omega_X^{[2]}) = 1$*. Then* X *is either simply connected or a smooth quotient of a complex torus* T *by a finite group* G *of a biholomorphic automorphisms.*

Remark 2.7

The above Proposition is connected with the Proposition A.1 of the article of D. Huybrechts and M. Nieper-Wisskirchen from 2011.

Complex torus quotient

By (complex) torus quotients we shall understand quotients $X := T/G$ where T is a complex torus and G a finite group acting by biholomorphisms on T.

Definition

Let ρ be a complex representation of the finite group G and χ_ρ its charakter. When ρ is irreducible, it can be of real, complex, or quaternionc type, depending on its Frobenius-Schur indicator $l(\rho) := \frac{1}{|G|}\Sigma_{g \in G}\chi_\rho(g^2)$ being 1, 0 or -1 respectively.

Lemma 3.1

Let $X = T/G$ be an n-dimensional torus quotient with induced analytic representation L. Then $H^0(X, \Omega_X^{[1]})$ is isomorphic to the space of $v \in \mathbb{C}^n$ fixed under $L(g)$ for all $g \in G$. Thus $h^0(X, \Omega_X^{[1]}) = (\chi_L|\chi_{\mathbb{I}_G}) = \frac{1}{|G|}\Sigma_{g \in G} trL(g)$ counts the multiplicity of the trivial representation $\mathbb{I}_G$ as a component in L.

Lemma 4.1

(Main Lemma) Let $X = T/G$ *be an n-dimensional torus quotient with induced analytic representation* L. *Then* $H^0(X, \Omega_X^{[2]})$ *is spanned by a symplectic form on* X *if and only if* L *is either an irreducible representation of quaternionic type or the direct sum of two complex conjugate irreducible representation of real or complex type. In every case it follows that* $h^1(X, \mathcal{O}_X) = 0$ *except when* X *is a 2-torus.*

Proof:

An irreducible representation L preserves a non-trivial alternating billinear form if and only if L is of quaternionic type; in this case this form is non-degenerate and unique up to scalar.
We suppose now that L is reducible representation $L = L_1 \oplus L_2$, where the $L_i : G \to GL_{\mathbb{C}}(V_i)$ are sub-representations of positive degrees and $\mathbb{C}^n = V_1 \oplus V_2$ is a decomposition invariant under $L(g)$ for all $g \in G$.

Then the decomposition $\bigwedge^2 \mathbb{C}^n \cong (\bigwedge^2 V_1) \oplus (V_1 \otimes V_2) \oplus (\bigwedge^2 V_2)$ is invariant under $\wedge^2 L(g)$ for all $g \in G$. When there is a G-invariant 2-form ω on X, its three components are also G-invariant. Hence $H^0(X, \Omega_X^2) = \mathbb{C}\omega$ implies that only one component is non-zero. As ω is symplectic it is non-degenerate, such that ω has to lie in the mixed part $V_1 \otimes V_2$. By the same argument applied to sub-representations, L_1, L_2 are seen to be irreducible. Clearly neither L_1 nor L_2 can be of quaternionic type, as then we had $h^0(X, \Omega_X^2) \geqslant 2$.

We have

$$1 = h^0(X, \Omega_X^{[2]}) = (\chi_{\wedge^2 L} | \chi_{\mathbb{I}_G})$$

$$= \frac{1}{2|G|} \Sigma_{g \in G} (\chi_{L_1}(g) + \chi_{L_2}(g))^2 - (\chi_{L_1}(g^2) + \chi_{L_2}(g^2))$$

$$= \frac{1}{2}(\chi_{L_1} | \chi_{\bar{L_1}}) + (\chi_{L_1} | \chi_{\bar{L_2}}) + \frac{1}{2}(\chi_{L_2} | \chi_{\bar{L_2}}) - \frac{1}{2}\iota(L_1) - \frac{1}{2}\iota(L_2))$$

$$= \begin{cases} (\chi_{L_1} | \chi_{\bar{L_2}}) & \text{if } L_1 \text{ and } L_2 \text{ are both real or both complex} \\ 0 & \text{else} \end{cases}$$

and conclude $L_1 = \bar{L_2}$.

By Lemma 3.1 we get $h^0(X, \Omega_X^{[1]}) = (\chi_L|\chi_{\mathbb{I}_G})$. This is zero unless L is reducible and L_1, L_2 are both trivial, which is precisely the case when T and X are both 2-tori.

Definition

A complex torus quotient $X = T/G$ is smooth if and only if every $g \in G$ acts fixed point free on T.

We call X a *smooth (complex) torus quotient.*

The real holonomy representation equals the restriction of scalars of the analytic representation L of G on $\mathbb{C}^n$.

Every complex representation $\rho : G \to GL(n, \mathbb{C})$ defines by restriction to $\mathbb{R}$ a real representation $\rho_{\mathbb{R}} : G \to GL(2n, \mathbb{R})$

$$\rho_{\mathbb{R}} = \begin{bmatrix} \mathcal{R}\rho & -Im\rho \\ Im\rho & \mathcal{R}\rho \end{bmatrix}.$$

We get $\mathbb{C} \otimes \rho_{\mathbb{R}} \simeq \rho \oplus \bar{\rho}$, hence two irreducible complex representations are $\mathbb{R}$-equivalent if and only if they are $\mathbb{C}$-equivalent up to a possible complex conjugation.

Definition

When $\mathbb{K}$ is a field, we call a representation ρ of G over $\mathbb{K}$ homogeneous if ρ is the direct sum of equivalent irreducible subrepresentation. When ρ is a complex representation, then its restriction of scalars $\rho_{\mathbb{R}}$ is over $\mathbb{R}$ homogeneous if and only if all irreducible complex subrepresentations of L are equivalent up to a possible complex conjugation.

Lemma 5.1

Let ρ be a rational representation of a finite group and $\mathbb{K} \supset \mathbb{Q}$ a field. When $\mathbb{K} \otimes \rho$ is homogeneous then ρ is homogeneous as well.

Theorem 5.2

Let $X = T/G$ be a smooth torus quotient with induced analytic representation L. If the restriction of scalars $L_{\mathbb{R}}$ is homogeneous then L is trivial and X is a complex torus.

Proof.

Follows from a paper of Hiss-Szczepański (1991) and Lutowski (2021). □

Corollary 5.3

Every smooth symplectic torus quotient with $h^0(X, \Omega_X^2) = 1$ is a complex 2-torus.

Proof.

Let $X = T/G$ be a symplectic torus quotient and $L : G \to GL(n, \mathbb{C})$ its analytic representation. From Lemma 4.1 L is either irreducible or the direct sum of two complex conjugate irreducible representations $L_i : G \to GL_{\mathbb{C}}(V_i), i = 1, 2$. In the second case, their restrictions of scalars $(L_i)_{\mathbb{R}}$ are equivalent as real representations, so $L_{\mathbb{R}} \cong (L_1)_{\mathbb{R}}^{\oplus^2}$. In particular $L_{\mathbb{R}}$ is homogeneous in both cases. By Theorem 5.2 it follows that X can only be smooth if it is a complex torus. For $n := \dim X$ this implies $1 = h^0(X, \Omega_X^{[2]}) = \frac{n(n-1)}{2}$ and thus $n = 2$. □

We are ready to prove the Theorem 2.5.

Let X be a compact Kähler manifold such that $H^0(X, \Omega_X^2) \cong \mathbb{C}$ is generated by a holomorphic symplectic form. Then $h^1(X, \mathcal{O}_X) = 0$ if and only if X is simply connected and therefore an irreducible symplectic manifold.

Proof.

If $h^1(X, \mathcal{O}_X) = 0$ then X is by Proposition 2.6 simply connected or a smooth torus quotient. In the latter case it would be a 2-torus by Corollary 5.3, for which $h^1(X, \mathcal{O}_X) = 2 \neq 0$. For the other direction, if X is simply connected, then $H^1(X, \mathbb{C}) = \pi_1(X)^{ab}$ is trivial and hence $h^1(X, \mathcal{O}_X) = 0$ by Hodge theory. □

Let $X = T/G$ be a smooth torus quotient and L its induced analytic representation. For each $g \in G$ to act fixed point free on T it is necessary that $L(g)$ has 1 as eigenvalue. Furthermore, finite groups G for which a smooth torus quotient $X = T/G$ with $h^1(X, \mathcal{O}_X) = 0$ exists are *primitive*, (H. Hiller-C.H. Sah-1986) It follows from Lemma 4.1 that G needed to be primitive if there was a non-trivial smooth torus quotient $X = T/G$ that is symplectic with $h^0(X, \Omega_X^2) = 1$.

Theorem 6.1

There are examples of irreducible representations L of each real, complex, and quaternionic type of primitive finite groups G such that all representing matrices have 1 as an eigenvalue. They give rise to examples of singular symplectic torus quotients $X = T/G$ with $h^1(X, \mathcal{O}_X) = 0$ and $h^0(X, \Omega_X^{[2]}) = 1$, but by Corollary 5.3 no set of translations $t(g)$ for $g \in G$ can be chosen to make the induced action of G on T free (again H. Hiller-C.H. Sah - 1986).

However, smooth symplectic quotient with $h^1(X, \mathcal{O}_X) = 0$ exist indeed. The smallest example comes from the direct sum of three non-equivalent irreducible representations of quaternionic type, hence $h^0(X, \Omega_X^{[2]}) = 3$ in that case.

Thank You.