Existence of spin structures on flat four manifilds

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Abstract

We prove that all but 3 of the 27 closed, orientable, flat, four dimensional manifolds have a spin structure.

Key words. Spin structure, flat manifold, Bieberbach group

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By flat manifold we understand a compact Riemannian manifold with sectional curvature equal to zero. Any such manifold M is an orbit space $\mathbb{R}^n/\Gamma$, where $\Gamma = \pi_1(M)$ is a Bieberbach group, i.e. a discrete, cocompact and torsion - free subgroup of the isometry group $E(n) = O(n) \ltimes \mathbb{R}^n$ of the n-dimensional euclidean space $\mathbb{R}^n$. Moreover, by Bieberbach theorems (cf. [2]) the translations subgroup $\Gamma \cap \mathbb{R}^n$ of Γ is a free abelian group of rank n of finite index. The finite group $\Gamma/(\Gamma \cap \mathbb{R}^n)$ is also the holonomy group of the manifold M. Let us recall that an oriented manifold M (a group Γ) has a spin structure if and only if there exists a homomorphism $\epsilon : \Gamma \to Spin(n)$ such that $\lambda_n \epsilon = h$. Here λ_n is the covering map from the group $Spin(n)$ to the group $SO(n)$ and h is the projection of Γ onto $SO(n)$ sometimes called a holonomy homomorphism. By $t_i, i = 1, 2, ..., n$ we shall denote the standard basis of $\mathbb{R}^n$.

In [12, Problem 15] is formulated the problem of classifying the finite groups that are the holonomy groups of flat oriented manifolds which admit a spin structure.

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Let G be a finite group. From the already known results (cf. [3], [6] and [7]) the following cases, related to the above problem can be distinguished.

1. Any oriented flat manifold M with holonomy group G has a spin structure (if G is a finite group of odd order (cf. [3, Proposition 1]), $G = \mathbb{Z}_2$ (cf. [6, Theorem 3.1,(3)], [7, Proposition 4.2]), or if G is a cyclic group of order $2^n, n \geq 3$ (cf. [6, Theorem 3.1, (3)]).

2. There are flat manifolds $M = \mathbb{R}^n/\Gamma$ and $M' = \mathbb{R}^n/\Gamma'$ with holonomy group G and the same holonomy homomorphism such that M has a spin structure but M' has none (it happens for groups of order four (cf. [6, Example 3.3], [7, Table 1, page 327] and Lemma 1)).

3. There exists a finite group G and a group representation $h : G \to GL(n, \mathbb{Z})$ such that any Bieberbach group with the above holonomy group and holonomy homomorphism has no a spin structure (it happens for $G = \mathbb{Z}_4$ (cf. [6, Theorem 3.1, (4)])).

Here $\mathbb{Z}_n$ denotes the cyclic group of order n and D_n denotes the dihedral group of order n.

It is well known (cf. [8, § 12]) that any oriented three manifold has a spin structure. In this paper we shall consider existence of spin structures on four dimensional flat oriented manifolds. We shall prove.

Theorem *All but 3 of the 27 four dimensional oriented flat manifolds have a spin structure. The holonomy groups of the manifolds which do not admit a spin structure are equal to* $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ *and* D_8.

Our methods are elementary and direct. Let Γ be a four dimensional oriented Bieberbach groups with maximal abelian subgroup $\mathbb{Z}^4$. We define a homomorphism $\epsilon : \mathbb{Z}^4 \to \mathbb{Z}_2$, which satisfies the conditions:

$$\epsilon(t_i) = \begin{cases} 1 & \text{if for any } \gamma \in \Gamma \text{ such that } (\gamma)^2 = t_i \ (\pm(\lambda_4^{-1}(h(\gamma)))^2 = 1 \\ -1 & \text{if for any } \gamma \in \Gamma \text{ such that } (\gamma)^2 = t_i \ (\pm(\lambda_4^{-1}(h(\gamma)))^2 = \text{-1}. \end{cases}$$

If ϵ exists, we have to extend it to a group homomorphism $\epsilon : \Gamma \to Spin(4)$, such that $\lambda_4 \epsilon = h$. We want to mention that an application of Proposition 2.1 of [6] about the existence of a spin structure on $M = \mathbb{R}^n/\Gamma$ being independent of the representation of Γ as a subgroup of $E(4)$ is crucial for our argument.

The above theorem was announced in 2000 by J. Ratcliffe and S. Tschantz, [10]. Some cases were considered in [11, Theorem 6]. One of our motivations was to give an explicit, written proof for all cases.

We would like to thank R.Lutowski for help in finding the isomorphism between the subgroup of the Bieberbach group with A_4 holonomy and Γ_5. The second author thanks Gerhard Hiss for discussions on the subject of this paper and is grateful to the Lehrstuhl D of RWTH Aachen for its hospitality during the work on this paper.
We would like to mention that most of the results were checked on GAP, [5].

1 Proof of the main result

From the Bieberbach theorem (cf.[2]) there are 27 four dimensional flat orientable manifolds (Bieberbach groups), up to affine equivalence (up to isomorphism) see [9].

It is not difficult to prove that the following Bieberbach groups of rank four have a spin structure: $\mathbb{Z}^4$ the fundamental group of the torus, whose holonomy group is trivial, the two groups with holonomy group $\mathbb{Z}_2$, [7, Proposition 4.2], the two groups with holonomy group $\mathbb{Z}_3$, the group with holonomy group isomorphic to $\mathbb{Z}_6$ and the three groups with holonomy group D_6 [3, Proposition 1]. Moreover, there are two Bieberbach groups with holonomy group $\mathbb{Z}_4$. In this case existence of the spin structure follows from [6, Theorem 3.1 (4) and Theorem 3.2].

Hence we still have to consider 16 Bieberbach groups of rank four: 9 with holonomy group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, 2 with holonomy group A_4, 4 with holonomy group D_8 and 1 with holonomy group D_{12}.

Lemma 1 *All but 2 of the 9 closed, oriented, flat four-manifolds with holonomy group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ have a spin structure. In five cases the number of spin structures is equal to 8, in one case 4 and in one case 16.*

Proof: We shall use notations from CARAT [9]. The first group Γ_1 from the family 22.1.1 is the product of a 3-dimensional group H and the integers. The group H is the fundamental group of the three dimensional Hantzsche-Wendt manifold, [12, Section 3] and admits four spin structures. Hence Γ_1 has eight spin structures. The second group from the same family is

$$\Gamma_2 = \text{gen}\{\gamma_A = ([-1,1,-1,1],(0,0,0,1/2)), \gamma_B = \\ ([1,-1,-1,1],(1/2,0,0,0)), \bar{t_1}, \bar{t_2}, \bar{t_3}, \bar{t_4}\} \subset E(4).$$

The bracket $[x_1,x_2,x_3,x_4]$ denotes a matrix $[a_{ij}]$, with $a_{ij} = 0$, for $i \neq j$ and $a_{ii} = x_i; i,j = 1,2,3,4$. Moreover $\bar{t_i} = (I, t_i)$, for $i = 1,2,3,4$. Let

$\pm s_M \in Spin(4)$ be mapped by λ_4 to a diagonal matrix $M \in SO(4) \cap GL(4,\mathbb{Z})$. By Lemma 2.3 of [6], we have that $(s_M)^2 = -1$, for $M \neq \pm I$. We have $(\gamma_A)^2 = \bar{t_4}$, $(\gamma_A\gamma_B)^2 = \bar{t_4}$, $(\gamma_B)^2 = \bar{t_1}$. Hence, we can define a family of sixteen maps $\epsilon : \Gamma_2 \to Spin(4)$ such that, $\epsilon((\gamma_A)) = \pm e_1e_3, \epsilon((\gamma_B)) = \pm e_2e_3, \epsilon(\bar{t_2}) = \pm 1, \epsilon(\bar{t_3}) = \pm 1, \epsilon(\bar{t_1}) = \epsilon(\bar{t_4}) = -1$. We can prove, by immediate calculations, that each of them is a homomorphism, which establishes a spin structure, where $e_ie_j \in Spin(4)$ are the standard generators for, $i,j = 1,...,4$, (cf. [4]).

The similar methods we apply to the group

$$\Gamma_3 = \text{gen}\{\gamma_C = ([-1,1,-1,1],(0,1/2,0,1/2)), \gamma_D = \\ ([1,-1,-1,1],(1/2,0,0,0)), \bar{t_1}, \bar{t_2}, \bar{t_3}, \bar{t_4}\}$$

and the group

$$\Gamma_4 = \text{gen}\{\gamma_E = ([-1,1,-1,1],(0,1/2,1/2,1/2)), \gamma_F = \\ ([1,-1,-1,1],(1/2,0,0,0)), \bar{t_1}, \bar{t_2}, \bar{t_3}, \bar{t_4}\}$$

from the family denoted 22.1.1. In the first case we have eight homomorphisms $\epsilon : \Gamma_3 \to Spin(4)$ such that, $\epsilon(\gamma_C) = \pm e_1e_3, \epsilon(\gamma_D) = \pm e_2e_3, \epsilon(\bar{t_3}) = \pm 1, \epsilon(\bar{t_1}) = \epsilon(\bar{t_4}) = -1, \epsilon(\bar{t_2}) = 1$ and $\lambda_4\epsilon = h$. In the second case we have the following eight spin structures: $\epsilon(\gamma_E) = \pm e_1e_3, \epsilon(\gamma_F) = \pm e_2e_3$ and $\epsilon(\bar{t_1}) = -1, \epsilon(\bar{t_2}) = \epsilon(\bar{t_3}) = -\epsilon(\bar{t_4}) = \pm 1$.

We still have to consider five groups. All of them do not have diagonal linear parts. We shall apply [6, Proposition 2.1]. Let us start from the group (22.1.12)

$$\Gamma_5 = \text{gen}\{\gamma_1, \gamma_2, \bar{t_1}, \bar{t_2}, \bar{t_3}, \bar{t_4}\}$$

where

$$\gamma_1 = \left(\begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 0 & -1 & -1 \\ 0 & 0 & -1 & 0 \\ -1 & -1 & -1 & 0 \end{bmatrix}, (-1/2,1/2,0,0) \right) \text{ and}$$

$$\gamma_2 = \left(\begin{bmatrix} -1 & 0 & 0 & -2 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, (-1/2,0,0,1/2) \right).$$

We shall conjugate the group Γ_5 inside the group $GL(4,\mathbb{R}) \ltimes \mathbb{R}^4$ by an element $(A_5, 0)$, where

$$A_5 = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

The Bieberbach group

$$\Gamma_5' = (A_5, 0)\Gamma_5(A_5^{-1}, 0) = \text{gen}\{A_5\gamma_1 A_5^{-1} = \gamma_1', A_5\gamma_2 A_5^{-1} = \gamma_2', f_1 = (I, A_5 t_1), f_2 = (I, A_5 t_2), f_3 = (I, A_5 t_3), f_4 = (I, A_5 t_4)\}$$

is a subgroup of $E(4)$ and has diagonal linear matrices. To finish the proof in that case it is enough to define $\epsilon : \Gamma_5' \to Spin(4)$ by formulas $\epsilon(\gamma_1') = \pm e_2 e_3, \epsilon(\gamma_2') = \pm e_1 e_2, \epsilon(f_2) = \epsilon(f_3) = \epsilon(f_4) = -\epsilon(f_1) = \pm 1$. Hence we have eight different spin structures on Γ_5.

The next group (22.1.2)

$$\Gamma_6 = \text{gen}\{\gamma_1, \gamma_2, \bar{t_1}, \bar{t_2}, \bar{t_3}, \bar{t_4}\}$$

where

$$\gamma_1 = \left(\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, (1/2, 1/2, 0, 0) \right) \text{ and}$$

$$\gamma_2 = \left(\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, (0, 1/2, 0, 0) \right).$$

As above, after conjugation by an element $(A_6, 0)$ where

$$A_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

we get the Bieberbach group

$$\Gamma_6' = (A_6, 0)\Gamma_6(A_6^{-1}, 0) = \text{gen}\{A_6\gamma_1 A_6^{-1} = \gamma_1', A_6\gamma_2 A_6^{-1} = \gamma_2', g_1 = (I, A_6 t_1), g_2 = (I, A_6 t_2), g_3 = (I, A_6 t_3), g_4 = (I, A_6 t_4)\}$$

with diagonal rotation matrices. We put $\epsilon(\gamma_1') = \pm e_1e_4, \epsilon(\gamma_2') = \pm e_1e_3$ and $\epsilon(g_1) = \epsilon(g_2) = -1, \epsilon(g_3) = \epsilon(g_4) = \pm 1$. It gives eight different spin structures.

Next we shall prove that the group (22.1.4 or 05/01/04/006 in [1])

$$\Gamma_7 = \text{gen}\{\gamma_1, \gamma_2, \bar{t_1}, \bar{t_2}, \bar{t_3}, \bar{t_4}\},$$

where

$$\gamma_1 = \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, (1/2, 1/2, 0, 0) \right) \text{ and}$$

$$\gamma_2 = \left(\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, (0, 0, -1/2, 1/2) \right) \text{ has no spin structure.}$$

After conjugation by an element $(A_6, 0)$ we get the Bieberbach group

$$\Gamma_7' = (A_6, 0)\Gamma_7(A_6^{-1}, 0) = \text{gen}\{A_6\gamma_1A_6^{-1} = \gamma_1', A_6\gamma_2A_6^{-1} = \gamma_2', g_1, g_2, g_3, g_4\}$$

where, $\gamma_1' = ([1, -1, 1 - 1], (1/2, 1/2, 0, 0)), \gamma_2' = ([-1, -1, 1, 1], (0, 0, 1, 0))$ and $\gamma_1'\gamma_2' = ([-1, 1, 1, -1], (1/2, 1/2, 1, 0))$.

Assume that there exists a homomorphism $\epsilon : \Gamma_7' \to Spin(4)$ of groups, such that $\lambda_4\epsilon = h$. We have $\epsilon(g_1) = -1, \epsilon(g_4 - g_3) = -1, \epsilon(g_2 + g_4 - g_3) = -1$. Hence $\epsilon(\gamma_1'g_4\gamma_1') = \epsilon(g_1 - g_3) = \epsilon((\gamma_1')^2)\epsilon(g_3) = -\epsilon((\gamma_1')^2)\epsilon(g_4) = -\epsilon(\gamma_1')\epsilon(g_4)\epsilon(\gamma_1') = -\epsilon(\gamma_1'g_4\gamma_1')$ and the spin stucture does not exist.

Let us consider a group (22.1.5)

$$\Gamma_8 = \text{gen}\{\gamma_1, \gamma_2, \bar{t_1}, \bar{t_2}, \bar{t_3}, \bar{t_4}\}$$

where

$$\gamma_1 = \left(\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}, (1/2, -1/2, 0, 0) \right) \text{ and}$$

$$\gamma_2 = \left(\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, (0, 0, 1/2, 0) \right).$$

After conjugation by an element $(A_8, 0)$ where

$$A_8 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

we get the Bieberbach group

$$\Gamma_8' = (A_8, 0)\Gamma_8(A_8^{-1}, 0) = \text{gen}\{A_8\gamma_1 A_8^{-1} = \gamma_1', A_8\gamma_2 A_8^{-1} = \gamma_2', h_1 = (I, A_8 t_1), h_2 = (I, A_8 t_2), h_3 = (I, A_8 t_3), h_4 = (I, A_8 t_4)\}$$

where

$$\gamma_1' = ([-1, 1, -1, 1], (1/2, -1/2, 0, -1/2)),$$
$$\gamma_2' = ([-1, 1, 1, -1], (0, 1/2, 1/2, 0))$$

and $\gamma_1'\gamma_2' = ([1, 1, -1, -1], (1/2, 0, -1/2, -1/2))$.

We define $\epsilon(\gamma_1^{'}) = \pm e_1 e_3, \epsilon(\gamma_2^{'}) = \pm e_1 e_4$ and $\epsilon(h_1) = \epsilon(h_2) = \epsilon(h_3) = \epsilon(h_4) = -1$. By immediate calculations we check that ϵ defines four spin structures on group Γ_8.

The second group, which has no spin structure, is the group (22.1.8 or 05/01/06/006 in [1])

$$\Gamma_9 = \text{gen}\{\gamma_1, \gamma_2, \bar{t_1}, \bar{t_2}, \bar{t_3}, \bar{t_4}\}$$

where

$$\gamma_1 = \left(\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & -1 & 0 \end{bmatrix}, (1/2, -1/2, 1/2, 0) \right) \text{ and}$$

$$\gamma_2 = \left(\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, (0, -1/2, 0, 1/2) \right).$$

After conjugation by an element $(A_9, 0)$ where

$$A_9 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix},$$

we get the Bieberbach group

$$\Gamma_9' = (A_9,0)\Gamma_9(A_9^{-1},0) = \text{gen}\{A_9\gamma_1 A_9^{-1} = \gamma_1', A_9\gamma_2 A_9^{-1}\} = \gamma_2', k_1 = (I, A_9 t_1), k_2 = (I, A_9 t_2), k_3 = (I, A_9 t_3), k_4 = (I, A_9 t_4)\}$$

where

$$\gamma_1' = ([-1,1,-1,1],(1/2,0,0,-1)), \gamma_2' = ([-1,1,1,-1],(0,-1,0,0))$$

and $\gamma_1'\gamma_2' = ([1,1,-1,-1],(1/2,-1,0,-1))$.

From definition $\epsilon(k_3-k_2) = -1, \epsilon(k_4-k_2) = -1$ and $\epsilon(k_1-k_2+k_4) = -1$. Hence $\epsilon(k_1) = 1, \epsilon(k_2) = 1, \epsilon(k_3) = -1$ and $\epsilon(k_4) = -1$ or $\epsilon(k_1) = 1, \epsilon(k_2) = -1, \epsilon(k_3) = 1$ and $\epsilon(k_4) = 1$. In any case we have $\epsilon(\gamma_2' k_2 \gamma_2') = \epsilon(k_3+k_4-k_2) = -\epsilon(\gamma_2')\epsilon(k_2)\epsilon(\gamma_2')$ and it proves our statement.

For the second proof of the last part of Lemma we shall use [4, Proposition on page 40], which says that the spin structures are classified by the first cohomology group with coeficients in $\mathbb{Z}_2$. With support of GAP [5] we can prove that $\mid H^1(\Gamma_i,\mathbb{Z}_2) \mid= 8$ for $i = 1,3,4,5,6$. Moreover $\mid H^1(\Gamma_2,\mathbb{Z}_2) \mid= 16$ and $\mid H^1(\Gamma_8,\mathbb{Z}_2) \mid= 4$.

$\square$

Next we have.

Lemma 2 *All but 1 of the 4 closed, oriented, flat four-manifolds with the holonomy group D_8 have a spin stucture. In two cases the number of spin structures is equal to 4 and in one case 8.*

Proof: We shall use the list of groups, from [9]. Let us introduce the matrices

$$D_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \text{ and } D_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Then the three Bieberbach groups from the family (29.1.1) of [9] have the following sets of generators in the group $E(4)$,

$$\Delta_1 = \text{gen}\{\alpha_1 = (D_1,(0,0,1/2,1/4)), \beta_2 = (D_2,(0,0,1/2,0)), \bar{t}_1,\bar{t}_2,\bar{t}_3,\bar{t}_4\},$$
$$\Delta_2 = \text{gen}\{\alpha_2 = (D_1,(1/2,0,0,1/4)), \beta_2, \bar{t}_1,\bar{t}_2,\bar{t}_3,\bar{t}_4\},$$
$$\Delta_3 = \text{gen}\{\alpha_3 = (D_1,(1/2,0,1/2,1/4)), \beta_2, \bar{t}_1,\bar{t}_2,\bar{t}_3,\bar{t}_4\}.$$

We define spin structures

$$\epsilon(\alpha_i) = \pm e_2 e_4, (i = 1, 2, 3),$$

$$\epsilon(\beta_2) = \pm\frac{1}{\sqrt{2}}(e_1 + e_2)e_4.$$

Moreover, for the group Δ_1 :

$$\epsilon(\bar{t}_1) = \epsilon(\bar{t}_2) = \pm 1$$

$$\epsilon(\bar{t}_3) = \epsilon(\bar{t}_4) = -1,$$

for the group Δ_2 :

$$\epsilon(\bar{t}_1) = \epsilon(\bar{t}_2) = \epsilon(\bar{t}_3) = \epsilon(\bar{t}_4) = -1$$

and for the group Δ_3 :

$$\epsilon(\bar{t}_1) = \epsilon(\bar{t}_2) = 1, \epsilon(\bar{t}_3) = \epsilon(\bar{t}_4) = -1.$$

The last Bieberbach group ((29.1.2) in [9] or 13/04/04/011 in [1]) of rank 4, with holonomy group D_8, has the set of generators.

$$\Delta_4 = \text{gen}\{(F_1, (0, -1/2, 0, 1/2)), (F_2, (1/4, 0, 1/2, 0)), \bar{t}_1, \bar{t}_2, \bar{t}_3, \bar{t}_4\},$$

where $F_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $F_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$,

Let us conjugate the group Δ_4 by the element $(A, 0) \in GL(4, \mathbb{R}) \ltimes \mathbb{R}^4$, where

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

We get $AF_1A^{-1} = D_1$ and $AF_2A^{-1} = D_2$. Hence

$$\Delta_4' = \{\gamma_1 = (D_1, (0, 0, -1, 0)), \gamma_2 = (D_2, (1/2, -1/2, 1/2, 1/4)), l_1 = (I, At_1) = (I, (0, 0, 0, 1)), l_2 = (I, At_2) = (I, (1, 1, 1, 0)), l_3 = (I, At_3) = (I, (1, -1, 1, 0)), l_4 = (I, At_4) = (I, (1, 1, -1, 0))\}.$$

We claim that this group has no spin structure. By contradiction, assume that $\epsilon : \Delta_4' \to Spin(4)$ defines a spin structure. Similar to the previous cases, we have:

$$\epsilon(\gamma_1) = \pm e_2 e_4, \epsilon(\gamma_2) = \pm \frac{1}{\sqrt{2}}(e_1 + e_2)e_4.$$

Hence

$$\epsilon(l_3) = \epsilon(\gamma_2^2) = -1,$$

$$\epsilon(l_3 + l_4) = \epsilon((\gamma_1 \gamma_2^2)^2) = -1 \Rightarrow \epsilon(l_4) = 1.$$

Moreover,

$$\epsilon(l_3 - l_2) = \epsilon((\gamma_2 \gamma_1 \gamma_2)^2) = -1 \Rightarrow \epsilon(l_2) = 1.$$

Finally

$$1 = \epsilon(l_4)\epsilon(l_2) = \epsilon(l_4 - l_2) = \epsilon(\gamma_1^2) = -1.$$

What is impossible.

□

For the Bieberbach group with holonomy group group D_{12} it is enough to observe that it has a subgroup isomorphic to the group Γ_2 from the Lemma 1. Hence it has a spin structure by Proposition 1 of [3]. Let us finally consider the 2 Bieberbach groups with holonomy group A_4. In [1] they are denoted by 24/01/02/004 and 24/01/04/004. Since the Sylow 2-subgroup of A_4 is equal to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, then it is enough to prove that the subgroups $\bar{\Gamma}, \bar{\Gamma}'$ of the above groups, with $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ holonomy, have a spin structure. In the first case it is obvious because $\bar{\Gamma} = \Gamma_1$. For the second group $\bar{\Gamma}'$ we can prove that it is isomorphic to Γ_5. (We thank R.Lutowski for help.) However, using calculations analogous to those in Lemma 1, we shall give a direct definition of the spin structures. In fact, from [1]

$$\bar{\Gamma}' = \text{gen}\{\gamma_1, \gamma_2, \bar{t_1}, \bar{t_2}, \bar{t_3}, \bar{t_4}\},$$

where

$$\gamma_1 = \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, (1/2, 1/2, 0, 1/2) \right) \text{ and}$$

$$\gamma_2 = \left(\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}, (1/2, 0, 1/2, 1/2) \right).$$

Then we can conjugate the group $\bar{\Gamma}'$ inside the group $GL(4, \mathbb{R}) \ltimes \mathbb{R}^4$ by the element $(X, 0)$, where

$$X = \begin{bmatrix} 2 & -1 & -1 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}.$$

The group

$$\bar{\Gamma}'' = (X, 0)\bar{\Gamma}'(X^{-1}, 0) = \text{gen}\{X\gamma_1 X^{-1} = \gamma_1', X\gamma_2 X^{-1} = \gamma_2', f_1 = (I, Xt_1), f_2 = (I, Xt_2), f_3 = (I, Xt_3), f_4 = (I, Xt_4)\}$$

is a subgroup of $E(4)$ and has diagonal linear matrices. Finally we shall define homomorphisms

$$\epsilon : \bar{\Gamma}'' \to Spin(4),$$

by the formulas $\epsilon(\gamma_1') = \pm e_2 e_3, \epsilon(\gamma_2') = \pm e_2 e_4$ and $\epsilon(f_1) = -1, \epsilon(f_2) = \epsilon(f_3) = -\epsilon(f_4) = \pm 1$. Hence we have eight different spin structures on $\bar{\Gamma}'$.

This finishes the proof of the main theorem.

□

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