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Properties of the combinatorial Hantzsche-Wendt groups

J. Popko, A. Szczepański *

Institute of Mathematics, University of Gdańsk, ul. Wita Stwosza 57, 80-308 Gdańsk, Poland

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The combinatorial Hantzsche-Wendt group $G_n = \langle x_1, ..., x_n \mid x_i^{-1}x_j^2x_ix_j^2, \forall i \neq j\rangle$ was defined by W. Craig and P.A. Linnell in [4]. For $n = 2$ it is a fundamental group of 3-dimensional oriented flat manifold with non cyclic holonomy group. We calculate the Hilbert-Poincaré series of $G_n, n \geq 1$ with $\mathbb{Q}$ and $\mathbb{F}_2$ coefficients. Moreover, we prove that the cohomological dimension of G_n is equal to $n+1$. Some other properties of this group are also considered.

1. Introduction

Let Γ_3 be the fundamental group of the oriented flat 3-manifold with non-cyclic holonomy, which was the first time defined by W. Hantzsche & H. Wendt and W. Nowacki in 1934, see [6], [13]. From [19, ch. 9], Γ_3 is a torsion free crystallographic group of a rank 3. Where, by crystallographic group of dimension n we understand a discrete and cocompact subgroup of the group $E(n) = O(n) \ltimes \mathbb{R}^n$ of isometries of the Euclidean space $\mathbb{R}^n$. From the Bieberbach theorems [19] any crystallographic group Γ of rank n defines a short exact sequence

$$1 \to \mathbb{Z}^n \to \Gamma \to H \to 1, \tag{1}$$

where $\mathbb{Z}^n$ is the free abelian subgroup of all translations of Γ and H is a finite group, called the holonomy group of Γ. In the case of Γ_3 the group $H = \mathbb{Z}_2 \oplus \mathbb{Z}_2$. As a subgroup of $E(3)$

* Corresponding author.
E-mail addresses: jpopko@mat.ug.edu.pl (J. Popko), matas@ug.edu.pl (A. Szczepański).

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$$\Gamma_3 = \text{gen}\{A = (\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, (1/2, 1/2, 0)), B = (\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, (0, 1/2, 1/2))\}.$$

The Hantzsche-Wendt groups/manifolds are also defined in higher odd dimensions, as fundamental groups of oriented flat manifolds of dimensions, $n \geq 3$ with holonomy group $(\mathbb{Z}_2)^{n-1}$. We shall denote them by Γ_n, see [19, ch. 9]. From the Bieberbach theorems there exist, for given n, a finite number $L(n)$ of Hantzsche-Wendt groups (HW groups), up to isomorphism. However, the number $L(n)$ growths exponentially, see [12, Theorem 2.8]. Let us define an example of the HW group Γ_n of dimension ≥ 3 which is a generalization of Γ_3.

Example 1. Let n be an odd number. Then

$$\Gamma_n = \text{gen}\{\gamma_i = (\begin{bmatrix} -1 & 0 & ... & 0 & 0 & 0 & ... & 0 & 0 \\ 0 & -1 & ... & 0 & 0 & 0 & ... & 0 & 0 \\ ... & & & & & & & & \\ 0 & 0 & ... & -1 & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & ... & 0 & 0 & -1 & ... & 0 & 0 \\ ... & & & & & & & & \\ 0 & 0 & ... & 0 & 0 & 0 & ... & -1 & 0 \\ 0 & 0 & ... & 0 & 0 & 0 & ... & 0 & -1 \end{bmatrix}, (0, ..., 0, 1/2, 1/2, 0, ..., 0))\},$$

where 1 is at the i-th place and the first $1/2$ is at the i-th place, $1 \leq i \leq n-1$.

In 1982, see [19], the second author proved that for odd $n \geq 3$, the manifolds $\mathbb{R}^n/\Gamma_n$ are rational homology spheres. Moreover, for $n \geq 5$ the commutator subgroup of the group Γ_n is equal to the translation subgroup [19, Theorem 9.3] and [17, Theorem 3.1]. Moreover, for $m \geq 7$ there exist many isospectral HW-manifolds non pairwise homeomorphic, [12, Corollary 3.6]. HW groups have an interesting connection with Fibonacci groups (see below) and the theory of quadratic forms over the field $\mathbb{F}_2$, [19, Theorem 9.5]. HW-manifolds have no Spin or Spin$^{\mathbb{C}}$-structures, [11] and [19, p. 109]. Finally HW manifolds are cohomological rigid that means two HW manifolds are homeomorphic if and only if their cohomology rings over $\mathbb{F}_2$ are isomorphic, [15].

G is called a unique product group if given two nonempty finite subset X, Y of G, then exists at least one element $g \in G$ which has a unique representation $g = xy$ with $x \in X$ and $y \in Y$. A unique product group is torsion free, though the converse is not true in general. The original motivation for studying unique product groups was the Kaplansky zero divisor conjecture, namely that if k is a field and G is a torsion free group, then kG is a domain. It was proved in 1988 [16] that the group G_2 is a nonunique product group. To prove it the author uses the combinatorial presentation ([14, Lemma 13.3.1, pp. 606–607])

$$\Gamma_3 = \langle x, y \mid x^{-1}y^2xy^2, y^{-1}x^2yx^2 \rangle. \tag{2}$$

However the counterexample to the Kaplansky unit conjecture was given in 2021 by G. Gardam [5]. Again the counterexample was found in the group ring $\mathbb{F}_2[\Gamma_3]$. The Kaplansky unit conjecture states that every unit in $K[G]$ is of the form kg for $k \in K \setminus \{0\}$ and $g \in G$.
In [4] the following generalization of Γ_3 is proposed.

Definition 1. By a combinatorial Hantzsche-Wendt group we shall understand a finitely presented group

$$G_n = \langle x_1, ..., x_n | x_i^{-1}x_j^2x_ix_j^2 \quad \forall \ \ i \neq j \rangle.$$

It is easy to see that, $G_0 = 1$ and $G_1 = \mathbb{Z}$. Moreover G_2 is the Hantzsche-Wendt group of dimension 3.

Let

$$\mathbb{Z}^n \simeq \mathbb{A}_n \simeq \langle x_1^2, x_2^2, ..., x_n^2 \rangle, \tag{3}$$

be a free abelian subgroup of G_n. In [4, Lemma 3.1] is proved that $\mathbb{Z}^n \lhd G_n$. Later we shall denote $\mathbb{A}_n$ by $\mathbb{Z}^n$. Moreover, $W_n = G_n/\mathbb{Z}^n = \langle x_1, ..., x_n \mid x_1^2, x_2^2, ..., x_n^2 \rangle \simeq *^n\ \mathbb{Z}_2$. Finally in [4, Theorem 3.3] it is proved that G_n is torsion free for all $n \geq 1$. This is also the corollary from Theorem 2. For any $1 \leq m \leq n$, G_m embeds in G_n and for $n \geq 2, G_n$ is a nonunique product group [4, Corollary 3.5]. Another interesting result of [4, Theorem 3.6] is the following. There is for $n \geq 3$ and odd a surjective homomorphism $\Phi_n : G_{n-1} \to \Gamma_n$. It is easy to see that $\Phi_n(\mathbb{Z}^{n-1})$ is a free abelian subgroup of the translation subgroup of Γ_n of a rank $n-1$. Since $\Gamma_n/\Phi_n(\mathbb{Z}^{n-1})$ is an infinite group and $Ker(\Phi_n) \cap \mathbb{Z}^{n-1} = 1$ then $Ker(\Phi_n)$ is an infinitely generated free group. (See [4, Theorem 3.6] and [7, p. 87].)

At that point we would like to mention the following related result, see [10]. Recall that the Fibonacci group $F(r, n)$ is defined by the presentation

$$F(r, n) = \langle a_0, ..., a_{n-1} | a_i a_{i+1} \cdots a_{i+r-1} = a_{i+r}, 0 \leq i \leq n-1 \rangle,$$

where the indices are understood modulo n. There exists a connection of these groups with our family G_n. We know that $F(2, 6)$ is isomorphic to Γ_3, and there is, for any $n \geq 3$ an epimorphism $\Psi_n : F(n-1, 2n) \to \Gamma_n$.

In the first part of a paper we shall show two models of BG_n (or $K(G_n, 1)$). They are a topological realization of two algebraic representations of G_n. The first model is an appropriate gluing of n copies of generalized fat Klein bottles. It corresponds to an isomorphism of G_n with $*^n_{\mathbb{Z}^n} K_n$ where K_n is a generalized Klein bottle crystallographic group amalgamated over the translation lattices. The second model is some Borel construction. It corresponds to the representation of G_n as an extension:

$$1 \to \mathbb{Z}^n \to G_n \to W_n \to 1. \tag{4}$$

From the first model we obtain that the cohomological dimension of G_n is equal to $n+1$ for $n > 1$.

In the second part we calculate the Hilbert-Poincaré series of $G_n, n \geq 1$ with $\mathbb{Q}$ and $\mathbb{F}_2$ coefficients and explain the algebra structure of the cohomology. Here our main tools will be the Lyndon-Hochschild-Serre (LHS) spectral sequence of the group extension (4). The case with $\mathbb{F}_2$ coefficients uses a multiplicative structure of LHS. In the $\mathbb{F}_2$ case it is enough to use $E_3^{\star,\star}$ groups, but for rational coefficients we only need the $E_2^{\star,\star}$-terms. (See formulas (12) and (16).)

In the last part we calculate some other invariants and properties of G_n. For example their abelianization and the Euler characteristic.

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2. Two models of BG_n

2.1. Gluing fat Klein bottles

We start with an example.

Example 2. Let K_- be the fundamental group of the Klein bottle and $\mathbb{Z}^2$ its maximal abelian subgroup of index two. It is well known (see [9, Chapter 8.2, p. 153]) that $\Gamma_3 \simeq K_- *_{\mathbb{Z}^2} K_-$.

A generalization of the above example gives us the following characterization of the combinatorial Hantzsche-Wendt group. Let $G_n^{(i)}$ denote the subgroup of G_n generated by $\{x_i\}$ and the abelian subgroup (3) $\mathbb{Z}^n$. We shall call it a generalized Klein bottle.

Proposition 1. *The natural group homomorphism*

$$*_{\mathbb{Z}^n} G_n^{(i)} \to G_n \tag{5}$$

is an isomorphism.

Proof. This follows from the definition and the structure of the free product with amalgamation. □

$G_n^{(i)}$ is a torsion free crystallographic group of dimension n and acts freely on $\mathbb{R}^n$ (in a way analogous to K_-) so has a classifying space which is an n dimensional closed flat manifold K^i (the generalized Klein bottle). A topological interpretation of the isomorphism (5) gives us a $n+1$ dimensional classifying space BG_n as (homotopically) gluing together n generalized Klein bottles $K^{(1)}, K^{(2)}, ..., K^{(n)}$ along a common n dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$. This space has dimension $n+1$ since we must convert maps $\mathbb{R}^n/\mathbb{Z}^n \to K^{(i)}$ to inclusions. More precisely it may be done as follows. Let us define an action of G_n on $\mathbb{R}^n$ by

$$x_i(v)_i = v_i + 1/2 \;\text{ and }\; x_i(v)_j = -v_j, j \neq i,$$

where $v = (v_1, v_2, ..., v_n)$ and an action on a segment $I = [-1, 1]$ by $x_i(t) = -t, t \in I$.

Definition 2. By a fat Klein bottle we shall understand the space $B_n^{(i)} := (\mathbb{R}^n \times I)/G_n^{(i)}$.

Let $S_n := \mathbb{R}^n/\mathbb{Z}^n$. Let us define maps $\alpha^{(i)} : S_n \to B_n^{(i)}$ by the formula $\alpha^{(i)}(v) = [(v, 1)]$.

Definition 3. By the space B_n we shall understand a *colim* of a diagram formed from maps $\alpha^{(i)}$, i.e.

$$B_n := colim_i \alpha^{(i)}.$$

Theorem 1. *The above space B_n is a classifying space for G_n.*

Proof. From the definition the action of the subgroup $G_n^{(i)}$ on $\mathbb{R}^n$ is free and the orbit space $K_n^{(i)}$ was called a generalized Klein bottle. Moreover, the fat Klein bottle is $(n+1)$ dimensional compact manifold with boundary and the projection on the first factor gives a bundle $B_n^{(i)} \to K_n^{(i)}$ with fiber I, hence in particular it is a homotopy equivalence. Finally, the map $\alpha^{(i)}$ is an embedding on the boundary of $B_n^{(i)}$. However, more geometrically we may write $B_n := \bigcup_i B_n^{(i)}$ treating the maps $\alpha^{(i)}$ as identifications (so $S_n \subset B_n^{(i)}$). In other words B_n is obtained from n copies of a fat Klein bottle by an appropriate identification of the boundaries of different copies. To finish our proof we observe that $\pi_1(B_n) \simeq G_n$ after van Kampen theorem. The space B_n is aspherical after JHC Whitehead's theorem in [20, Theorem 5]. □

Corollary 1. *For $n > 1$ the cohomological dimension of G_n is equal to $n+1$.*

Proof. For brevity we write $B = B_n$ and $S = S_n$. From the properties of B we have, that $\mathrm{cd}G_n \leq n+1$. Let H denote cohomology with $\mathbb{F}_2$ coefficients. We have an exact sequence

$$H^n(S) \to H^{n+1}(B,S) \to H^{n+1}(B).$$

Since $\dim H^n(S) = 1$ and $\dim H^{n+1}(B,S) = n$, then $\dim H^{n+1}(B) \geq n-1$. Hence $\mathrm{cd}G_n \geq n+1$ for $n > 1$. □

Remark 1. The space B_n is for $n = 1$ a Möbius band, for $n = 2$ a closed manifold (a classical 3-dimensional Hantzsche-Wendt manifold). However for $n > 2$ it is nonmanifold, since there is singularity along S_n.

2.2. Borel construction (homotopy quotient)

Let G be a discrete group and let $p_G : EG \to BG$ be the universal G bundle. The assignment $G \mapsto p_G$ may be done functorial in the group G and respecting products. If X is some G-space then the space

$$X_G := (EG \times X)/G$$

is called the Borel construction on X, [1, p. 10]. Here G acts on $EG \times X$ diagonally. Let $f_X : X_G \to BG$ be the quotient map. It is a fibration with fiber X. It is easy to see that, if X is aspherical then X_G is also aspherical.

Definition 4 *(morphisms between maps).* If $f : X_1 \to X_2$ and $g : Y_1 \to Y_2$ then a morphism from f to g is a pair (m_1, m_2) where $m_i : X_i \to Y_i$ and $gm_1 = m_2 f$.

The operation of taking pullback along ψ is denoted by $\psi^\star$. We shall write $f \simeq m_2^\star(g)$ if $(m_1, m_2) : f \to g$ and m_1 is an isomorphism on fibers.

Let $\xi, \eta \in W_2$ be generators of order 2 and $D := \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The abelianization of W_2 defines

$$1 \to \mathbb{Z} \simeq (\langle \xi\eta \rangle^2) \to W_2 \xrightarrow{\alpha} D \to 1. \tag{6}$$

Let Σ be the unit circle on the complex plane. Define an action of D on Σ by formulas

$$\xi(z) = \bar{z} \ \text{ and } \ \eta(z) = -\bar{z}.$$

Denote a resulting D-space by U. In the above language we have a map $f_U : U_D \to BD$ and we can observe that $\pi_1(f_U) \simeq \alpha$.

We define (for $i = 1, 2, ..., n$) homomorphisms $\phi_i : G_n \to W_2$

$$\phi_i(x_i) = \xi\eta \ \text{ and } \ \phi_i(x_j) = \xi \ \text{ for } \ j \neq i. \tag{7}$$

Then $\phi := (\phi_1, ..., \phi_n)$ gives a homomorphism from $G_n \to (W_2)^n$. Let $q_n : G_n \to W_n$ be the canonical surjection. The homomorphism ϕ factorizes and we obtain a map $(\phi, \psi) : q_n \to \alpha^n$ and ϕ is an isomorphism on fibers. We have:

Lemma 1. $q_n \simeq \psi^\star \alpha^n$.

Proof. With support of (6) we have the following commutative diagram

$$
\begin{tikzcd}
1 \arrow[r] & \mathbb{Z}^n \arrow[r, "i"] \arrow[d, "\simeq"] & G_n \arrow[r, "q_n"] \arrow[d, "\phi"] & W_n \arrow[r] \arrow[d, "\psi"] & 1 \\
1 \arrow[r] & \mathbb{Z}^n \arrow[r, "i_1"] & (W_2)^n \arrow[r, "\alpha^n"] & (\mathbb{Z}_2 \oplus \mathbb{Z}_2)^n \arrow[r] & 1
\end{tikzcd},
$$

Diagram 1

where i, i_1 are inclusions. □

Define a W_n action on the space Σ^n

$$x_i(z)_i = -z_i \text{ and } x_i(z)_j = \bar{z}_j \text{ for } j \neq i, \tag{8}$$

where $z = (z_1, z_2, ..., z_n)$.
Denote the resulting W_n-space by T_n.

Proposition 2.

$$\pi_1(f_{T_n}) \simeq q_n$$

in particular $\pi_1((T_n)_{W_n}) \simeq G_n$.

Proof. The action on T_n is obtained by composing the product D^n action with the homomorphism ψ (i.e. $w(x) = \psi(w)(z)$ for $w \in W_n$). Hence, from naturality we have a map of fibrations

$$(\hat{\psi}_1, \hat{\psi}_2) = \hat{\psi} : f_{T_n} \to f_{U^n}.$$

Applying π_1 we get

$$\pi_1(\hat{\psi}) : \pi_1(f_{T_n}) \to \pi_1(f_{U^n}).$$

The map $\hat{\psi}$ gives an isomorphism (identity) on fibers so the map $\pi_1(\hat{\psi})$ also gives an isomorphism on fibers by an application of the long exact sequence for fibrations. We have (for codomain components) $(\hat{\psi})_2 = B\psi$ so $\pi(\hat{\psi})_2 = \pi(B\psi) = \psi$. Hence

$$\pi_1(f_{T_n}) \simeq \psi^\star \pi_1(f_{U^n}).$$

But

$$\psi^\star \pi_1(f_{U^n}) \simeq \psi^\star \pi_1(f_U))^n \simeq \psi^\star \alpha^n \simeq q_n. \quad \square$$

Theorem 2.

$$(T_n)_{W_n} = K(G_n, 1).$$

Proof. The space $(T_n)_{W_n}$ is aspherical because T_n is. And it has the appropriate fundamental group by Proposition 2. □

Let $B = \bigvee_1^n \mathbb{R}P(\infty)$. The space $B = K(W_n, 1)$, cf. [20]. Let $E \to B$ be the universal covering. Then

Corollary 2. *We have the fibration*

$$T^n \to (T^n)_{W_n} \to E/W_n, \tag{9}$$

where a W_n action on E is by deck transformation. □

Remark 2. The W_n action on T^n is highly noneffective. The kernel of it is the commutator subgroup of W_n, which by the Kurosh subgroup theorem, is a free group of rank $1 + (n-2)2^{n-1}$.

See [3, Exercise 3, p. 212] and the proof of Proposition 6.

3. Cohomologies of G_n

In this part we shall calculate a cohomology of the group G_n with $\mathbb{Q}$ coefficients (Theorem 3) and $\mathbb{F}_2$ coefficients (Theorem 4). We shall apply the Leray-Serre spectral sequence of the fibration (9) and equivalently Lyndon-Hochschild-Serre spectral sequence for the short exact sequence of groups (4)

$$1 \to \mathbb{Z}^n \to G_n \xrightarrow{\nu} W_n \to 1.$$

3.1. Hilbert-Poincaré series

Definition 5. ([7, p. 230]) Let M be a topological space. For a fixed coefficient field k, define the Poincaré series of M the formal power series

$$P(x,k) = \Sigma_i a_i x^i$$

where a_i is the dimension of $H^i(M,k)$ as a vector space over k, assuming this dimension is finite for all i.

Theorem 3. *The rational Hilbert-Poincaré series of the space*

$$K(G_n, 1) = T^n \times_{W_n} E$$

is equal to

$$P_n(x,\mathbb{Q}) = ((1+x)(1 + \frac{(1-(-1)^n))}{2}x^n + x(\frac{n-2}{2}(1+x)^{n-1} - \frac{n}{2}(1-x)^{n-1})). \tag{10}$$

In particular, $P_0(x,\mathbb{Q}) = 1, P_1(x,\mathbb{Q}) = x+1, P_2(x,\mathbb{Q}) = x^3+1$).

Proof. We start with Lemma.

Lemma 2. *For $p > 1, H^p(W_n,\mathbb{Q}) = 0$.*

Proof. We have a short exact sequence of groups related to the abelianization

$$1 \to \mathbb{F}_k \to W_n \to (\mathbb{Z}_2)^n \to 1, \tag{11}$$

where $\mathbb{F}_k$ is a non abelian free group of a rank $k = 1 + (n-2)2^{n-1}$. Hence for $q > 1, H^q(\mathbb{F}_k, M) = 0$ for any $\mathbb{F}_k$-module M. Similar for any $p \geq 1, H^p((\mathbb{Z}_2)^n, N) = 0$ for any $(\mathbb{Z}_2)^n$-rational vector space N. Applying a Leray-Serre spectral sequence to (11) we have for $i \geq 2, H^i(W_n, S) = 0$. Where S is a W_n-rational vector space. □

Corollary 3. *For $p > 1, q \geq 0, E_2^{p,q} = H^p(W_n, H^q(\mathbb{Z}^n, \mathbb{Q})) = 0$ and the differentials $d_i = 0$ for $i \geq 2$. Moreover, $E_2^{0,q}$ and $E_2^{1,q}, q \geq 0$ are two non trivial columns of the spectral sequence.*

The Hilbert-Poincaré polynomial (10) is the sum $f_0 + f_1$, where

$$f_p = x^p \Sigma_i \dim(E_2^{p,i}) x^i. \tag{12}$$

Let us start to calculate dimensions of $E_2^{p,q} = H^p(W_n, H^q(\mathbb{Z}^n, \mathbb{Q}))$ for $p = 0, 1$ and $q \geq 0$. We shall use a W_n action on $H^q(\mathbb{Z}^n, \mathbb{Q}) = \Lambda^q(\mathbb{Q}^n)$, which follows from an action W_n on T^n, see (8).

We introduce sequences $\epsilon \in \{-1, 1\}^n$. Denote by $(-1)\epsilon = -\epsilon = (-\epsilon_1, -\epsilon_2, ..., -\epsilon_n)$. Moreover, for $A \subset \{1, 2, ..., n\}$ the sequence e^A has -1 exactly on the positions from A. Finally let $1 = (1, 1, ..., 1) := e^{\varnothing}$ and $|\epsilon| := \Sigma_i \frac{1-\epsilon_i}{2}$ (the number of -1 in the sequence).

By $\mathbb{Q}_\epsilon$ we shall understand the rational numbers $\mathbb{Q}$ with the structure of a W_n-module such that the k-th generator of W_n acts as multiplication by $\epsilon_k, 1 \leq k \leq n$. In this language $H^1(\mathbb{Z}^n, \mathbb{Q}) \simeq \Sigma_i \mathbb{Q}_{-e^{\{i\}}}$ as W_n-module. Moreover, $H^*(\mathbb{Z}^n, \mathbb{Q}) \simeq \Lambda^*(\mathbb{Q}^n)$ is a sum of some $\mathbb{Q}_\epsilon$. Let

$$h^i(\epsilon) = \dim H^i(W_n, \mathbb{Q}_\epsilon).$$

From the definition

$$h^0(\epsilon) = \begin{cases} 1 & \text{if } \epsilon = 1 \\ 0 & \text{if } \epsilon \neq 1 \end{cases} \text{ and } h^1(\epsilon) = \begin{cases} 0 & \text{if } \epsilon = 1 \\ |\epsilon| - 1 & \text{if } \epsilon \neq 1 \end{cases}.$$

Using (12) and the above formulas gives us

$$f_i = x^i \sum_{A \subset \{1,2,...,n\}} x^{|A|} h^i((-1)^{|A|} e^A), i = 1, 2.$$

Hence for $n \geq 0$:

$$\begin{gathered} f_0 = 1 + \frac{1-(-1)^n}{2} x^n \\ f_1 = x^{n+1}(n-1)\frac{1+(-1)^n}{2} + \\ +x\, \Sigma_{0<k<n} x^k \binom{n}{k} ((k-1)\frac{1+(-1)^k}{2} + (n-k-1)\frac{1-(-1)^k}{2}). \end{gathered}$$

Lemma 3. *Formula (10) from Theorem 3 is equal to $f_0 + f_1$.*

Proof. We shall use two formulas:

$$\Sigma_{0<k<n} \binom{n}{k} x^k = (1+x)^n - 1 - x^n = g(x)$$

and

$$\Sigma_{0<k<n} k \binom{n}{k} x^k = nx(1+x)^{n-1} - nx^n = f(x).$$

On the beginning we shall prove that

$$S = \Sigma_{0<k<n} x^k \binom{n}{k}((k-1)\frac{1+(-1)^k}{2} + (n-k-1)\frac{1-(-1)^k}{2})$$

=[1]

$$\Sigma_{0<k<n} x^k \binom{n}{k}(k(-1)^k + \frac{n-2}{2} - \frac{n}{2}(-1)^k) =$$
$$\Sigma_{0<k<n} x^k (-1)^k \binom{n}{k} + \frac{n-2}{2}\Sigma_{0<k<n} x^k \binom{n}{k} - \frac{n}{2}\Sigma_{0<k<n}(-1)^k x^k \binom{n}{k} =$$
$$f(-x) + \frac{n-2}{2} - \frac{n}{2} g(-x) =$$
$$\frac{n-2}{2}(1+x)^n - \frac{n}{2}(1+x)(1-x)^{n-1} + (1 - n\frac{1+(-1)^n}{2})x^n + 1.$$

Since

$$f_1 = x^{n+1}(n-1)\frac{1+(-1)^n}{2} + xS$$

then

$$f_1 = x + \frac{1-(-1)^n}{2}x^{n+1} + (1+x)x(\frac{n-2}{2}(1+x)^{n-1} - \frac{n}{2}(1-x)^{n-1})$$

and

$$f_0 + f_1 =$$
$$(1+x)(1 + \frac{1-(-1)^n}{2}x^n + x(\frac{n-2}{2}(1+x)^{n-1} - \frac{n}{2}(1-x)^{n-1})). \quad \square$$

As a complement to the above results we present an observation about the algebra structure of $H^*(G_n, \mathbb{Q})$.

Corollary 4. *If $x, y \in H^*(G_n, \mathbb{Q})$ are such that deg$(x) > 0$ and deg$(y) > 0$ then $xy = 0$.*

Proof. For $n = 1$ it is obvious. Let us assume $n \geq 2$. From the proof of Theorem 3 $H^1(G_n, \mathbb{Q}) = 0$ and $H^s(G_n, \mathbb{Q}) = 0$ for $s > n + 1$. So, if $\deg(x) \geq n$ or $\deg(y) \geq n$ then $xy = 0$. Hence we can assume that $\deg(x) < n$ and $\deg(y) < n$. From the proof of Theorem 3 we know that the appropriate Serre spectral sequence converging to $H^*(G_n, \mathbb{Q})$ has the property:

$$(\star) \quad E_\infty^{p,q} \neq 0 \Longrightarrow (p,q) \in \{(0,0),(0,n)\} \cup \{(1,i) : i \leq n\}.$$

Let (F_i) denote the filtration of $H^*(G_n, \mathbb{Q})$ associated with the spectral sequence. From $(\star)$ and $\deg(x) < n$ and $\deg(y) < n$ it follows that $x, y \in F_1$. Consequently $xy \in F_2$. But again from $(\star)$ $F_2 = 0$. $\square$

A calculation of the cohomology with $\mathbb{F}_2$ coefficients needs different tools. We shall also apply the Lyndon-Hochschild-Serre spectral sequence for the short exact sequence of groups (4)

$$1 \to \mathbb{Z}^n \to G_n \xrightarrow{\nu} W_n \to 1.$$

In this case we have $E_3^{*,*} \neq 0$ and we shall use multiplicative structures.

[1] $k(-1)^k + \frac{n-2}{2} - \frac{n}{2}(-1)^k = (k-1)\frac{1+(-1)^k}{2} + (n-k-1)\frac{1-(-1)^k}{2}$.

Theorem 4.

$$P_n(x, \mathbb{F}_2) = (1 + x)(1 + (n - 1)x(1 + x)^{n-1}). \tag{13}$$

In particular, $P_0(x, \mathbb{F}_2) = 1, P_1(x, \mathbb{F}_2) = x + 1, P_2(x, \mathbb{F}_2) = 1 + 2x + 2x^2 + x^3)$.

Proof. There are canonical isomorphisms over $\mathbb{F}_2$

$$E_2^{p,q} = H^p(W_n, H^q(\mathbb{Z}^n, \mathbb{F}_2)) \xleftarrow{\sim} H^p(W_n, \mathbb{F}_2) \otimes H^q(\mathbb{Z}^n, \mathbb{F}_2),$$
$$H^*(\mathbb{Z}_2^n, \mathbb{F}_2) = \mathbb{F}_2[z_1, z_2, ..., z_n],$$

where $z_i \in H^1(\mathbb{Z}_2^n, \mathbb{F}_2) = \mathrm{Hom}(\mathbb{Z}_2^n, \mathbb{F}_2)$ is the projection on the i-coordinate, $i = 1, 2, ..., n$ and

$$H^*(W_n, \mathbb{F}_2) = H^*(\mathbb{Z}_2^n, \mathbb{F}_2)/\{z_i z_j | i \neq j\}. \tag{14}$$

Let $g_1, ..., g_n \in H^1(\mathbb{Z}^n, \mathbb{F}_2)$ be a dual basis to $x_1^2, ..., x_n^2$. We shall denote by $\Lambda(g_1, ..., g_n)$ the exterior algebra over $\mathbb{F}_2$ generated by $g_1, ..., g_n$ $(H^*(\mathbb{Z}^n, \mathbb{F}_2))$.[2] To begin we shall prove:

Proposition 3. *Let* (E_r, d_r) *be the above spectral sequence and on* E_3 *we use the total grading.* $E_3^{0,0} = \langle 1 \rangle, E_3^{0,q} = 0, q > 0$ *and* $E_3^{p,q} = 0$ *for* $p > 2$*. Moreover* $d_2 \neq 0$ *and* $d_i = 0$ *for* $i \geq 3$*.*

Proof. To prove that $E_3^{0,q} = 0, q > 0$ it is enough to show that the kernel of $d_2^{0,q}$ is trivial. By contradiction assume that $\exists\ 0 \neq \omega \in E_2^{0,q}, q > 0$ and $d_2^{0,q}(\omega) = 0$. Then $\exists\ i$, s.t. $\omega = g_i\alpha + \beta$ and α, β do not depend on g_i. If $d_2^{0,q}(\omega) = 0$ then from the properties of the transgression $0 = d_2^{0,q}(\omega) = z_i\alpha + \gamma$, where γ is a linear combination of elements from the set $\{z_s^2 : s \neq i\}$ with coefficients in $\Lambda(g_1, ..., g_n)$. Hence $\alpha = 0$ a contradiction.
From Lemmas 5 and 6 cycles of the differential d_2 are linear combinations of elements $z_i^k\omega$ (for some i, k) and $\omega \in \Lambda(g_1, g_2, ..., g_n)$ does not include g_i. Hence for $k \geq 3$ $z_i^k\omega = d_2^{k,s}(z_i^{k-2}g_i\omega)$ and for $p \geq 3, E_3^{p,q} = 0$. □

Lemma 4. *In the spectral sequence of the extension (6)*

$$0 \to \mathbb{Z} \to W_2 \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to 0,$$

$\bar{d}_2(g) = z_1z_2$*, where* g *is a generator of* $H^1(\mathbb{Z}, \mathbb{F}_2)$*.*

Proof. From the above $z_1z_2 \in H^*(W_2, \mathbb{F}_2)$ is equal to zero. Applying the five-term exact sequence (see [8, pp. 16, 57]) we get the exact sequence

$$H^1(\mathbb{Z}, \mathbb{F}_2) \xrightarrow{\bar{d}_2} H^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{F}_2) \xrightarrow{f^*} H^2(W_2, \mathbb{F}_2)$$

and $\mathrm{Im}\bar{d}_2 = \mathrm{Ker} f^*$. Hence $\bar{d}_2(g) = z_1z_2$. □

Lemma 5. *Let* $g_1, ..., g_n \in H^1(\mathbb{Z}^n, \mathbb{F}_2)$ *be a dual basis to* $x_1^2, ..., x_n^2$*. For the spectral sequence of the exact sequence of groups*

$$0 \to \mathbb{Z}^n \to G_n \to W_n \to 0,$$

using naturality and properties of homomorphisms (7) $\phi_i : G_n \to W_2$ *we obtain* $d_2(g_i) = z_i^2$*.*

[2] $\Lambda^*(g_1, ..., g_n) \simeq H^*(\mathbb{Z}^n, \mathbb{F}_2)$.

Proof. We have a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \mathbb{Z}^n \arrow[r, "i"] \arrow[d, "\alpha_i"] & G_n \arrow[r, "q_n"] \arrow[d, "\phi_i"] & W_n \arrow[r] \arrow[d, "\gamma_i"] & 1 \\ 1 \arrow[r] & \mathbb{Z} \arrow[r, "i_1"] & W_2 \arrow[r, "\alpha"] & \mathbb{Z}_2 \oplus \mathbb{Z}_2 \arrow[r] & 1 \end{tikzcd}.$$

Diagram 2

Here, α_i and γ_i are defined by ϕ_i. From naturality $d_2 \circ \alpha_i^* = \gamma_i^* \circ \bar{d}_2$, (see Diagram 3) where α_i^*, γ_i^* are the induced maps on cohomology.

$$\begin{tikzcd} \bar{E}_2^{0,1} \simeq H^1(\mathbb{Z}, \mathbb{F}_2) \arrow[r, "\bar{d}_2"] \arrow[d, "\alpha_i^*"] & \bar{\mathrm{E}}_2^{2,0} \simeq \mathrm{H}^2(\mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{F}_2) \arrow[d, "\gamma_i^*"] \\ E_2^{0,1} \simeq H^1(\mathbb{Z}^n, \mathbb{F}_2) \arrow[r, "d_2"] & \mathrm{E}_2^{2,0} \simeq \mathrm{H}^2(\mathrm{W_n}, \mathbb{F}_2) \end{tikzcd}$$

Diagram 3

We have

$$g_i = \alpha_i^*(g).$$

Hence

$$d_2(g_i) = d_2(\alpha_i^*(g)) = \gamma_i^*(\bar{d}_2(g)) \overset{\text{Lemma 4}}{=} \gamma_i^*(z_1 z_2) = \gamma_i^*(z_1)\gamma_i^*(z_2).$$

Moreover, let $\overline{\gamma_i} : W_n/[W_n, W_n] \to \mathbb{Z}_2 \oplus \mathbb{Z}_2$ be induced by γ_i, then $\overline{\gamma_i}(\lambda_1, \lambda_2, \cdots, \lambda_n) = (\Sigma_s \lambda_s, \lambda_i)$. Hence,

$$\gamma_i^*(z_1) = \Sigma_s z_s \text{ and } \gamma_i^*(z_2) = z_i.$$

Finally, since in $H^*(W_n)$ $z_i z_j = 0$ for $i \neq j$, we get

$$d_2(g_i) = (\Sigma_s z_s) z_i = z_i^2. \quad \square$$

The next observation is the following.

Lemma 6. *Let $k > 0$, then, $d_2(z_i^k \omega) = 0$ if and only if ω does not depend on g_i.*

Proof. Let us write $\omega = g_i \alpha + \beta$ where $\alpha, \beta \in \Lambda$ do not depend on g_i. In fact, by definition $d_2(g_i \alpha + \beta) = z_i^2 \alpha + \gamma$, where γ is a linear combination of elements from the set $\{z_s^2 : s \neq i\}$ with coefficients from Λ. Hence, because for $i \neq s$, $z_i z_s = 0$ (14)

$$d_2(z_i^k(g_i \alpha + \beta)) = z_i^k d_2(g_i \alpha + \beta) = z_i^k(z_i^2 \alpha + \gamma) = z_i^{k+2} \alpha. \tag{15}$$

Assume $d_2(z_i^k \omega) = 0$. We can write $\omega = g_i \alpha + \beta$, where α and β are independent from g_i. From (15) $0 = d_2(z_i^k \omega) = z_i^k \alpha$. Hence $\alpha = 0$ and $\omega = \beta$ and so ω does not include g_i. Finally, if ω does not include g_i substituting $\alpha = 0$ and $\omega = \beta$ to the formula (15) we get $d_2(z_i^k \omega) = 0$. $\square$

Corollary 5. *Let $k > 0$ and $\upsilon = \Sigma_s z_s^k \omega_s \in E_2^{k,s}$, where $\forall\ s\ \omega_s \in \Lambda_s$ then*

$$d_2(\upsilon) = 0 \Longleftrightarrow \forall s \ \ \omega_s \ \ \textit{does not include} \ \ g_s.$$

Proof. ($\Leftarrow$) Follows from the above Lemma 6. ($\Rightarrow$) For any i if $d_2(\upsilon) = 0$ then also $d_2(z_i\upsilon) = 0$. But $z_i\upsilon = z_i^{k+1}\omega_i$. Again, from Lemma 6 it follows that ω_i does not include g_i. □

Remark 3. Let $Z_2^{i,j} = \mathrm{ker}d_2^{i,j}$ and $B_2^{i,j} = \mathrm{Im}d_2^{i,j}$. Let M be an $\mathbb{F}_2$-vector space and a trivial G-module, then

$$\dim H^i(G, M) = \dim H^i(G, \mathbb{F}_2) \dim M.$$

Moreover $\dim H^i(\mathbb{Z}^n, \mathbb{F}_2) = \binom{n}{i}$, $\dim H^i(W_n, \mathbb{F}_2) = n$ (for $i > 0$), $\dim E_2^{p,q} = n\binom{n}{q}$ (for $p > 0$).

Summing up the generating function for $H^*(G_n, \mathbb{F}_2)$ is a sum of three components: $f_0 + f_1 + f_2$ where

$$f_p = \Sigma_i \mathrm{dim}(E_3^{p,i})x^{p+i}. \tag{16}$$

Lemma 7. *From the properties of the differentials d_2 we have:*

I. $f_0 = 1$;
II. $f_1 = nx(1 + x)^{n-1}$;
III. $f_2 = nx^2(1 + x)^{n-1} - x((1 + x)^n - 1)$.

Proof. By an application of the proof of Proposition 3 $f_0 = 1$.

From the above $d_2(z_i^k\omega) = 0$ if and only if ω does not include g_i. Hence $\mathrm{dim}Z_2^{k,s} = n\binom{n-1}{s}$ and $f_1 = nx(1 + x)^{n-1}$, cf. Corollary 5.

For $p = 2$ we have $E_3^{2,i} = \mathrm{Ker}d_2^{2,i}/\mathrm{Im}d_2^{0,i+1}$. Moreover, for $i > 0, d_2^{0,i}$ is a monomorphism. This follows from the proof of Proposition 3. Hence, dim $\mathrm{Im}d_2^{0,i} = \mathrm{dim}E_2^{0,i} = \binom{n}{i}$. Summing up $f_2 = nx^2(1 + x)^{n-1} - x((1 + x)^n - 1)$ and the Lemma is proved. □

Example 3. We have $\mathrm{dim}Z_2^{k,s} = n\binom{n-1}{s}$. In fact, a basis of $Z_2^{k,s}$ is the set $\{z_i^k\omega\}$, where $1 \le i \le n$ and ω is a Grassmann monomial of degree s on elements $\{g_1, ..., g_n\} \setminus \{g_i\}$.
For $n = 4$ the basis of $Z_2^{2,2}$ has 12 elements:

$$z_1^2g_2g_3, z_1^2g_2g_4, z_1^2g_3g_4, z_2^2g_1g_3, z_2^2g_1g_4, z_2^2g_3g_4, z_3^2g_1g_2, z_3^1g_2g_4, z_3^2g_2g_4, z_4^2g_1g_2, z_4^2g_1g_3, z_4^2g_2g_3$$

the basis of $E_2^{0,3}$ has the following 4 elements:

$$y_1 = g_1g_2g_3, y_2 = g_1g_2g_4, y_3 = g_1g_3g_4, y_4 = g_2g_3g_4.$$

Since $d_2^{0,3}$ is a monomorphism the basis of $B_2^{2,2}$ has four elements:

$$\begin{aligned}
d_2^{0,3}(y_1) &= z_1^2g_2g_3 + z_2^2g_1g_3 + z_3^2g_1g_2, \\
d_2^{0,3}(y_2) &= z_1^2g_2g_4 + z_2^2g_1g_4 + z_4^2g_1g_2, \\
d_2^{0,3}(y_3) &= z_1^2g_3g_4 + z_3^2g_1g_4 + z_4^2g_1g_3, \\
d_2^{0,3}(y_4) &= z_2^2g_3g_4 + z_3^2g_2g_4 + z_4^2g_2g_3.
\end{aligned}$$

Hence $\mathrm{dim}E_3^{2,2} = 12 - 4 = 8$.

Finally

$$f_0 + f_1 + f_2 = 1 + nxb + nx^2 b - xb(1+x) + x =$$
$$= x + 1 + nxb(1+x) - xb(1+x) = (x+1)(1 + nxb - xb) =$$
$$= (1+x)(1 + (n-1)x(1+x)^{n-1}).$$

Here $b = (1+x)^{n-1}$.

Finally, we would like to present some grading of $H^*(G_n, \mathbb{F}_2)$. We start with a definition.

Definition 6. Define the bigraded algebra $\mathcal{E}^{(n)}$ over $\mathbb{F}_2$ by (a direct sum of vector space):

$$\mathcal{E}^{(n)} = \mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2$$

where $\mathcal{E}_i$ are given by:

- $\mathcal{E}_0 = \langle 1 \rangle$
- $\mathcal{E}_1$ is spanned (i.e. is a free vector space) by symbols: $z_i g_A$ where $1 \leq i \leq n$ and $A \subset \{1, 2, ..., n\}, i \notin A$
- $\mathcal{E}_2 = \mathcal{E}_2'/\mathcal{R}$ where $\mathcal{E}_2'$ is spanned by symbols $z_i^2 g_A$ with restrictions as above and $\mathcal{R} = \text{span}\{r_A : A \subset \{1, 2, ..., n\}, A \neq \emptyset\}$ where $r_A = \Sigma_{i \in A} z_i^2 g_{A \setminus \{i\}}$

Bidegrees are given by:

$$\text{bideg}(1) = (0,0),\ \text{bideg}(z_i g_A) = (1, |A|),\ \text{bideg}(z_i^2 g_A) = (2, |A|)$$

Multiplication is given by:

- 1 acts in obvious way;
- $(z_i g_A)(z_i g_B) = z_i g_{A \cup B}$ if $A \cap B = \emptyset$;
- all other products are zero.

The above definition summarizes explicitly the description of the bigraded algebra structure of E_3, namely

Proposition 4. *If (E_r, d_r) is the spectral sequence of the short exact sequence (4) then*

$$E_3 \simeq \mathcal{E}^{(n)}$$

as bigraded algebras.

Example 4 *(Bigraded algebra $H^*(G_2, \mathbb{F}_2)$).* There are elements: $a, b, A, B, w \in H^*(G_2, \mathbb{F}_2)$ such that:

- $a, b \in H^1(G_2, \mathbb{F}_2), A, B \in H^2(G_2, \mathbb{F}_2), w \in H^3(G_2, \mathbb{F}_2)$;
- $aA = bB = w$ and all other products of elements from $\{a, b, A, B, w\}$ are zero;
- $\{a, b, A, B, w\}$ is a basis of $H^*(G_2, \mathbb{F}_2)$.

Using Definition 6 we have:

$$\mathcal{E}_0 = \langle 1 \rangle,$$

$$\mathcal{E}_1 = \langle z_1 g_\emptyset, z_2 g_\emptyset, z_1 g_{\{2\}}, z_2 g_{\{1\}} \rangle,$$
$$\mathcal{E}_2' = \langle z_1^2 g_\emptyset, z_2^2 g_\emptyset, z_1^2 g_{\{2\}}, z_2^2 g_{\{1\}} \rangle,$$
$$\mathcal{R} = \langle z_1^2 g_\emptyset, z_2^2 g_\emptyset, z_1^2 g_{\{2\}} + z_2^2 g_{\{1\}} \rangle.$$

So

$$(1, z_1 g_\emptyset, z_2 g_\emptyset, z_1 g_{\{2\}}, z_2 g_{\{1\}}, [z_1^2 g_{\{2\}}])$$

is a basis of $\mathcal{E}^{(2)}$, where $[\xi]$ denotes the class of ξ.

If $(1, a, b, A, B, w)$ are elements of $H^*(G_2, \mathbb{F}_2)$ which correspond (in this order) to elements of the above basis then they satisfy the conditions stated above.

Example 5. Let $X = P_2 \bigvee P_2 \bigvee S^3$ where S^3 is the 3-dimensional sphere and P_2 is the 2-dimensional real projective space. Let $Y = BG_2$. Then the Poincaré polynomials of X and Y over $\mathbb{F}_2$ and over $\mathbb{Q}$ are the same but $H^*(X, \mathbb{F}_2)$ and $H^*(Y, \mathbb{F}_2)$ are not isomorphic as algebras.

4. Additional observation

Proposition 5.

1. *For $n > 1, (G_n)_{ab} \simeq \mathbb{Z}_4^n$;*
2. *For $n > 1$ the center of G_n is trivial;*
3. *The Euler characteristic and the first Betti number of G_n are equal to zero.*

Proof. 1. Follows from a direct calculation.
2. From (4) and the fact that the center of W_n is trivial we have an inclusion $Z(G_n) \subset \mathbb{Z}^n$. Let $v \in Z(G_n)$. From the above $v = \Pi_i (x_i^2)^{\alpha_i}$ for some $\alpha_i \in \mathbb{Z}$. Using relations in G_n, we have

$$x_1 v x_1^{-1} = (x_1^2)^{\alpha_1} \Pi_{i \geq 2} (x_i^2)^{-\alpha_i}.$$

Since $v = x_1 v x_1^{-1}$, then $\alpha_i = 0$ for $i \geq 2$. Similar $x_2 v x_2^{-1} = v$, which gives us $\alpha_1 = 0$.
3. From the properties of the Euler characteristic of a fibration (9) (see [18, p. 481]) $\chi(K(G_n, 1)) = \chi(T^n)\chi(E/W_n) = 0 \cdot \chi(E/W_n) = 0$. The conclusion about the Betti number follows directly from Theorem 1. □

Proposition 6. *Let $n \geq 2$. The short exact sequence of groups (4) defines a representation*

$$h : W_n \to GL(n, \mathbb{Z}), \forall x \in W_n \;\; h(x)(e_i) = \bar{x} e_i \bar{x}^{-1},$$

where $e_i \in \mathbb{Z}^n$ is the standard basis $i = 1, 2, ..., n$ and $\nu(\bar{x}) = x$. However, $K = Kerh \neq 0$, because $[W_n, W_n] \subset Kerh$. In particular, K is a finitely generated free group of rank $1 + (n-2)2^{2[n/2]-1}$.

Proof. From definition of h we have an extension $K \to W_n \to \mathbb{Z}_2^s$, where $s = 2[\frac{n}{2}]$ and $[x]$ is the largest integer not exceeding x. Again by the definition of h the commutator subgroup W_n' of W_n has index 1 for n even and index 2 for n odd in the group K. Hence $s = 2[\frac{n}{2}]$. Computing (fractional) Euler characteristics we get (see [2, Corollary 5.6, p. 245]) from the Euler characteristic formula $e(K) = e(W_n)/e(\mathbb{Z}_2^s) = e(W_n)2^s = (1 - \frac{n}{2})2^s$. Which gives the announced rank.
Analogously $e(W_n') = (1 - \frac{n}{2})2^n$ and $e(K/W_n') = 2^{s-n}$. □

Remark 4. Let n be an even number then

$$P_n(x, \mathbb{Q}) = P_n(x, \mathbb{F}_2) \mod 2.$$

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