Algebra Colloquium **7**:3 (2000) 295–303

Algebra Colloquium

Generalized Neuwirth Groups and Seifert Fibered Manifolds*

Andrzej Szczepański
Institute of Mathematics, University of Gdańsk
ul. Wita Stwosza 57, 80-952, Gdańsk, Poland
E-mail: matas@paula.univ.gda.pl

Andrei Vesnin
Sobolev Institute of Mathematics
pr. Koptyuga 4, Novosibirsk 630090, Russia
E-mail: vesnin@math.nsc.ru

Received 7 March 1999
Revised 8 July 1999

Communicated by A.C. Kim

Abstract. The topological properties of the generalized Neuwirth groups , ^k_n are discussed. For example, we demonstrate that the group , ^k_n is the fundamental group of the Seifert fibered space Σ_n^k. Moreover, we discuss some other invariants and algebraic properties of the above groups.

2000 Mathematics Subject Classification: 20F34, 57M05, 57M60

Keywords: Seifert manifolds, flat manifolds, spine complex

1 Introduction

Let $\mathbb{F}_n$ be the free group on free generators $x_1, x_2, \ldots, x_n$. Let $\theta : \mathbb{F}_n \to \mathbb{F}_n$ be the automorphism such that $\theta(x_i) = x_{i+1}$ for $i = 1, 2, \ldots, n-1$ and $\theta(x_n) = x_1$. For any word $w \in \mathbb{F}_n$, we define a group $G_n(w) = \mathbb{F}_n/R$, where R is the normal closure in $\mathbb{F}_n$ of the set $\{w, \theta(w), \ldots, \theta^{n-1}(w)\}$ (cf. [10]). A group G is said to have a *cyclic presentation* if $G = G_n(w)$ for some n and $w \in \mathbb{F}_n$. Among the well-studied examples of cyclically presented groups are the *Fibonacci groups* $F(r, n) = G_n(x_1 x_2 \cdots x_r x_{r+1}^{-1})$, where $n \geq 3$ and $r \geq 2$ (cf. [10]).

*This work was supported by Polish grant (BW-5100-5-0259-9) and the Russian Foundation for Basic Research (grant number 98-01-00699).

Our aim is to study a class of cyclically presented groups which are the fundamental groups of 3-manifolds. We note that it is impossible to find an algorithm for determinig whether an arbitrary finite presentation presents a fundamental group of a 3-manifold (cf. [19]).

In [14], Neuwrith considers the groups $\Gamma_n = F(n-1, n)$ $(n \geq 3)$ as fundamental groups $\pi_1(\overline{M}_n)$ of closed orientable 3-manifolds. It was proven in [4] that the *Neuwirth manifold* $\overline{M}_n$ is PL-homeomorphic to the Seifert fibered space

$$\Sigma_n = (0\ o\ 0 \mid -1;\ \underbrace{(2,1),\ (2,1),\ \ldots,(2,1)}_{n \text{ times}}).$$

In this paper, we introduce a family of cyclically presented groups $\Gamma_n^k = G_n(x_1 x_2 \cdots x_{n-1} x_n^{-k})$ with $n \geq 3$ and $k \geq 1$. We call these groups *generalized Neuwirth groups.* We show that Γ_n^k is the fundamental group of the closed orientable 3-manifold M_n^k. Moreover, we prove that M_n^k is homeomorphic to the Seifert fibred space

$$\Sigma_n^k = (0\ o\ 0 \mid -1;\ \underbrace{(k+1,1),\ (k+1,1),\ \ldots,(k+1,1)}_{n \text{ times}})$$

in respect to notations from [12]. As a corollary, we show that the groups Γ_n^k are automatic for $(n,k) \neq (4,1)$ and give a formula for the Casson–Walker–Lescop invariant of the manifolds Σ_n^k.

2 An Example

Let Γ_3^2 be a generalized Neuwirth group

$$\Gamma_3^2 = \langle x_1, x_2, x_3 \mid x_1 x_2 = x_3^2,\ x_2 x_3 = x_1^2,\ x_3 x_1 = x_2^2 \rangle.$$

Proposition 2.1. *The group Γ_3^2 is the fundamental group of a compact orientable flat 3-manifold.*

Proof. Let P_3^2 be the unit cube in the Euclidean 3-space $\mathbb{E}^3$ with notations of vertices according to Fig. 1.

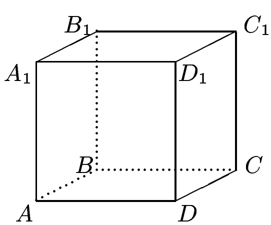

Fig. 1. The cube P_3^2.

Consider the orientation-preserving Euclidean isometries which identify pairs of faces of the polyhedron P_3^2 as follows:

$$\begin{aligned} a &: A_1ABB_1 \to B_1C_1CB, \\ b &: A_1B_1C_1D_1 \to D_1DCC_1, \\ c &: A_1D_1DA \to ABCD. \end{aligned}$$

Edges of P_3^2 are divided in three classes of equivalent:

$$\begin{aligned} x_1 &: A_1A \xrightarrow{a} B_1C_1 \xrightarrow{b} DC \xrightarrow{c^{-1}} AD \xrightarrow{c^{-1}} A_1A, \\ x_2 &: A_1B_1 \xrightarrow{b} D_1D \xrightarrow{c} BC \xrightarrow{a^{-1}} B_1B \xrightarrow{a^{-1}} A_1B_1, \\ x_3 &: A_1D_1 \xrightarrow{c} AB \xrightarrow{a} C_1C \xrightarrow{b^{-1}} D_1C_1 \xrightarrow{b^{-1}} A_1D_1. \end{aligned}$$

Therefore, by the Poincare Theorem [20], the group generated by a, b, c has the polyhedron P_3^2 as its fundamental domain, and has the presentation $\langle a, b, c \mid ab = c^2,\ bc = a^2,\ ca = b^2\rangle$, which is isomorphic to $,{}_3^2$. Since all dihedral angles of P_3^2 are equal to $\pi/2$ and all cycles of edges are of length 4, we conclude that the quotient space $\mathbb{E}^3/,{}_3^2$ is a flat 3-manifold. □

Let us denote $M_3^2 = \mathbb{E}^3/,{}_3^2$. We recall [22] that there are only six compact orientable flat 3-manifolds. Computing the first homology group $H_1(M_3^2) = \mathbb{Z}_3 \times \mathbb{Z}$, we see that M_3^2 is the flat manifold $\mathcal{G}_3$ in notations from [22]. According to [15, p. 138], $M_3^2 = (0\ o\ 0 \mid -1;\ (3,1),(3,1),(3,1))$ and it can be obtained as the T^2-bundle over S^1 with matrix $\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$ of the characteristic map of order 3.

Proposition 2.2. *Let* $\mathrm{Out}(,{}_3^2)$ *be the outer automorphism group of* $,{}_3^2$. *Then* $\mathrm{Out}(,{}_3^2)$ *is a dihedral group of order* 12.

Proof. (See [9] for an alternative description.) Since $,{}_3^2$ is the fundamental group of the flat 3-manifold $\mathcal{G}_3$, we have the short exact sequence

$$0 \to \mathbb{Z}^3 \to ,{}_3^2 \to \mathbb{Z}_3 \to 0.$$

From the definition (cf. [22]), the action of $\mathbb{Z}_3$ on $\mathbb{Z}^3$ is given by a matrix

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let N denote the normalizer in $GL(3, \mathbb{Z})$ of the above matrix. Then N acts in a natural way on $H^2(\mathbb{Z}_3, \mathbb{Z}^3)$. Let $\alpha \in H^2(\mathbb{Z}_3, \mathbb{Z}^3)$ denote the cohomology class giving rise to the above extension, and N_α its stabilizer in N. Then $\mathbb{Z}_3$ is a normal subgroup of N_α. By Theorem V.1.1 of [5, p. 172], we have the short exact sequence

$$0 \to H^1(\mathbb{Z}_3, \mathbb{Z}^3) \to \mathrm{Out}(,{}_3^2) \to N_\alpha/\mathbb{Z}_3 \to 0.$$

It is not difficult to see that $H^1(\mathbb{Z}_3, \mathbb{Z}^3) \simeq \mathbb{Z}_3$ and $N_\alpha/\mathbb{Z}_3 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (cf. [2]). In order to obtain our result, it is enough to remark that the group $\mathrm{Out}(,{}^2_3)$ is not abelian.

Indeed, let us consider two isometries of the cube P_3^2, i.e., the rotation ρ of order 3 about the line passing through vertices A_1 and C, and the rotation σ of order 2 about the line passing through midpoints of edges AB and C_1D_1. These symmetries induce automorphisms of the group $,{}^2_3$ such that $\rho : a \mapsto b$, $b \mapsto c$, $c \mapsto a$, and $\sigma : a \mapsto c^{-1}$, $b \mapsto b^{-1}$, $c \mapsto a^{-1}$. It is easy to check that the automorphisms ρ and σ do not commute and are not inner. □

Note that, under the action of the group $,{}^2_3$, we obtain the tesselation of $\mathbb{E}^3$ by the right-angled unit cube P_3^2. Applying the results of Grayson [7] on the growth function of the tesselation of the Euclidean space by a right-angled cube, we have the following:

Proposition 2.3. *The growth function for the group $,{}^2_3$ in respect to the set of generators $S = \{x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}\}$ is*

$$f_S(t) = \left(\frac{1+t}{1-t}\right)^3 = 1 + 6t + 18t^2 + 38t^3 + 66t^4 + 102t^5 + \cdots .$$

3 The Properties of the Groups Γ_n^k

It was shown in [14] that $,{}^1_n = ,{}_n$ is the fundamental group of a 3-manifold. In this section, we will demonstrate that the same is true for the groups $,{}^k_n$ with $k \geq 2$.

Let us recall that a *squashable complex* C on the 2-sphere $S^2 = \partial B^3$ is a combinatorial complex together with a grouping of its 2-cells in oppositely oriented identifiable pairs (cf. [18]).

The identification squashes the ball-sphere pair (B^3, S^2) into a combinatorial complex pair $(M, K) = (B^3/\sim, S^2/\sim)$. The resulting quotient map $q = q(C) : (B^3, S^2) \to (M, K)$ is called a *squashing map* associated with C. The quotient complex M is, in general, an orientable closed connected pseudo-manifold in the sense of [17]. It was shown in [17] that M is a closed orientable 3-manifold if and only if the Euler characteristic of M vanishes. Furthermore, the embedded 2-complex is a *spine* of M, i.e., M minus the open 3-cell $q(\mathring{B}^3)$ collapses onto K. If K has a single 0-cell, then K is the canonical 2-complex associated with a finite presentation

$$\langle x_1, x_2, \ldots, x_n \mid r_1, r_2, \ldots, r_n \rangle$$

of the fundamental group $\pi_1(K) \cong \pi_1(M)$. The 1-skeleton $K^{(1)}$ of K is a bouquet of n oriented circles, also denoted by $x_1, x_2, \ldots, x_n$. The 2-cells c_j of K correspond bijectively to the relators r_j which determine loops in K as the corresponding attaching maps $h_j : \partial B^2 \to K^{(1)}$.

Consider the complex P_n^k with $n(n+k-1)$ edges and $2n$ faces, each of which is an $(n+k-1)$-gon. The combinatorial structure of P_n^k is clear from Fig. 2, where the complex P_5^3 for the group , $_5^3$ is pictured. Let A_i and A_i', $i = 1, \dots, n$, be faces with edges numerated by

$$\{i, i+1, \dots, i+n-2, \underbrace{i+n-1, \dots, i+n-1}_{k \text{ times}}\}$$

such that the corresponding relation r_i is $x_i x_{i+1} \cdots x_{i+n-2} = x_{i+n-1}^k$ with subscripts reduced modulo n. As we see, the enumerations of edges of faces of A_i and A_i' induce opposite orientations of faces.

Denote by M_n^k the complex obtained under the squashing map associated with P_n^k. As we see, the pseudo-manifold M_n^k has one 0-cell, n 1-cells, n 2-cells and one 3-cell. Hence, its Euler characteristic vanishes and M_n^k is a manifold.

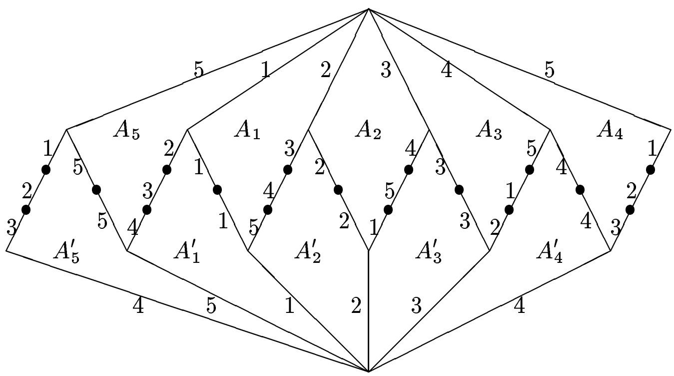

Fig. 2. The polyhedron P_5^3.

Thus, we obtain the following result.

Theorem 3.1. *For $n \geq 3$ and $k \geq 1$, the group , $_n^k$ is the fundamental group of the orientable closed 3-manifold M_n^k.*

We recall that a Seifert fibered 3-manifold is completely determined by a system of invariants (cf. [4, p. 180] and [12, Chpt. 4])

$$(\epsilon\, g\, \epsilon' \mid b;\ (\alpha_1, \beta_1), (\alpha_2, \beta_2), \dots, (\alpha_r, \beta_r)).$$

Here, (i) g is the genus of the closed connected surface (called the orbit-manifold) obtained from the Seifert manifold by identifying each fiber to a point, (ii) $\epsilon = \epsilon' = 0$ if the Seifert manifold and its orbit-manifold are orientable, (iii) $b = -(e_0 + \sum_{i=1}^r \beta_i/\alpha_i) \in \mathbb{Z}$, where e_0 is the rational Euler number of the Seifert fibration, (iv) the (α_j, β_j)'s are pairs of coprime integers with $0 < \beta_j < \alpha_j$ such that $\beta_j/\alpha_j \in (\mathbb{Q}/\mathbb{Z})^*$ ($\mathbb{Q}$ is the field of rational numbers and $(\mathbb{Q}/\mathbb{Z})^*$ is the group $\mathbb{Q}/\mathbb{Z}$ minus zero) characterize the holonomy of the exceptional fibers.

The fundamental group of a Seifert fiber space with the above invariants is well known (cf. [12] and [15, p. 91]). A standard presentation for its oriented case is

$$\langle a_1, b_1, \ldots, a_g, b_g, q_1, \ldots, q_r, h \mid q_1 q_2 \cdots q_r [a_1, b_1] \cdots [a_g, b_g] = h^b, \\ [a_i, h] = [b_i, h] = 1, \ [q_i, h] = 1, \ q_i^{\alpha_i} h^{\beta_i} = 1 \ \forall \ i \, \rangle.$$

Let us denote by Σ_n^k the Seifert 3-manifold with invariants

$$(0 \, o \, 0 \mid -1; \, \underbrace{(k+1, 1), \ (k+1, 1), \ \ldots, (k+1, 1)}_{n \text{ times}} \,).$$

From above, we have

$$\pi_1(\Sigma_n^k) = \langle q_1, q_2, \ldots, q_n, h \mid q_1 q_2 \cdots q_n = h^{-1} \\ [q_i, h] = 1, \, q_i^{k+1} h = 1 \ \forall \ i \, \rangle. \tag{1}$$

Since $,{}^k_n = G_n(x_1 x_2 \cdots x_{n-1} x_n^{-k})$, we see

$$,{}^k_n = \langle x_1, x_2, \ldots, x_n \mid x_1 x_2 \cdots x_{n-1} = x_n^k, \\ x_2 x_3 \cdots x_n = x_1^k, \, \ldots, \, x_n x_1 \cdots x_{n-2} = x_{n-1}^k \, \rangle. \tag{2}$$

Lemma 3.2. *The group* $,{}^k_n$ *is isomorphic to the group* $\pi_1(\Sigma_n^k)$.

Proof. To obtain the presentation (2) from (1), we identify $x_i = q_i$ for $i = 1, \ldots, n$. From $q_i^{k+1} = h^{-1}$, we have $h = x_1^{-(k+1)} = \cdots = x_n^{-(k+1)}$. By the relation $q_1 \cdots q_n = h^{-1}$, we obtain $x_1 \cdots x_n = x_n^{k+1}$, so $x_1 \cdots x_{n-1} = x_n^k$. Thus, $x_2 \cdots x_n = x_1^{-1} x_n^{k+1} = x_1^k$. By the same way, we obtain all relations in (2).

Conversely, to obtain the presentation (1) from (2), we identify $q_i = x_i$ for $i = 1, \ldots, n$ and $h = x_n^{-(k+1)}$. Therefore, $q_1 \cdots q_n = h^{-1}$. By the relations $x_2 \cdots x_{n-1} = x_1^{-1} x_n^k$ and $x_2 \cdots x_{n-1} = x_1^k x_n^{-1}$, we have $x_1^{k+1} = x_n^{k+1} = h^{-1}$. Analogously, we obtain $h = x_1^{-(k+1)} = \cdots = x_n^{-(k+1)}$. Therefore, $[q_i, h] = 1$ and $q_i^{k+1} h = 1$. □

By the standard calculations, one can find the first integral homology group of M_n^k (here, $\mathbb{Z}_0 = \mathbb{Z}$).

Lemma 3.3. *The group* $H_1(M_n^k, \mathbb{Z})$ *is generated by* $n-1$ *elements and is isomorphic to*

$$\underbrace{\mathbb{Z}_{k+1} \oplus \mathbb{Z}_{k+1} \oplus \cdots \oplus \mathbb{Z}_{k+1}}_{n-2 \text{ times}} \oplus \mathbb{Z}_{(k+1)\,|n-k-1|}.$$

The next theorem generalizes Theorem 4 of [4].

Theorem 3.4. *The manifold M_n^k with $n \geq 3$ and $k \geq 1$ is homeomorphic to the Seifert fibered 3-manifold Σ_n^k.*

Proof. We recall that if the fundamental groups of two large Seifert manifolds are isomorphic, then the manifolds are homeomorphic [15, p. 97]. From the definition of the Seifert manifold Σ_n^k, it is *large* (see [15, p. 91]) for $n \geq 3$ and $k \geq 2$ with one exceptional case $(n,k) = (3,1)$ for which the proof is given in [4]. By Lemma 3.2, the groups $\pi_1(\Sigma_n^k)$ and $\pi_1(M_n^k) \cong$, $_n^k$ are isomorphic. To prove the theorem, we will demonstrate that the manifold M_n^k is a large Seifert manifold. By Lemma 3.3, for its Heegaard genus, we have $h(M_n^k) \geq n-1$. Furthermore, the Heegaard genus of Σ_n^k is exactly $n-1$ (see [1]). According to [15, p. 92], the group , $_n^k$ is a non-trivial free product with amalgamation and the subgroup generated by the element $h = x_1^{-(k+1)} = \cdots = x_n^{-(k+1)}$ is the unique maximal cyclic normal subgroup of , $_n^k$. Further, $\langle h \rangle$ is infinite and is contained in the center of , $_n^k$, hence , $_n^k$ has a non-trivial center.

Since M_n^k is prime and different from $S^1 \times S^2$ (use $h(M_n^k) > 3$), the manifold M_n^k is irreducible (i.e., any embedded 2-sphere in M_n^k bounds a 3-cell). By Theorem 6.2 of [3, p. 77], it follows that M_n^k is homeomorphic to a Seifert fibered manifold with orientable orbit-manifold. Since M_n^k is irreducible and , $_n^k$ is infinite, the manifold M_n^k is aspherical, i.e., $\pi_j(M_n^k) = 0$ for $j \geq 2$ (see Lemma 1.1.5 of [21]). Thus, M_n^k is a $K($, $_n^k, 1)$-space with infinite torsion-free fundamental group. Hence, M_n^k is large [15, p. 91]. Therefore, the large Seifert manifolds Σ_n^k and M_n^k are homeomorphic. □

Corollary 3.5. *The groups* , $_n^k$ *are automatic for all* $(n,k) \neq (4,1)$.

Proof. Recall that the geometry on a Seifert fibred manifold is completely determined by the Euler characteristic of the base orbifold and the Euler number of the fibration [16]. Calculating these values for manifolds Σ_n^k, one can see that Σ_3^1 is a spherical manifold, Σ_3^2 is a Euclidean manifold, Σ_4^1 is a nilpotent manifold, and for $k > 2$, the manifolds Σ_{k+1}^k are $(\mathbb{H}^2 \times \mathbb{R})$-manifolds, and all others are $\widetilde{SL_2}$-manifolds.

Hence, it is enough to apply Theorem 12.2.5 in [6, p. 296]. □

At the end of this section, due to results of Lescop [11], we calculate the Casson–Walker–Lescop λ-invariant of the above considered manifolds.

Corollary 3.6. *Denote by λ_n^k the Casson–Walker–Lescop invariant of the manifold M_n^k. Then*

$$\lambda_n^{p-1} = \begin{cases} \frac{p^{n-2}}{24}\left(n^2p^2 - n(3p^3 + p^2 + 2p - 1) + 2p^2(p^2+2)\right) & \text{if } p < n, \\ \frac{n^{n-1}}{24}(n^2 - 2n - 1) & \text{if } p = n, \\ -\frac{p^{n-2}}{24}\left(n^2p^2 - n(3p^3 - 5p^2 + 2p - 1) + 2p^2(p^2 - 3p + 2)\right) & \text{if } p > n. \end{cases}$$

Proof. By [11, p. 97], the invariant λ of a Seifert fibered space

$$(0\ o\ 0 \mid b;\ (\alpha_i, \beta_i),\ i = 1, \dots, n)$$

is given by the following formula:

$$\lambda = \Big(\frac{s(e)}{24}\big(2 - n + \sum_{i=1}^{n} \frac{1}{\alpha_i^2}\big) + \frac{|e|\,e}{24} - \frac{e}{8} - \frac{|e|}{2} \sum_{i=1}^{n} s(\beta_i, \alpha_i)\Big) \prod_{i=1}^{n} \alpha_i,$$

where we use the Euler number of the fibration $e = b + \sum_{i=1}^{n} \beta_i/\alpha_i$ and the Dedekind sum

$$s(q,p) = \sum_{i=1}^{|p|} \Big(\Big(\frac{i}{p}\Big)\Big)\Big(\Big(\frac{qi}{p}\Big)\Big) \quad \text{with} \quad ((x)) = \begin{cases} 0 & \text{if } x \in \mathbb{Z}, \\ x - [x] - \frac{1}{2} & \text{otherwise,} \end{cases}$$

where $[x]$ denotes the integer part of x and

$$s(e) = \begin{cases} \text{sign}(e) & \text{if } e \neq 0, \\ -1 & \text{if } e = 0. \end{cases}$$

For $M_n^k = \Sigma_n^k$, we have $b = -1$, $\alpha_i = k+1$, $\beta_i = 1$, $e = -1 + \frac{n}{k+1}$, and

$$s(1, k+1) = \sum_{i=1}^{k} \big(\frac{i}{k+1} - \frac{1}{2}\big)^2 = \frac{k(k-1)}{12(k+1)}.$$

Let us introduce a notation $p = k+1$. If $p = n$, then $e = 0$ and

$$\lambda_n^{n-1} = -\frac{1}{24}\big(2 - n + \frac{1}{n}\big)\, n^n.$$

If $e \neq 0$ (i.e., $p \neq n$), then

$$\lambda_n^{p-1} = \Big(\frac{\epsilon}{24}\big(2 - n + \frac{n}{p^2}\big) + \frac{\epsilon(n-p)^2}{24} - \frac{n-p}{8} - \frac{\epsilon(n-p)(p-1)(p-2)}{24p}\Big) p^n,$$

where $\epsilon = \text{sign}(e)$. The result follows. □

Corollary 3.7. *If $\overline{M}_n$ with $n \geq 3$ is a Neuwirth manifold, then*

$$\lambda(\overline{M}_n) = \frac{n^{n-3}}{12}\,(4n^2 - 31n + 48).$$

As it was remarked above, the manifold M_4^1 is the unique nil-manifold in the family introduced by Neuwirth [14], and we have $\lambda_4^1 = -2$. For the spherical manifold M_3^1, we have $\lambda_3^1 = -3/4$. The manifold M_3^2 is flat and $\lambda_3^2 = 3/4$. Since the manifolds M_n^k are two-fold branched coverings of the 3-dimensional sphere branched over Montesinos links, the invariants

λ_n^k for rational homology spheres M_n^k can also be calculated from Jones' polynomials and the signatures of Montesinos links (see [13]).

References

[1] M. Boileau, H. Zieschang, Heegaard genus of closed orientable Seifert 3-manifolds, *Invent. Math.* **76** (1984) 455–468.
[2] H. Brown, R. Bülow, J. Neubüser, H. Wondratschek, H. Zassenhaus, *Crystallographic Groups of Four-dimensional Space*, Wiley, New York, 1978.
[3] G. Burde, H. Zieschang, *Knots*, Walter de Gruyter, Berlin-New York, 1985.
[4] A. Cavicchioli, Neuwirth manifolds and colourings of graphs, *Aequationes Math.* **44** (1992) 168–187.
[5] L.S. Charlap, *Bieberbach Groups and Flat Manifolds*, Springer-Verlag, New York-Berlin, 1986.
[6] D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, W. Thurston, *Word Processing in Groups*, Jones and Bartlett, Boston, 1992.
[7] M.A. Grayson, Geometry and growth in three dimensions, Ph.D. Thesis, Princeton University, 1983.
[8] J. Hempel, 3-*Manifolds,* Annals of Mathematics Studies, Vol. 86, Princeton University Press, Princeton, 1976.
[9] J. Hillman, Flat 4-manifolds groups, *New Zeland J. Math.* **24** (1995) 23–40.
[10] D. Johnson, *Topics in the Theory of Group Presentations,* London Mathematical Society, Lecture Note Series, Vol. 42, Cambridge University Press, 1980.
[11] C. Lescop, *Global Surgery Formula for the Casson–Walker Invariant,* Annals of Mathematics Studies, Vol. 140, Princeton University Press, Princeton, 1996.
[12] J.M. Montesinos, *Classical Tessellations and Three Manifolds,* Universitext, Springer-Verlag, 1987.
[13] D. Mullins, The generalized Casson invariants for 2-fold branched coverings of S^3 and the Jones polynomial, *Topology* **32** (1993) 419–438.
[14] L. Neuwirth, An algorithm for the construction of 3-manifolds from 2-complexes, *Proc. Camb. Phil. Soc.* **64** (1968) 603–613.
[15] P. Orlik, *Seifert Manifolds*, Lecture Notes in Mathematics, Vol. 291, Springer-Verlag, 1972.
[16] P. Scott, The geometries of 3-manifolds, *Bull. London Math. Soc.* **15** (1983) 401–487.
[17] H. Seifert, W. Threllfall, *Lehrbuch der Topologie*, Teubner, Leipzig, 1929.
[18] A.J. Sieradski, Combinatorial squashings, 3-manifolds, and the third homotopy of groups, *Invent. Math.* **84** (1986) 121–139.
[19] J. Stallings, On the recursiveness of sets of presentations of 3-manifold groups, *Fund. Math.* **51** (1962/63) 191–194.
[20] E.B. Vinberg, O.V. Shvartsman, Discrete groups of motions of spaces of constant curvature, in: *Encycl. Math. Sc., Geometry II*, Springer-Verlag, Berlin-Heidelberg-NewYork, 1993, pp. 139–248.
[21] F. Waldhausen, On irreducible 3-manifolds which are sufficiently large, *Ann. Math.* **87** (1968) 56–88.
[22] J.A. Wolf, *Spaces of Constant Curvature*, McGraw-Hill, New York, 1972.