Simplicial Volume and Bounded Cohomology

1 Simplicial Volume

Simplicial volume is a topological invariant that was introduced by Gromov in his proof of Mostow Rigidity theorem. In particular, there is a quite simple proof that for closed orientable hyperbolic manifolds it is proportional to the hyperbolic volume divided by the supremum to the volume of the geodesic simplex. From that, we immediately see that the volume of a hyperbolic manifold is topological invariant.

First step in toward the definition of simplicial volume is an ℓ^1-norm a space of singular chains. So, let X be a topological space and let k be a non-negative integer. The real vector space of singular k-chains shall be denoted $C_k(X)$. Now, let $c = \sum_i \alpha_i \sigma_i \in C_k(X)$ be an arbitrary element, where $\alpha_i \in \mathbb{R}$ and σ_i are singular k-simplices. The norm shall be defined $\|c\| := \sum_i |\alpha_i|$. This norm induces a semi-norm on the level of homology, which can be interpreted as a distance of a homology class from the subspace of boundaries. The precise formula is

$$\|\alpha\| = \inf\{\|z\| \mid [z] = \alpha\},$$

where $\alpha \in H_k(X)$.

Definition 1 *Let M be a closed, orientable manifold. Then the simplicial volume $\|M\|$ is defined in the following way*

$$\|M\| = \|[M]\|,$$

where $[M]$ denotes a fundamental class of M.

The simplest manifold we can imagine is a circle S^1. It is easy to see that $\|S^1\| = 0$, simply consider simplices σ_k which wind arould the circle k times.

Moreover, the simplicial volume of a sphere or torus of any dimension is zero. It is a conseqence of the following theorem

Theorem 2 *Let $f : M_1 \to M_2$ be a continuous map between closed orientable n-dimensional manifolds M_1 and M_2, then*

$$\|M_1\| \geq |\deg f| \cdot \|M_2\|$$

Proof. First, notice that the induced map $f_\bullet : C_n(M_1) \to C_n(M_2)$ does not increase the norm. Now let $z \in C_n(M_1)$ be a cycle representing a fundamental class of M_1, we have $\|f_\bullet(z)\| \leq \|z\|$. Taking infimum over representatives of a fundamental class of M_1 we get

$$\inf\{\|(\deg f)^{-1} f_\bullet(z)\| \mid [z] = [M_1]\} \leq |\deg f|^{-1} \|M_1\|.$$

By noticing that $(\deg f)^{-1} f_\bullet(z)$ represents a fundamental class of M_2, we get our result.

□

Moreover, the for the special case when f is a covering map, we have the following theorem

Theorem 3 *Let $f : M_1 \to M_2$ be a covering map of a finite degree d between closed orientable n-dimensional manifolds M_1 and M_2, then*

$$\|M_1\| = d \cdot \|M_2\|$$

Proof. Inequality $\|M_1\| \geq d \cdot \|M_2\|$ is true by the previous theorem. To prove the other inequality, notice that thre is the *transfer mapping* $g : C_n(M_2) \to C_n(M_1)$, that is defined on an arbitrary simplex σ with the formula $g(\sigma) = \sum_i \tilde{\sigma}_i$, where $\tilde{\sigma}_i : \Delta^n \to M_1$ denote lifts of simplex σ.

Let $z \in C_n(M_2)$ represent the fundamental class of M_2. Then, we see that $g(z)$ represents the fundamental class of M_1. Indeed, suppose $g(z)$ represents $\alpha \cdot [M_1]$ for some $\alpha \in \mathbb{R}$. Then $f(g(z))$ represents $d \cdot \alpha \cdot [M_2]$. But, $f \circ g$ equals d times the identity on M_2, hence $\alpha = 1$.

Moreover, we can see that for every $c \in C_n(M_2)$ we have $\|g(c)\| = d \cdot \|c\|$. So if we take infimum over representatives $z \in C_n(M_2)$ of the fundamental class of M_2, we get

$$\inf\{\|g(z)\| \mid [z] = [M_2]\} = d \cdot \|M_1\|.$$

But the left hand side in the above formula is greater than $\|M_1\|$, thus we got our result.

□

This definition can be a motivation to define the simplicial volume for nonorientable manifolds.

Definition 4 *Let M be a closed, nonorientable manifold and let $\tilde{M} \to M$ its orientable double cover. Then we define*

$$\|M\| := \frac{1}{2}\|\tilde{M}\|.$$

Simplicial volume of orientable closed surfaces can be calculated almost explicitly. The case of a sphere and torus, as we mentioned above, is clear by Theorem 2.

So consider the case of the surfaces Σ_g of genus $g > 1$. Each of these surfaces can be equipped with a Riemannian metric of constant curvature -1. Now, take a cycle $z = \sum_i \alpha_i \sigma_i$ representing a fundamental class of Σ_g. We can perform a process of *straightening*, described in a detail in [2, Chapter 6], substituting each σ_i with a geodesic simplex τ_i with the same vertices. Let Ω be a volume form on Σ_g, we have the following inequality

$$\left|\int_{\Sigma_g} \Omega\right| = \left|\int_{\Sigma_i \alpha_i \tau_i} \Omega\right| \leq \sum_i |\alpha_i| \left|\int_{\tau_i} \Omega\right| \leq \pi \sum_i |\alpha_i|.$$

The last part of the above inequality is a consequence of the fact that volume of a geodesic simplex is bouded by π. Moreover, by the Gauss-Bonnet theorem the volume of Σ_g (which is the leftmost integral in the above inequality) equals $-2\pi\chi(\Sigma_g)$. From that, we get

$$-2\chi(\Sigma_g) \leq \left\|\sum_i \alpha_i \tau_i\right\|.$$

The straightening process is norm decreasing, so when calculating the simplicial volume we can consider ony *straight cycles*. Thus, we get lower bound for the simplicial volume

$$-2\chi(\Sigma_g) \leq \|\Sigma_g\|.$$

This lower bound actually is the simplicial volume. To see this, we consider Σ_g as a $4g$-gon with an aproperiate gluing. Now, choose a vertex of this $4g$-gon and connect it to $4g-3$ remainging vertices with a geodesic segment. This gives us triangulation of Σ_g with $4g-2$ triangles. This triangulation defines a representative of the fundamental class of Σ_g – just take each triangle with coefficeient 1. Hence, we have the upper bound for the simplicial volume

$$\|\Sigma_g\| \leq 4g - 2 = -2\chi(\Sigma_g) + 2.$$

Now, consider d-sheeted covering $\Sigma \to \Sigma_g$. We know that $\chi(\Sigma) = d \cdot \chi(\Sigma_g)$, and by Theorem 3 we have $\|\Sigma\| = d \cdot \|\Sigma_g\|$. Thus, applying the above inequality to Σ we obtain

$$\|\Sigma_g\| \leq -2\chi(\Sigma_g) + 2d^{-1}.$$

Because d can be arbitrarily large, we get our result

$$\|\Sigma_g\| = -2\chi(\Sigma_g).$$

So we calculated the simplicial volume for every orientable closed surface.

2 Bounded Cohomology

Let X be a topological space. By $C^*(X)$ we shall denote a chain-complex of simplicial cochains. Let $c \in C^*(X)$, we say that c is *bounded* there is a constant C such that for every singular simplex $\sigma \in C(\Delta^k, X)$ we have $c(\sigma) < C$. The group of bounded k-cochains shall be denoted by $C_b^k(X)$.

We define ℓ^∞-norm for bounded cochains:

$$\|c\|_\infty = \sup_{\sigma \in C(\Delta^k, X)} |c(\sigma)|.$$

Notice that coboundary of a bounded chain is bounded. Indeed, let $c \in C_b^k(X)$, then

$$|(\delta c)(\sigma)| = \left| \sum_{i=0}^{k} (-1)^i c(\partial_i \sigma) \right| \leq (k+1)\|c\|_\infty.$$

Thus $C_b^*(X)$ with coboundary defined as a restriction of δ to bounded cochains is a chain-complex. Bounded cohomology $H_b^*(X)$ are defined as homology of this chain-complex. It is equipped with a seminorm denoted also by $\|\cdot\|_\infty$ induced by the one on the level of cochains.

Moreover, let $f : X \to Y$ be a continuous map. Restriction of the induced mapping to bounded cochains $f_b^* : H_b^*(Y) \to H_b^*(X)$ is a bounded map. Indeed, take k-simplex σ, we have $|(f_b^k c)(\sigma)| = |c(f_b^k \sigma)| \leq \|c\|_\infty$ hence $\|f_b^k c\|_\infty \leq \|c\|_\infty$ and thus $\|f_b^k\|_\infty \leq 1$. Hence f_b^* is bounded on the level of cochains. Finally, it is not hard to see that f_b^* has to be bounded on the level of cohomology, and its norm is less then one.

There is a canonical homomorphism $H_b^*(X) \to H^*(X;\mathbb{R})$ induced by the inclusion of bounded cochains. There is a result by Thurston about this homomorphism

Theorem 5 *(Thurston) Let X be a closed manifold which admits negatively curved Riemannian metric, then the canonical homomorphism $H_b^*(X) \to H^*(X;\mathbb{R})$ is suriective.*

2.1 Simplicial volume from the dual point of view

The simplicial volume can be calculated using the semi-norm on bouded cohomology groups. This allows us to prove many interesting results. In particluar, the fact that simplicial volume of manifolds with amenable fundamental groups vanish (cf. Section 2.2).

In this section we prove the following result

Theorem 6 *Let X be a topological space. Take $\alpha \in H_k(X)$, where k is a nonnegative integer. Then we have:*

1. $\|\alpha\| = 0$ *if and only if for every* $\phi \in H_b^k(X)$ *we have* $\langle \phi, \alpha \rangle = 0$,
2. *if* $\|\alpha\| \neq 0$, *then* $\|\alpha\| = \sup\left\{ \frac{1}{\|\phi\|_\infty} \mid \phi \in H_b^k(X), \langle \phi, \alpha \rangle = 1 \right\}$.

Proof. Let us start with taking $z \in C_k(X)$ that represents homology class α. First, assume that $z \in \overline{B_k(X)}$ (here the overline denotes closure in Gromov norm). It is equivalent to assuming $\|\alpha\| = 0$.

Every cocycle equals zero on $B_k(X)$. Moreover, bouded cocycles are continuous with respect to ℓ^1-norm on $C_k(X)$. Thus, for every bounded cocycle f we have $\langle f, z \rangle = 0$, that equivalent to saying that for every $\phi \in H_b^k(X)$ we have $\langle \phi, \alpha \rangle = 0$.

On the other hand, if $z \notin \overline{B_k(X)}$, we can define a functional g on $\overline{B_k(X)} \oplus \langle z \rangle$, such that $\ker g = \overline{B_k(X)}$ and $g(z) = 1$. By the Hahn-Banach theorem g has a bounded extension to $C_k(X)$, we shall denote it by g also. From that, we see that there exists a bounded cohomology class ϕ, represented by g, such that $\langle \phi, \alpha \rangle \neq 0$. Hence, we proved that the norm of α equals zero if and only if its product with every bounded cohomology class equals zero.

Now, let us consider the case when $\|\alpha\| \neq 0$. We shall use the functional g that was defined before. There exists a sequence $(b_n)_{n=1}^\infty$ of elements of

$B_k(X)$ such that $\|z+b_n\|$ converges to $\|\alpha\|$. Thus, for every $b \in \overline{B_k(X)}$ we have $\|z+b_n\| \leq \|z+b\|$ if n is large enough. From that,

$$g\left(\frac{z+b_n}{\|z+b_n\|}\right) \geq g\left(\frac{z+b}{\|z+b\|}\right),$$

if n is large enough. Moreover, every element of norm one in $\overline{B_k(X)} \oplus \langle z \rangle$ can be written as $(z+b')/\|z+b'\|$ for some element $b' \in \overline{B_k(X)}$. If we take a sequence of elements in $\overline{B_k(X)} \oplus \langle z \rangle$ approximating the norm of g, and then use the above inequality in the limit we will get:

$$\lim_{n\to\infty} g\left(\frac{z+b_n}{\|z+b_n\|}\right) = \lim_{n\to\infty} \|z+b_n\|^{-1} = \|\alpha\|^{-1} \geq \|g\|_\infty.$$

The norm of g in the above inequality was in fact calculated in the direct sum $\overline{B_k(X)} \oplus \langle z \rangle$, but knowing that g was extended to $C_k(X)$ by using Hahn-Banach theorem, we know that it is equal to ℓ^∞-norm. Hence, we proved that

$$\|\alpha\| \leq \sup\left\{\frac{1}{\|\phi\|_\infty} \mid \phi \in H_b^k(X), \langle \phi, \alpha \rangle = 1\right\}$$

Now let us take an arbitrary bounded cocycle such that $f(z) = 1$. We see that $f(z/\|z\|) = 1/\|z\|$, therefore the operator norm of $1/\|f\|_\infty \leq \|z\|$. From that, we get $1/\|\phi\|_\infty \leq \|\alpha\|$, by taking first infimum over representatives of α and then infimum over cocycles f that represent a bounded cohomology class $\phi \in H_b^k(X)$ (here we use the fact that for an arbitrary coboundary δh we have $\langle \delta h, z \rangle = 0$). Now, taking supremum over ϕ we get:

$$\|\alpha\| \geq \sup\left\{\frac{1}{\|\phi\|_\infty} \mid \phi \in H_b^k(X), \langle \phi, \alpha \rangle = 1\right\}.$$

And this ends the proof of our theorem.

□

2.2 Bounded cohomology for amenable groups

A very important result is a criterion for amenability of groups. Bounded cohomology for a group π is defined as a bounded cohomology of its $K(\pi, 1)$. Then we have a theorem

Theorem 7 *(Hirsch-Thurston) If a group π is amenable, then*

$$H_b^k(\pi) = 0, \quad \textit{for } k > 0$$

This theorem is a consequence of

Theorem 8 *(Trauber's vanishing theorem) Let $f : Y \to X$ be a regular covering with amenable Galois group π. Then the induced map $f_b^* : H_b^*(X) \to H_b^*(Y)$ is iniective and isometric.*

Proof. Let $\mu : \ell_\infty(\pi) \to \mathbb{R}$ denote the invariant mean that exists on π by the assumption. On the level of cochains we define a chain-maping $m^* : C_b^*(Y) \to C_b^*(X)$ by the formula

$$(m^k c)(\sigma) = \mu\{\pi \ni \gamma \mapsto c(\gamma\tilde{\sigma})\}$$

where $c \in C_b^*(Y)$ and $\tilde{\sigma}$ is a lift of the simplex σ. The value of the right hand side is indepentent on the lift since the mean μ is invariant. Moreover, we can see that

$$|\mu\{\pi \ni \gamma \mapsto c(\gamma\tilde{\sigma})\}| \leq \|\mu\|_\infty \|\{\pi \ni \gamma \mapsto c(\gamma\tilde{\sigma})\}\|_\infty \leq \|\mu\|_\infty \|c\|_\infty$$

because μ has norm equal one by definition we see that $\|m^k\|_\infty \leq 1$.

The values of an induced mapping $f^* : C_b^*(X) \to C_b^*(Y)$ are cochains that assign to a simplex σ upstairs value of its projection downstairs. As a consequence it is an invariant cochain in Y. Conversely, every invariant cochain arises that way. From that, we see that the composition $m^* \circ f^*$ equals identity on $H_b^*(X)$. Thus, f^* is an injection on the level of cohomology.

Moreover, we know that $\|m^k\|_\infty \leq 1$ and $\|f^k\|_\infty \leq 1$ and its composition has norm one. Thus $\|f^k\|_\infty = 1$.

□

Proof of Theorem 7. Consider the universal covering of $K(\pi, 1)$, it is a space homotopically equivalent to a point, and thus it has trivial bounded cohomology groups. By Theorem 8 bounded cohomology groups of $K(\pi, 1)$ are subgroups of it, hence they are also trivial.

□

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