If I were a rich density

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Abstract upper density

Examples of upper densities

- Asymptotic density: $\overline{d}(A) = \limsup_{n\to\infty} \frac{|A\cap\{1,\dots,n\}|}{n}$
- Logarithmic density: $\overline{\delta}(A) = \limsup_{n\to\infty} \frac{\sum_{k\in A\cap\{1,\dots,n\}} \frac{1}{k}}{\sum_{k\leqslant n} \frac{1}{k}}$
- Uniform density (aka Banach density):
  $\overline{u}(A) = \limsup_{n\to\infty} \max_{k\in\mathbb{N}} \frac{|A\cap\{k+1\dots,k+n\}|}{n}$

Definition

An abstract upper density on $\mathbb{N}$ is a function $\delta : \mathcal{P}(\mathbb{N}) \to [0,1]$ that satisfies the following properties:

1. $\delta(\mathbb{N}) = 1$,
2. if $F \subseteq \mathbb{N}$ is finite then $\delta(F) = 0$,
3. if $A \subseteq B$ then $\delta(A) \leqslant \delta(B)$,
4. $\delta(A\cup B) \leqslant \delta(A) + \delta(B)$.

Abstract upper densities and ideals

Proposition

If $\delta : \mathcal{P}(\mathbb{N}) \to [0,1]$ is an abstract upper density, then

$$\mathcal{Z}_\delta = \{A \subseteq \mathbb{N} : \delta(A) = 0\}$$

is an ideal on $\mathbb{N}$ i.e.

1. if $A, B \in \mathcal{Z}_\delta$ then $A \cup B \in \mathcal{Z}_\delta$,
2. if $A \subseteq B$ and $B \in \mathcal{Z}_\delta$ then $A \in \mathcal{Z}_\delta$,
3. $\mathcal{Z}_\delta$ contains all finite subsets of $\mathbb{N}$,
4. $\mathbb{N} \notin \mathcal{Z}_\delta$.

Abstract upper densities and ideals

Proposition

Let $\mathcal{I}$ be an ideal. The function $\delta : \mathcal{P}(\mathbb{N}) \to [0,1]$ given by

$$\delta(A) = \begin{cases} 0 & \text{if } A \in \mathcal{I}, \\ 1 & \text{otherwise} \end{cases}$$

is an abstract upper density and $\mathcal{I} = \mathcal{Z}_\delta$.

Proof

Straightforward.

Question (G. Gerkos, 2013)

Let $\mathcal{I}$ be an ideal. Does there is a "nice" abstract upper density δ such that $\mathcal{Z}_\delta = \mathcal{I}$, where "nice" would mean the properties of the familiar densities consider in number theory?

Nice = translation invariance

Definition

- Translation invariant density: $\delta(A+k) = \delta(A)$ for all A and k
- Translation invariant ideal: $A+k \in \mathcal{I}$ for all $A \in \mathcal{I}$ and k

Proposition

Let $\mathcal{I}$ be a translation invariant ideal. The function

$$\delta(A) = \begin{cases} 0 & \text{if } A \in \mathcal{I}, \\ 1 & \text{otherwise} \end{cases}$$

is a translation invariant abstract upper density and $\mathcal{I} = \mathcal{Z}_\delta$.

Proof

Straightforward.

Nice = richness

Definition

Rich density: for every $r \in [0,1]$ there is $A \subseteq \mathbb{N}$ with $\delta(A) = r$.

Theorem (M. Di Nasso–R. Jin, 2018)

If $\mathcal{I}$ is a summable ideal then there is a rich abstract upper density δ with $\mathcal{I} = \mathcal{Z}_\delta$.

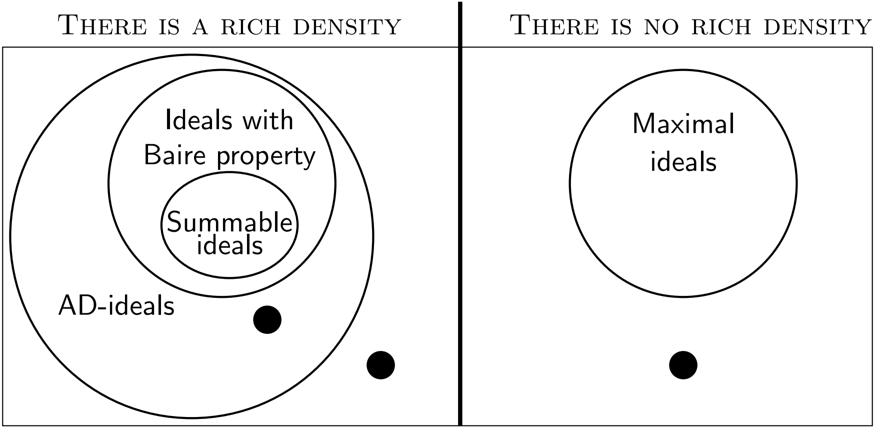
Definition of a summable ideal

There is $f : \mathbb{N} \to [0,\infty)$ such that

$$\mathcal{I} = \{A \subseteq \mathbb{N} : \sum_{n\in A} f(n) < \infty\}.$$

$\mathrm{Fin} = \{A : A \text{ is finite}\}$ and $\mathcal{I}_{1/n} = \{A : \sum_{n\in A} \frac{1}{n} < \infty\}$ are summable ideals.

Nice = richness
What we know



Nice = richness; case of AD-ideals

Definition

$\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ is $\mathcal{I}$ almost disjoint family ($\mathcal{I}$-AD family) if $A \notin \mathcal{I}$ and $A \cap B \in \mathcal{I}$ for any distinct $A, B \in \mathcal{A}$.

Theorem

If there exists an $\mathcal{I}$-AD family of cardinality $\mathfrak{c}$, then there is a rich abstract upper density δ such that $\mathcal{Z}_\delta = \mathcal{I}$.

Sketch of the proof

- Extend $\mathcal{A}$ to a maximal $\mathcal{I}$-AD-family.
- Enumerate: $\mathcal{A} = \{A_\alpha : \alpha < \mathfrak{c}\}$ and $(0,1) = \{r_\alpha : \alpha < \mathfrak{c}\}$.
- Define: $\delta(A) = \sup\{r_\alpha : A_\alpha \cap A \notin \mathcal{I}\}$
- δ is an abstract upper density
- δ is rich [use $\mathcal{I}$-almost disjointness of $\mathcal{A}$]
- $\mathcal{Z}_\delta = \mathcal{I}$ [use maximality of $\mathcal{A}$]

Nice = richness; case of ideals with Baire property

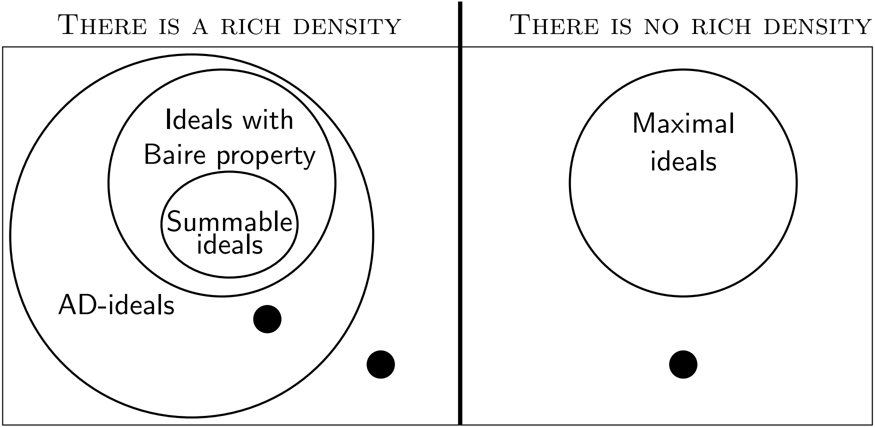
Corollary

If $\mathcal{I}$ has the Baire property, then there is a rich abstract upper density δ such that $\mathcal{Z}_\delta = \mathcal{I}$. In particular, summable ideals.

Sketch of the proof

- Talagrand: $\exists k_1 < k_2 < \ldots (\exists^\infty_n [k_n, k_{n+1}) \subseteq A \implies A \notin \mathcal{I})$
- Let: $I_n = [k_n, k_{n+1})$
- Take: Fin-AD family $\mathcal{A} \subseteq \mathcal{P}(\mathbb{N})$ of cardinality $\mathfrak{c}$.
- Define: $C_A = \bigcup_{n\in A} I_n$ for every $A \in \mathcal{A}$
- Let: $\mathcal{C} = \{C_A : A \in \mathcal{A}\}$
- $\mathcal{C}$ is of cardinality $\mathfrak{c}$ [because $\mathcal{A}$ has cardinality $\mathfrak{c}$]
- $C_A \in \mathcal{I}^+$ for every $A \in \mathcal{A}$ [use Talagrand's characterization]
- $\mathcal{C}$ is $\mathcal{I}$-AD family [use $\mathcal{I}$-almost disjointness of $\mathcal{A}$]

Nice = richness: what we know



Nice = richness; density without AD family

Example

Let

- $\mathcal{J}$ be a maximal ideal,
- $\mathcal{I} = \{\emptyset\} \otimes \mathcal{J}$,
- $\delta(A) = \sum_{A_n \notin \mathcal{J}} \frac{1}{2^n}$.

Then

- $\mathcal{I}$ is an ideal,
- δ is an abstract upper density,
- δ is rich [use binary expansion],
- $\mathcal{Z}_\delta = \mathcal{I}$,
- there is no $\mathcal{I}$-AD family of cardinality $\mathfrak{c}$ (even uncountable) [use maximality of $\mathcal{J}$].

Nice = richness; AD family but without Baire property

Example

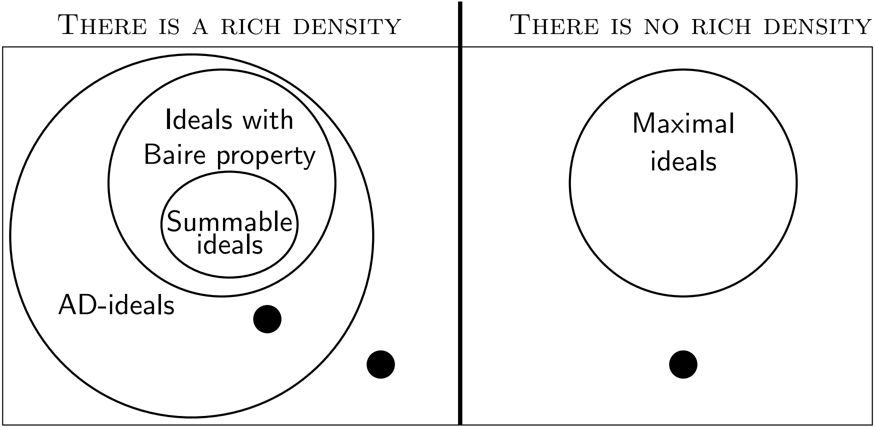
Let

- $\mathcal{J}$ be a maximal ideal,
- $\mathcal{I} = \mathrm{Fin} \otimes \mathcal{J}$.

Then

- $\mathcal{I}$ is an ideal,
- there is $\mathcal{I}$-AD family of cardinality $\mathfrak{c}$ [use Fin-AD family of cardinality $\mathfrak{c}$]
- $\mathcal{I}$ does not have the Baire property [use maximality of $\mathcal{J}$ and Plewik theorem saying that intersection of countably many maximal ideals does not have the Baire property]

Nice = richness: what we know



Nice = richness; ideals without density

Theorem

If $\mathcal{I}$ is a maximal ideal, then there is no rich abstract upper density δ with $\mathcal{Z}_\delta = \mathcal{I}$.

Proof

- If $A \in \mathcal{I}$ then $\delta(A) = 0$.
- If $A \notin \mathcal{I}$, then $\mathbb{N} \setminus A \in \mathcal{I}$ [use maximality]
- hence

$$1 = \delta(\mathbb{N}) \leqslant \delta(A) + \delta(\mathbb{N} \setminus A) = \delta(A) + 0 = \delta(A) \leqslant 1$$

- so $\delta(A) = 1$.
- Thus δ takes only 2 values, so it is not rich.

Nice = richness; non-maximal ideals without density

Example

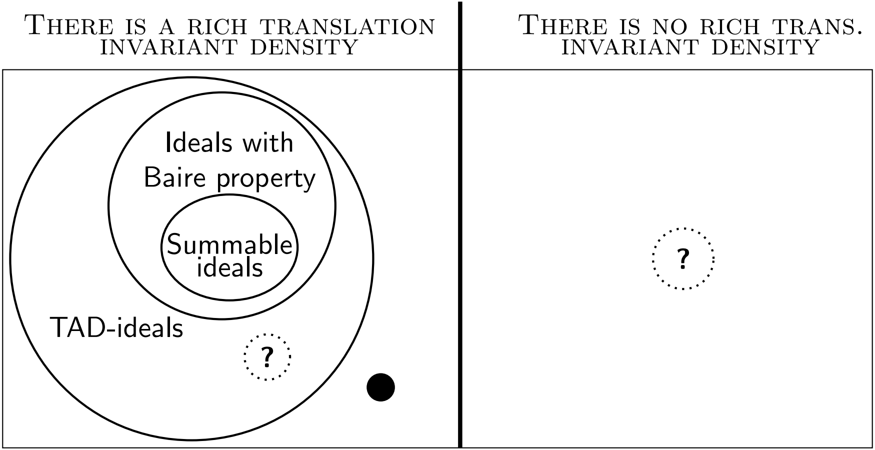
Let

- $\mathcal{I}_1, \mathcal{I}_2$ be maximal ideals,
- $\mathcal{I} = \mathcal{I}_1 \oplus \mathcal{I}_2$.

Then

- $\mathcal{I}$ is an ideal
- $\mathcal{I}$ is non-maximal [since $\mathcal{P}(\mathbb{N}) \oplus \mathcal{I}_2$ is a larger ideal]
- If $A \notin \mathcal{I}$, then $A \cap (\{1\} \times \mathbb{N}) \notin \mathcal{I}_1$ or $A \cap (\{2\} \times \mathbb{N}) \notin \mathcal{I}_2$.
- Say $A \cap (\{1\} \times \mathbb{N}) \notin \mathcal{I}_1$.
- Then $\delta(A \cap (\{1\} \times \mathbb{N})) = \delta(\{1\} \times \mathbb{N})$ [since $\mathcal{I}_1$ is maximal]
- Since $\delta(A) \geqslant \delta(A \cap (\{1\} \times \mathbb{N})) = \delta(\{1\} \times \mathbb{N}) \neq 0$, so
- $\mathrm{ran}(\delta) = \{0\} \cup [\delta(\{1\} \times \mathbb{N}), 1]$.
- Thus δ is not rich.

Nice = richness + translation invariance: what we (don't) know



Nice = richness + translation invariance: non TAD-ideal with density

Example

Let

- Let $\mathcal{J}$ be a maximal ideal,
- $\delta(A) = \mathcal{J}\text{–}\lim \frac{|A \cap \{1,\ldots,n\}|}{n}$

Then

- δ is well defined [since $\mathcal{J}$ is maximal]
- δ is an abstract upper density (it is finitely additive measure)
- δ is translation invariant.
- δ is rich [since asymptotic density is so]
- Let $\mathcal{I} = \mathcal{Z}_\delta$.
- There is no $\mathcal{I}$-AD family of cardinality $\mathfrak{c}$ [since finite measures satisfy ccc]