Two $\mathfrak{b}$ or not two $\mathfrak{b}$?

cardinal numbers *inspired* by convergence of sequences of functions

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The talk is based on a joint work with Adam Kwela.

Quasinormal convergence of sequences of functions

$f, f_n : X \to \mathbb{R}$, $n = 1, 2, \ldots$

Pointwise convergence

$$(f_n) \xrightarrow{P} f \iff \forall \varepsilon > 0 \, \forall x \in X \, \exists k \, \forall n > k \, (|f_n(x) - f(x)| < \varepsilon)$$

Quasinormal convergence

$$(f_n) \xrightarrow{QN} f \Leftrightarrow \exists (\varepsilon_n) \to 0 \, \forall x \in X \, \exists k \, \forall n > k \, (|f_n(x) - f(x)| < \varepsilon_n)$$

Relationship between pointwise and quasinormal convergence

- $(f_n) \xrightarrow{QN} f \implies (f_n) \xrightarrow{P} f$ [always "yes"]
- $(f_n) \xrightarrow{P} f \not\Longrightarrow (f_n) \xrightarrow{QN} f$ [in general "no"]

E.g. if $X = \mathbb{R}$, there are even continuous functions f_n and f such that $(f_n) \xrightarrow{P} f$ but $(f_n) \overset{QN}{\not\to} f$

QN-spaces

Definition (Bukovský-Recław-Repický, 1991)

X is a QN-space if

$$(f_n) \xrightarrow{P} f \implies (f_n) \xrightarrow{QN} f$$

for any continuous functions $f, f_n : X \to \mathbb{R}$

Theorem (Bukovský-Recław-Repický, 1991)

- continuous image of a QN-space is a QN-space
- QN-spaces are not hereditary,
- . . . many properties of QN-spaces

Minimal size of non-QN-spaces

Definition

X is a QN-space if

$$(f_n) \xrightarrow{P} f \implies (f_n) \xrightarrow{QN} f$$

for any continuous functions $f, f_n : X \to \mathbb{R}$

Definition (Bukovský-Recław-Repický, 1991)

$$\text{non(QN-space)} = \min\{|X| : X \text{ in } \underline{\text{not}} \text{ a QN-space}\}$$

Theorem

$$\aleph_0 < \text{non(QN-space)} \leq \mathfrak{c}$$

Purely combinatorial characterization of non(QN-space)

$$\text{non(QN-space)} = \min\{|X| : X \text{ in } \underline{\text{not}} \text{ a QN-space}\}$$

Theorem (Bukovska (≥); Bukovský-Recław-Repický (≤); 1991)

$$\text{non(QN-space)} = \mathfrak{b}$$

Definition (The bounding number)

$$\mathfrak{b} = \min\Big\{|\mathcal{F}| : \mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}} \land \mathcal{F} \text{ is unbounded from above with respect to the ordering } \leq^*\Big\}$$

$$= \min\Big\{|\mathcal{F}| : \neg\Big(\exists (b_n) \in \mathbb{N}^{\mathbb{N}} \, \forall (a_n) \in \mathcal{F} \, ((a_n) \leq^* (b_n))\Big)\Big\},$$

where $(a_n) \leq^* (b_n) \iff \exists k \, \forall n > k \, (a_n \leq b_n)$.

Ideals on $\mathbb{N}$

Definition

A family $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is an ideal on $\mathbb{N}$ if

1. $\emptyset \in \mathcal{I}$ and $\mathbb{N} \notin \mathcal{I}$,
2. $A \subseteq B \in \mathcal{I} \implies A \in \mathcal{I}$,
3. $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$,
4. $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$.

Example

1. $\mathrm{Fin} = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$
2. $\mathcal{I}_{1/n} = \left\{A \subseteq \mathbb{N} : \sum_{n \in A} \frac{1}{n} < \infty\right\}$
3. $\mathcal{I}_d = \left\{A \subseteq \mathbb{N} : \lim_{n \to \infty} \frac{|A \cap \{1,\ldots,n\}|}{n} = 0\right\}$

Ideal convergence

Definition

A sequence (x_n) of reals is convergent to $a \in \mathbb{R}$ if

$$\forall \varepsilon > 0 \, \exists k \, \forall n > k \; (|x_n - a| < \varepsilon)$$

equivalently:

$$\forall \varepsilon > 0 \, \exists A \in \mathrm{Fin} \, \forall n \in \mathbb{N} \setminus A \; (|x_n - a| < \varepsilon) \, .$$

Definition

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$.
A sequence (x_n) of reals is $\mathcal{I}$-convergent to $a \in \mathbb{R}$ if

$$\forall \varepsilon > 0 \, \exists A \in \mathcal{I} \, \forall n \in \mathbb{N} \setminus A \; (|x_n - a| < \varepsilon) \, .$$

In symbols: $(x_n) \xrightarrow{\mathcal{I}} a$

Ideal pointwise and quasinormal convergence

Let $\mathcal{I}, \mathcal{J}, \mathcal{K}$ be ideals on $\mathbb{N}$.
Let $f, f_n : X \to \mathbb{R}$, $n = 1, 2, \ldots$

Ideal pointwise convergence

$$(f_n) \xrightarrow{\mathcal{K}} f \iff \forall x \in \mathbb{R} \left((f_n(x)) \xrightarrow{\mathcal{K}} f(x) \right) \iff$$

$$\forall x \in X \left(\forall \varepsilon > 0\, \exists A \in \mathcal{K}\, \forall n \in \mathbb{N} \setminus A\, (|f_n(x) - f(x)| < \varepsilon) \right)$$

Ideal quasinormal convergence

$$(f_n) \xrightarrow{QN(\mathcal{I},\mathcal{J})} f \iff$$

$$\exists (\varepsilon_n) \xrightarrow{\mathcal{J}} 0\, \forall x \in X\, \exists A \in \mathcal{I}\, \forall n \in \mathbb{N} \setminus A\, (|f_n(x) - f(x)| < \varepsilon_n)$$

In general there are no relations:

- $(f_n) \xrightarrow{QN(\mathcal{I},\mathcal{J})} f \;\not\Longrightarrow\; (f_n) \xrightarrow{\mathcal{K}} f$
- $(f_n) \xrightarrow{\mathcal{K}} f \;\not\Longrightarrow\; (f_n) \xrightarrow{QN(\mathcal{I},\mathcal{J})} f$

When does ideal quasinormal convergence imply poinwise?

Theorem (Staniszewski, 2017)

The following conditions are equivalent:

1. $(f_n) \xrightarrow{QN(\mathcal{I},\mathcal{J})} f \implies (f_n) \xrightarrow{\mathcal{K}} f$
2. $\mathcal{I} \subseteq \mathcal{K}$ and $\mathcal{J} \subseteq \mathcal{K}$.

• The above characterization is expressed without mentioning the cardinality of X.
• A characterization of the reversed implication:

$$(f_n) \xrightarrow{\mathcal{K}} f \implies (f_n) \xrightarrow{QN(\mathcal{I},\mathcal{J})} f$$

will depend on the cardinality of X as it was in the ordinary case.

Minimal size of non ideal-QN-space

Definition

X is a QN($\mathcal{I}$,$\mathcal{J}$,$\mathcal{K}$)-space if

$$(f_n) \xrightarrow{\mathcal{K}} f \implies (f_n) \xrightarrow{QN(\mathcal{I},\mathcal{J})} f$$

for any continuous functions $f, f_n : X \to \mathbb{R}$

Definition

$$\text{non(QN}(\mathcal{I},\mathcal{J},\mathcal{K})\text{-space)} =$$

$$\min\left(\{|X| : X \text{ in } \underline{\text{not}} \text{ a QN}(\mathcal{I},\mathcal{J},\mathcal{K})\text{-space}\} \cup \{\infty\}\right)$$

Theorem (Staniszewski, 2017)

There are ideals for $\mathcal{I}, \mathcal{J}, \mathcal{K}$ with non(QN($\mathcal{I}$,$\mathcal{J}$,$\mathcal{K}$)-space) $= \infty$
i.e. for any space X every $\mathcal{K}$-pointwise convergent sequence
$f_n : X \to \mathbb{R}$ is automatically QN($\mathcal{I}$,$\mathcal{J}$)-convergent.

Combinatorial characterization of non(QN($\mathcal{I}$,$\mathcal{J}$,$\mathcal{K}$)-space)

Theorem (Staniszewski, 2017)

• If $\mathcal{K} \subseteq \mathcal{I}$, then

$$\text{non}(\text{QN}(\mathcal{I},\mathcal{J},\mathcal{K})\text{-space}) = \mathfrak{b}_S(\mathcal{I},\mathcal{J},\mathcal{K}),$$

where $\mathfrak{b}_S(\mathcal{I},\mathcal{J},\mathcal{K})$ is a cardinal defined in a purely combinatorial manner (we'll see the definition in a moment).

• If $\mathcal{K} \not\subseteq \mathcal{I}$, there are some partial results about non(QN($\mathcal{I}$,$\mathcal{J}$,$\mathcal{K}$)) but mainly for ideals with the hereditary Baire property (thus: there is still something to explore).

The cardinal $\mathfrak{b}_S(\mathcal{I}, \mathcal{J}, \mathcal{K})$

Notation

$\mathcal{P}_{\mathcal{I}}$ is the family of all partitions $\{A_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ such that $A_n \in \mathcal{I}$ for each $n \in \omega$.

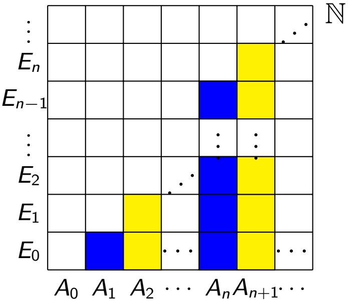
Definition (Staniszewski, 2017)

$$\mathfrak{b}_S(\mathcal{I}, \mathcal{J}, \mathcal{K}) = \min\left\{|\mathcal{E}| : \mathcal{E} \subseteq \mathcal{P}_{\mathcal{K}} \wedge \forall \{A_n\} \in \mathcal{P}_{\mathcal{J}} \, \exists \{E_n\} \in \mathcal{E} \bigcup_{n\in\mathbb{N}} \left(A_{n+1} \cap \bigcup_{i \leq n} E_i\right) \notin \mathcal{I}\right\}.$$

At the next slide we have a picture that can help us to grasp the definition ...

The cardinal $\mathfrak{b}_S(\mathcal{I},\mathcal{J},\mathcal{K})$

$$\mathfrak{b}_S(\mathcal{I},\mathcal{J},\mathcal{K}) = \min\Big\{|\mathcal{E}| : \mathcal{E}\subseteq \mathcal{P}_{\mathcal{K}} \land \forall\{A_n\}\in\mathcal{P}_{\mathcal{J}}\,\exists\{E_n\}\in\mathcal{E}\ \bigcup_{n\in\mathbb{N}}\Big(A_{n+1}\cap\bigcup_{i\le n}E_i\Big)\notin\mathcal{I}\Big\}.$$



$$\bigcup_{n\in\mathbb{N}}\Big(A_{n+1}\cap\bigcup_{i\le n}E_i\Big) =$$

$A_1 \cap E_0 \cup$

$A_2 \cap (E_0 \cup E_1) \cup$

$\ldots$

$A_n \cap (E_0 \cup \cdots \cup E_{n-1}) \cup$

$A_{n+1} \cap (E_0 \cup \cdots \cup E_{n-1} \cup E_n) \cup \ldots$

Is $\mathfrak{b}_S(\mathcal{I},\mathcal{J},\mathcal{K})$ a bounding number, anyway?

Theorem (Bukovska ($\geq$); Bukovský-Recław-Repický ($\leq$); 1991)

$$\mathrm{non}(\text{QN-space}) = \mathfrak{b},$$

where
$\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \mathbb{N}^\mathbb{N} \wedge \mathcal{F}$ is unbounded from above wrt $\leq^*\}$.

Theorem (Staniszewski, 2017)

- If $\mathcal{K} \subseteq \mathcal{I}$, then

$$\mathrm{non}(\text{QN}(\mathcal{I},\mathcal{J},\mathcal{K})\text{-space}) = \mathfrak{b}_S(\mathcal{I},\mathcal{J},\mathcal{K}),$$

where

$$\mathfrak{b}_S(\mathcal{I},\mathcal{J},\mathcal{K}) = \min\left\{|\mathcal{E}| : \mathcal{E} \subseteq \mathcal{P}_\mathcal{K} \wedge \forall\{A_n\} \in \mathcal{P}_\mathcal{J}\, \exists\{E_n\} \in \mathcal{E} \; \bigcup_{n\in\mathbb{N}}\left(A_{n+1} \cap \bigcup_{i\leq n} E_i\right) \notin \mathcal{I}\right\}.$$

Yes, $\mathfrak{b}_S(\mathcal{I},\mathcal{J},\mathcal{K})$ is, indeed, a bounding number!

Theorem (F.-Kwela)

$$\mathfrak{b}_S(\mathcal{I},\mathcal{J},\mathcal{K}) = \mathfrak{b}(\geq_{\mathcal{I}} \cap (\mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}}))$$

Definition (Vojtáš, 1993)

For a relation $R \subseteq X \times Y$,

$$\mathfrak{b}(R) = \min\left(\{|B| : B \subseteq \mathrm{dom}(R) \wedge B \text{ is } R\text{-unbounded}\} \cup \{\infty\}\right)\},$$

where $B \subseteq \mathrm{dom}(R)$ is <u>R-unbounded</u> if

$$\neg\left(\exists y \in \mathrm{ran}(R)\, \forall x \in B\, ((x,y) \in R)\right)$$

Example

$$\mathfrak{b}(\leq^*) = \mathfrak{b},$$

where $\leq^* = \{((a_n),(b_n)) \in \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N} : (a_n) \leq^* (b_n)\} \subseteq \mathbb{N}^\mathbb{N} \times \mathbb{N}^\mathbb{N}$.

$\mathfrak{b}(\leq^{\mathcal{I}})$ and $\mathfrak{b}(\geq^{\mathcal{I}})$

Definition

For an ideal $\mathcal{I}$ and sequences $(a_n), (b_n) \in \mathbb{N}^{\mathbb{N}}$,

$$(a_n) \leq^{\mathcal{I}} (b_n) \iff \exists A \in \mathcal{I} \, \forall n \in \mathbb{N} \setminus A \, (a_n \leq b_n) \, .$$

Theorem (Farkas-Soukup, 2009)

If $\mathcal{I}$ has the Baire property, $\quad \mathfrak{b}(\leq^{\mathcal{I}}) = \mathfrak{b}$.

Definition

For an ideal $\mathcal{I}$ and sequences $(a_n), (b_n) \in \mathbb{N}^{\mathbb{N}}$,

$$(a_n) \geq^{\mathcal{I}} (b_n) \iff \exists A \in \mathcal{I} \, \forall n \in \mathbb{N} \setminus A \, (a_n \geq b_n) \, .$$

Then $\quad \mathfrak{b}(\geq^{\mathcal{I}}) = \infty, \quad$ as every set is $\geq^{\mathcal{I}}$-bounded by the sequence $(b_n) = (0, 0, 0, \dots)$

The relation $\geq_{\mathcal{I}} \cap (\mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}})$

Theorem (F.-Kwela)

$$\mathfrak{b}_S(\mathcal{I}, \mathcal{J}, \mathcal{K}) = \mathfrak{b}(\geq_{\mathcal{I}} \cap (\mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}}))$$

Definition

For an ideal $\mathcal{I}$,

$$\begin{aligned}\mathcal{D}_{\mathcal{I}} &= \{(a_n) \in \mathbb{N}^{\mathbb{N}} : (a_n) \text{ is constant only on sets from } \mathcal{I}\} \\ &= \{(a_n) \in \mathbb{N}^{\mathbb{N}} : \forall c \in \mathbb{N}\,(\{n \in \mathbb{N} : a_n = c\} \in \mathcal{I})\}\end{aligned}$$

Definition

$$\geq_{\mathcal{I}} \cap (\mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}}) = \{((a_n), (b_n)) \in \mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}} : (a_n) \geq^{\mathcal{I}} (b_n)\}$$

non(QN($\mathcal{I}$,$\mathcal{J}$,$\mathcal{K}$)-space) $= \mathfrak{b}(\geq_{\mathcal{I}} \cap (\mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}}))$

$\mathcal{D}_{\mathcal{I}} = \{(a_n) \in \mathbb{N}^{\mathbb{N}} : (a_n) \text{ is constant only on sets from } \mathcal{I}\}$

$\geq_{\mathcal{I}} \cap (\mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}}) = \{((a_n),(b_n)) \in \mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}} : (a_n) \geq^{\mathcal{I}} (b_n)\}$

Theorem (Staniszewski)

If $\mathcal{K} \subseteq \mathcal{I}$, non(QN($\mathcal{I}$,$\mathcal{J}$,$\mathcal{K}$)-space) $= \mathfrak{b}_S(\mathcal{I},\mathcal{J},\mathcal{K})$

Theorem (F.-Kwela)

$\mathfrak{b}_S(\mathcal{I},\mathcal{J},\mathcal{K}) = \mathfrak{b}(\geq_{\mathcal{I}} \cap (\mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}}))$

Corollary (independently, Repický)

If $\mathcal{K} \subseteq \mathcal{I}$, non(QN($\mathcal{I}$,$\mathcal{J}$,$\mathcal{K}$)-space) $= \mathfrak{b}(\geq_{\mathcal{I}} \cap (\mathcal{D}_{\mathcal{K}} \times \mathcal{D}_{\mathcal{J}}))$

Let's examin $\mathfrak{b}_S(\mathcal{I}, \mathcal{J}, \mathcal{K})$

Question 1

What are the possible values of $\mathfrak{b}_S(\mathcal{I}, \mathcal{J}, \mathcal{K})$?

Question 2

What are relationships between $\mathfrak{b}_S(\mathcal{I}, \mathcal{J}, \mathcal{K})$ for various triples $(\mathcal{I}, \mathcal{J}, \mathcal{K})$?

To make life easier we considered only triples such that at most one ideal is not equal Fin i.e. we have 8 triples:

$$(F, F, F), \quad (F, F, \mathcal{I}), (F, \mathcal{I}, F), (\mathcal{I}, F, F),$$

$$(F, \mathcal{I}, \mathcal{I}), (\mathcal{I}, F, \mathcal{I}), (\mathcal{I}, \mathcal{I}, F), \quad (\mathcal{I}, \mathcal{I}, \mathcal{I}),$$

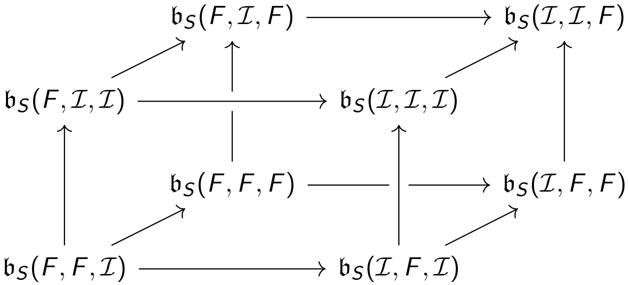
where $F = \mathrm{Fin}$.

$\mathfrak{b}_S(\mathcal{I},\mathcal{J},\mathcal{K})$ is piecewise monotone

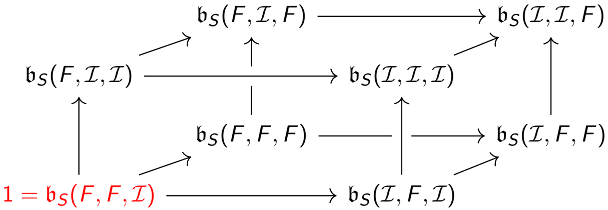
Theorem

- If $\mathcal{I} \subseteq \mathcal{I}'$, then $\mathfrak{b}_S(\mathcal{I},\mathcal{J},\mathcal{K}) \leq \mathfrak{b}_S(\mathcal{I}',\mathcal{J},\mathcal{K})$.
- If $\mathcal{J} \subseteq \mathcal{J}'$, then $\mathfrak{b}_S(\mathcal{I},\mathcal{J},\mathcal{K}) \leq \mathfrak{b}_S(\mathcal{I},\mathcal{J}',\mathcal{K})$.
- If $\mathcal{K} \subseteq \mathcal{K}'$, then $\mathfrak{b}_S(\mathcal{I},\mathcal{J},\mathcal{K}) \geq \mathfrak{b}_S(\mathcal{I},\mathcal{J},\mathcal{K}')$.

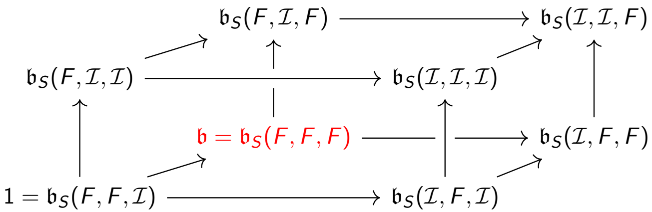
Corollary



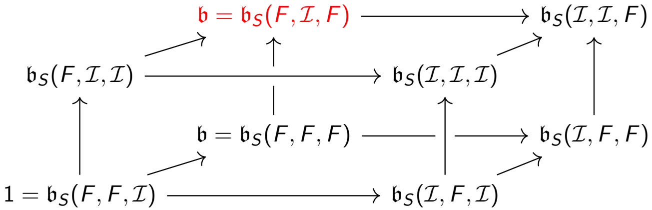
Known values in the diagram: $\mathfrak{b}_S(F, F, \mathcal{I})$



Known values in the diagram: $\mathfrak{b}_S(F, F, F)$

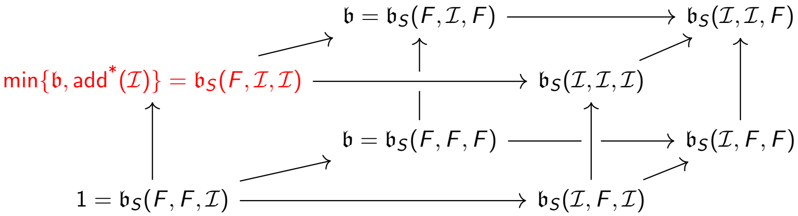


Known values in the diagram: $\mathfrak{b}_S(F,\mathcal{I},F)$



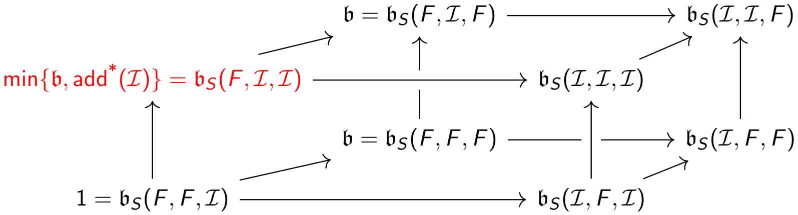
Known values in the diagram: $\mathfrak{b}_S(F,\mathcal{I},\mathcal{I})$

Theorem (F.-Kwela)



$$\text{add}^*(\mathcal{I}) = \min(\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \land \neg [\exists A \in \mathcal{I} \forall F \in \mathcal{F} (F \setminus A \in \mathrm{Fin})]\}$$

Known values in the diagram: $\mathfrak{b}_S(F,\mathcal{I},\mathcal{I})$



$\mathrm{add}^*(\mathcal{I}) = \min(\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \land \neg [\exists A \in \mathcal{I} \forall F \in \mathcal{F} \, (F \setminus A \in \mathrm{Fin})]\}$

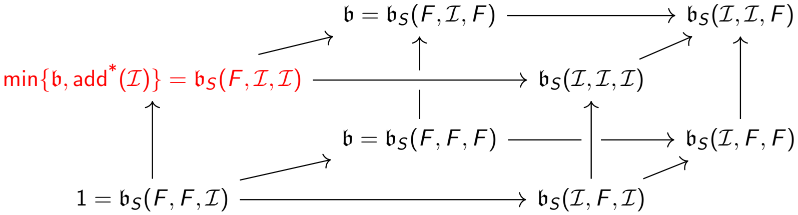
Corollary

$\mathfrak{b}_S(F,\mathcal{I},\mathcal{I}) = \aleph_0$ for non P-ideals.

$\mathcal{I}$ is a P-ideal if $\forall A_1, A_2, \dots \in \mathcal{I} \, \exists B \in \mathcal{I} \, \forall n \, (A_n \setminus B \in \mathrm{Fin})$.

Proof: For non-P-ideals, $\mathrm{add}^*(\mathcal{I}) = \aleph_0 < \mathfrak{b}$.

Known values in the diagram: $\mathfrak{b}_S(F, \mathcal{I}, \mathcal{I})$



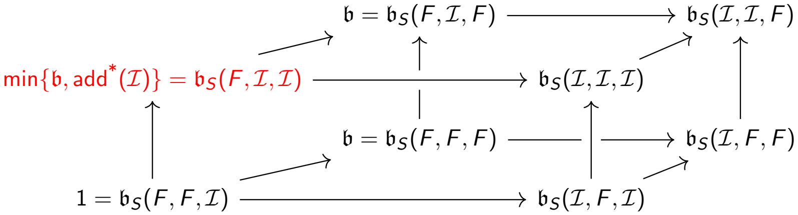
$$\mathrm{add}^*(\mathcal{I}) = \min(\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \land \neg [\exists A \in \mathcal{I} \forall F \in \mathcal{F} (|F \setminus A| < \aleph_0)]\}$$

Corollary

$$\mathfrak{b}_S(F, \mathcal{I}_{1/n}, \mathcal{I}_{1/n}) = \mathfrak{b}_S(F, \mathcal{I}_d, \mathcal{I}_d) = \mathrm{add}(\mathcal{N})$$

$\mathrm{add}(\mathcal{N})$ = the smallest number of Lebesgue null sets with non-null union.

Proof: It's known that $\mathrm{add}^*(\mathcal{I}_{1/n}) = \mathrm{add}^*(\mathcal{I}_d) = \mathrm{add}(\mathcal{N}) \leq \mathfrak{b}$.



$$\text{add}^*(\mathcal{I}) = \min(\{|\mathcal{F}| : \mathcal{F} \subseteq \mathcal{I} \wedge \neg\,[\exists A \in \mathcal{I}\, \forall F \in \mathcal{F}\,(|F \setminus A| < \aleph_0)]\}$$

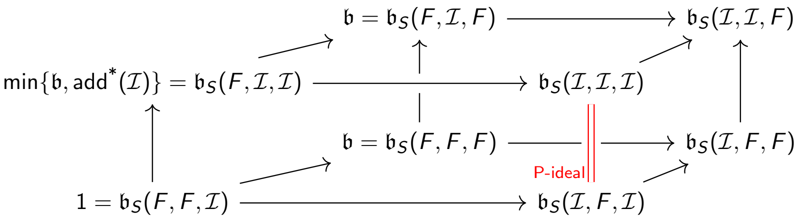
Corollary

$\mathfrak{b}_S(\mathrm{Fin}, \emptyset \otimes \mathrm{Fin}, \emptyset \otimes \mathrm{Fin}) = \mathfrak{b}$

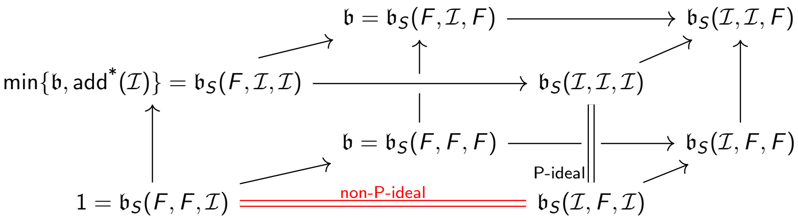
$\emptyset \otimes \mathrm{Fin} = \{A \subseteq \mathbb{N} \times \mathbb{N} : \forall n\,\{k : (n,k) \in A\} \in \mathrm{Fin}\}.$

Proof: It's known that $\text{add}^*(\emptyset \otimes \mathrm{Fin}) = \mathfrak{b}$.

Known values in the diagram: $\mathfrak{b}_S(\mathcal{I}, F, \mathcal{I})$

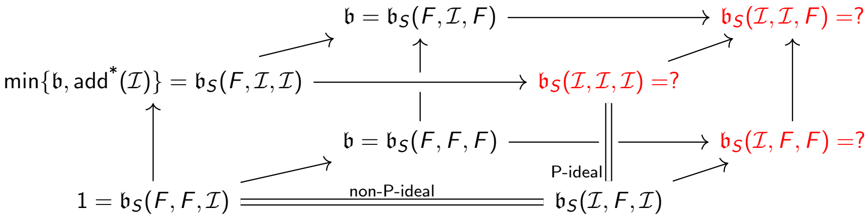


$\mathcal{I}$ is a P-ideal if $\forall A_1, A_2, \dots \in \mathcal{I}\, \exists B \in \mathcal{I}\, \forall n\, (A_n \setminus B \in \mathrm{Fin})$.



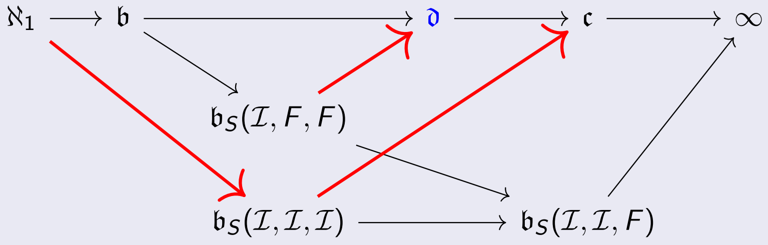
$\mathcal{I}$ is a P-ideal if $\forall A_1, A_2, \dots \in \mathcal{I}\, \exists B \in \mathcal{I}\, \forall n\, (A_n \setminus B \in \mathrm{Fin})$.

Difficult to get to know values in the diagram



The remaining three cardinals in the diagram

Theorem (F.-Kwela-Staniszewski)



$$\mathfrak{d} = \min\left\{|\mathcal{F}| : \mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}} \wedge \mathcal{F} \text{ is cofinal in } (\mathbb{N}^{\mathbb{N}}, \leq^*)\right\}$$
$$= \min\left\{|\mathcal{F}| : \forall (b_n) \in \mathbb{N}^{\mathbb{N}}\, \exists (a_n) \in \mathcal{F}\, ((b_n) \leq^* (a_n))\right\}.$$

The remaining three cardinals for P-ideals

$\mathcal{I}$ is a P-ideal if $\forall A_1, A_2, \dots \in \mathcal{I}\, \exists B \in \mathcal{I}\, \forall n\, (A_n \setminus B \in \mathrm{Fin})$

Theorem (F.-Kwela)

(1) If $\mathcal{I}$ is a P-ideal,

$$\mathfrak{b} \leq \mathfrak{b}_S(\mathcal{I}, F, F) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F) \leq \mathfrak{d}$$

(2) If $\mathcal{I}$ is a P-ideal with the Baire property (e.g. $\mathcal{I} = \mathcal{I}_{1/n}, \mathcal{I}_d$),

$$\mathfrak{b} = \mathfrak{b}_S(\mathcal{I}, F, F) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F) \leq \mathfrak{d}$$

(3) If $\mathfrak{b} < cf(\mathfrak{d})$ and $\mathfrak{d} = \mathfrak{c}$, then there exists a P-ideal $\mathcal{I}$ such that

$$\mathfrak{b} < \mathfrak{b}_S(\mathcal{I}, F, F) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F) = cf(\mathfrak{d}) \leq \mathfrak{d}$$

The remaining three cardinals for weak P-ideals

Definition

$\mathcal{I}$ is a P-ideal if $\forall A_1, A_2, \dots \in \mathcal{I}\, \exists B \in \mathcal{I}\, \forall n\, (A_n \setminus B \in \mathrm{Fin})$

Equivalently (taking $C = \mathbb{N} \setminus B$): $\mathcal{I}$ is a P-ideal if

$$\forall A_1, A_2, \dots \in \mathcal{I}\, \exists C \in \mathcal{I}^*\, \forall n\, (A_n \cap C \in \mathrm{Fin})$$

where $\mathcal{I}^* = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \notin \mathcal{I}\}$

Definition

$\mathcal{I}$ is a weak P-ideal if

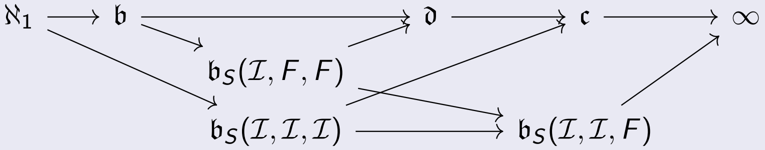
$$\forall A_1, A_2, \dots \in \mathcal{I}\, \exists C \in \mathcal{I}^+\, \forall n\, (A_n \cap C \in \mathrm{Fin}),$$

where $\mathcal{I}^+ = \{A \subseteq \mathbb{N} : A \notin \mathcal{I}\}$

Note: $\mathcal{I}^* \subseteq \mathcal{I}^+$.

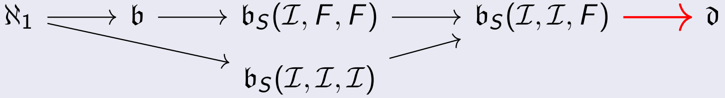
The remaining three cardinals for weak P-ideals

Diagram for an arbitrary ideal $\mathcal{I}$



Theorem (F.-Kwela)

If $\mathcal{I}$ is a weak P-ideal,



The remaining three cardinals for weak P-ideals

Theorem (F.-Kwela)

If $\mathcal{I}$ is a weak P-ideal,

$$\begin{array}{ccccccccc} \aleph_1 & \longrightarrow & \mathfrak{b} & \longrightarrow & \mathfrak{b}_S(\mathcal{I},F,F) & \longrightarrow & \mathfrak{b}_S(\mathcal{I},\mathcal{I},F) & \longrightarrow & \mathfrak{d} \\ & \searrow & & & & \nearrow & & & \\ & & & & \mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) & & & & \end{array}$$

(1) If $\mathcal{I} = \mathcal{ED} = \{A \subseteq \mathbb{N}\times\mathbb{N} : \exists M\, \forall^\infty n\, (|\{k : (n,k)\in A\}| \leq M)\}$,

$$\aleph_1 = \mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) \leq \mathfrak{b} = \mathfrak{b}_S(\mathcal{I},F,F) = \mathfrak{b}_S(\mathcal{I},\mathcal{I},F)$$

(2) If $\mathcal{I} = \mathrm{nwd} = \{A \subseteq \mathbb{Q} : A \text{ is nowhere dense in } \mathbb{R}\}$,

$$\aleph_1 \leq \mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) = \mathrm{add}(\mathcal{M}) \leq \mathfrak{b} = \mathfrak{b}_S(\mathcal{I},F,F) = \mathfrak{b}_S(\mathcal{I},\mathcal{I},F)$$

(3) If $\mathfrak{b} < cf(\mathfrak{d})$, there exists $\mathcal{I}$ with

$$\aleph_1 \leq \mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) \leq \mathfrak{b} = \mathfrak{b}_S(\mathcal{I},F,F) < \mathfrak{b}_S(\mathcal{I},\mathcal{I},F) = cf(\mathfrak{d})$$

Question for weak P-ideals

If $\mathcal{I}$ is a weak P-ideal,

$$\aleph_1 \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{b}_S(\mathcal{I},F,F) \longrightarrow \mathfrak{b}_S(\mathcal{I},\mathcal{I},F) \longrightarrow \mathfrak{d}$$

$$\aleph_1 \longrightarrow \mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) \longrightarrow \mathfrak{b}_S(\mathcal{I},\mathcal{I},F)$$

Question

Is it consistent that there is a weak P-ideal $\mathcal{I}$ such that

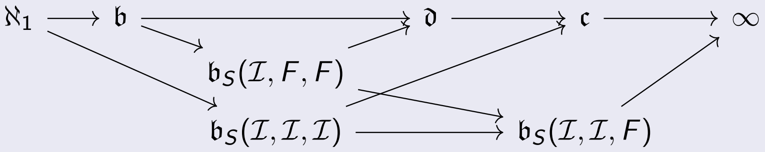
$$\mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) < \mathfrak{b}_S(\mathcal{I},F,F) < \mathfrak{b}_S(\mathcal{I},\mathcal{I},F)?$$

Question

Is it consistent that there is a weak P-ideal $\mathcal{I}$ such that

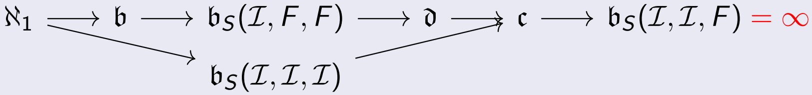
$$\mathfrak{b}_S(\mathcal{I},F,F) < \mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) < \mathfrak{b}_S(\mathcal{I},\mathcal{I},F)?$$

Diagram for an arbitrary ideal $\mathcal{I}$



Theorem (F.-Kwela)

If $\mathcal{I}$ is not a weak P-ideal,



The remaining three cardinals for non weak P-ideals

Theorem (F.-Kwela)

If $\mathcal{I}$ is not a weak P-ideal,

$$\aleph_1 \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{b}_S(\mathcal{I},F,F) \longrightarrow \mathfrak{d} \longrightarrow \mathfrak{c} \longrightarrow \mathfrak{b}_S(\mathcal{I},\mathcal{I},F) = \infty$$

$$\aleph_1 \longrightarrow \mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) \longrightarrow \mathfrak{c}$$

(1) If $\mathcal{I} = \mathrm{Fin}^2 = \{A \subseteq \mathbb{N}\times\mathbb{N} : \forall^\infty n\,\{k : (n,k)\in A\} \in \mathrm{Fin}\}$,

$$\mathfrak{b} = \mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) = \mathfrak{b}_S(\mathcal{I},F,F)$$

(2) If $\mathfrak{b} < cf(\mathfrak{d})$, there exists a non-weak P ideal $\mathcal{I}$ with

$$\aleph_1 \leq \mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) = \mathfrak{b} < \mathfrak{b}_S(\mathcal{I},F,F) = cf(\mathfrak{d})$$

(3) If $\mathfrak{b} < cf(\mathfrak{d})$ and $\mathfrak{d} = \mathfrak{c}$, there exists a non-weak P ideal $\mathcal{I}$ with

$$\aleph_1 \leq \mathfrak{b}_S(\mathcal{I},F,F) = \mathfrak{b} < \mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) = cf(\mathfrak{d})$$

Can $\mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I})$ be bigger than $\mathfrak{d}$?

If $\mathcal{I}$ is a weak P-ideal,

$$\aleph_1 \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{b}_S(\mathcal{I},F,F) \longrightarrow \mathfrak{b}_S(\mathcal{I},\mathcal{I},F) \longrightarrow \mathfrak{d}$$

$$\aleph_1 \longrightarrow \mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) \longrightarrow \mathfrak{b}_S(\mathcal{I},\mathcal{I},F)$$

If $\mathcal{I}$ is not a weak P-ideal,

$$\aleph_1 \longrightarrow \mathfrak{b} \longrightarrow \mathfrak{b}_S(\mathcal{I},F,F) \longrightarrow \mathfrak{d} \longrightarrow \mathfrak{c} \longrightarrow \mathfrak{b}_S(\mathcal{I},\mathcal{I},F) = \infty$$

$$\aleph_1 \longrightarrow \mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) \longrightarrow \mathfrak{c}$$

In all previous examples we had $\mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) \leq \mathfrak{d}$.

Theorem (F.-Kwela)

If $\mathcal{I}$ is an analytic ideal, then $\mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) \leq \mathfrak{d}$.

Can $\mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I})$ be bigger than $\mathfrak{d}$?

If $\mathcal{I}$ is a weak P-ideal,

$$\begin{array}{ccccccccc} \aleph_1 & \longrightarrow & \mathfrak{b} & \longrightarrow & \mathfrak{b}_S(\mathcal{I},F,F) & \longrightarrow & \mathfrak{b}_S(\mathcal{I},\mathcal{I},F) & \longrightarrow & \mathfrak{d} \\ & \searrow & & & & \nearrow & & & \\ & & & & \mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) & & & & \end{array}$$

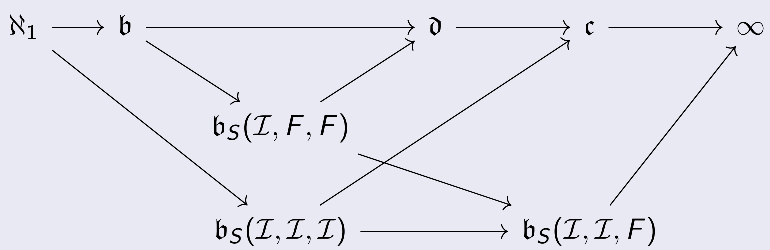
If $\mathcal{I}$ is not a weak P-ideal,

$$\begin{array}{ccccccccccc} \aleph_1 & \longrightarrow & \mathfrak{b} & \longrightarrow & \mathfrak{b}_S(\mathcal{I},F,F) & \longrightarrow & \mathfrak{d} & \longrightarrow & \mathfrak{c} & \longrightarrow & \mathfrak{b}_S(\mathcal{I},\mathcal{I},F) = \infty \\ & \searrow & & & & & & \nearrow & & & \\ & & & & \mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) & & & & & & \end{array}$$

Theorem (F.-Kwela)

If $\mathfrak{d} < cf(\mathfrak{c}) \leq \mathfrak{u} = \mathfrak{c} = \mathfrak{c}^{\mathfrak{d}}$, then there exists an ideal $\mathcal{I}$ with

$$\mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) > \mathfrak{d}.$$

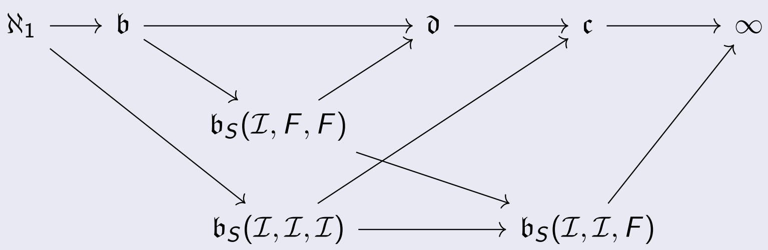


Theorem (F.-Kwela)

If $\mathcal{I}$ has the Baire property,

$$\mathfrak{b} = \mathfrak{b}_S(\mathcal{I}, F, F)$$

(1) If $\mathcal{I} = \mathcal{I}_d$, $\mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) = \mathfrak{b}_S(\mathcal{I}, F, F) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F) = \mathfrak{b}$
(2) If $\mathcal{I} = \mathcal{ED}$, $\mathfrak{b}_S(\mathcal{I}, \mathcal{I}, \mathcal{I}) = \aleph_1 \leq \mathfrak{b}_S(\mathcal{I}, F, F) = \mathfrak{b}_S(\mathcal{I}, \mathcal{I}, F) = \mathfrak{b}$



Theorem (F.-Kwela)

If $\mathcal{I}$ is Π^0_4 (e.g. F_σ or $F_{\sigma\delta}$ ideal),

$$\mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) \leq \mathfrak{b} = \mathfrak{b}_S(\mathcal{I}, F, F) = \mathfrak{b}_S(\mathcal{I},\mathcal{I}, F)$$

(1) If $\mathcal{I} = \mathcal{I}_d$, $\mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) = \mathfrak{b}_S(\mathcal{I}, F, F) = \mathfrak{b}_S(\mathcal{I},\mathcal{I}, F) = \mathfrak{b}$

(2) If $\mathcal{I} = \mathcal{ED}$, $\mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) = \aleph_1 \leq \mathfrak{b}_S(\mathcal{I}, F, F) = \mathfrak{b}_S(\mathcal{I},\mathcal{I}, F) = \mathfrak{b}$

Definition

(λ, κ) gap in a poset $(P, <)$ is a pair of sequences

$$\{p_\alpha : \alpha < \lambda\} \quad \text{and} \quad \{q_\beta : \beta < \kappa\}$$

of elements of P such that

$$p_0 < p_1 < \cdots < p_\alpha < \cdots < q_\beta < \cdots < q_1 < q_0$$

and there is no element r with

$$p_\alpha < r < q_\beta \qquad \text{for all } \alpha \text{ and } \beta.$$

Example of (ω, ω) gap in $(\mathbb{Q}, <)$

p_n – increasing sequence of rationals with limit $\sqrt{2}$
q_n – decreasing sequence of rationals with limit $\sqrt{2}$

Gaps in $(\mathcal{P}(\mathbb{N}), \subseteq^*)$

Theorem (Hadamard, 1894)

There is <u>no</u> (ω, ω) gap in $(\mathcal{P}(\mathbb{N}), \subseteq^*)$, where

$$A \subseteq^* B \iff A \setminus B \in \mathrm{Fin}.$$

Theorem (Hausdorff, 1909)

There exists (ω_1, ω_1) gap in $(\mathcal{P}(\mathbb{N}), \subseteq^*)$.

Theorem (Rothberger, 1941)

There exists $(\omega, \mathfrak{b})$ gap in $(\mathcal{P}(\mathbb{N}), \subseteq^*)$ and there is <u>no</u> (ω, λ) gap for $\lambda < \mathfrak{b}$.

$$\mathfrak{b} = \min\{\lambda : \exists (\omega, \lambda) \text{ gap in } (\mathcal{P}(\mathbb{N}), \subseteq^*)\}$$

The Rothberger numbers

$$\mathfrak{b} = \min\{\lambda : \exists(\omega, \lambda) \text{ gap in } (\mathcal{P}(\mathbb{N}), \subseteq^*)\}$$

Definition (Brendle-Mejía, 2014)

$$\mathfrak{b}_R(\mathcal{I}) = \min\left(\{\lambda : \exists(\omega, \lambda)\text{gap in } (\mathcal{P}(\mathbb{N}), \subseteq^{\mathcal{I}})\} \cup \{\infty\}\right),$$

where

$$A \subseteq^{\mathcal{I}} B \iff A \setminus B \in \mathcal{I}.$$

Example

If $\mathcal{I}$ is a maximal ideal, $\mathfrak{b}_R(\mathcal{I}) = \infty$.

Theorem (Todorčević, 1998)

If $\mathcal{I}$ is an F_σ ideal, then $\mathfrak{b}_R(\mathcal{I}) \leq \mathfrak{b}$.

Characterization of Rothberger numbers á la Staniszewski

$\mathcal{P}_\mathcal{I}$ is the family of all partitions $\{A_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ such that $A_n \in \mathcal{I}$ for each n.

Definition (Staniszewski, 2017)

$$\mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) = \min\left\{|\mathcal{E}| : \mathcal{E} \subseteq \mathcal{P}_\mathcal{I} \wedge \forall\{A_n\} \in \mathcal{P}_\mathcal{I}\, \exists\{E_n\} \in \mathcal{E} \bigcup_{n\in\mathbb{N}}\left(A_{n+1} \cap \bigcup_{i\leq n} E_i\right) \notin \mathcal{I}\right\}.$$

Theorem (F.-Kwela)

$$\mathfrak{b}_R(\mathcal{I}) = \min\left\{|\mathcal{E}| : \mathcal{E} \subseteq \mathcal{P}^\star_\mathcal{I} \wedge \forall\{A_n\} \in \mathcal{P}_\mathcal{I}\, \exists\{E_n\} \in \mathcal{E} \bigcup_{n\in\mathbb{N}}\left(A_{n+1} \cap \bigcup_{i\leq n} E_i\right) \notin \mathcal{I}\right\}.$$

where $\mathcal{P}^\star_\mathcal{I}$ is somehow defined subfamily of $\mathcal{P}_\mathcal{I}$, and consequently:

$$\mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) \leq \mathfrak{b}_R(\mathcal{I})$$

Question: Two $\mathfrak{b}$ or not two $\mathfrak{b}$?

$$b_S(\mathcal{I},\mathcal{I},\mathcal{I}) \stackrel{?}{=} \mathfrak{b}_R(\mathcal{I})$$

Answer

In general, two $\mathfrak{b}$.

Example

If $\mathcal{I}$ is a maximal ideal,

$$\mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) \leq \mathfrak{c} < \infty = \mathfrak{b}_R(\mathcal{I}).$$

Theorem (F.-Kwela)

It is consistent that there exists $\mathcal{I}$ with

$$\mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) < \mathfrak{b}_R(\mathcal{I}) \leq \mathfrak{c}$$

Two $\mathfrak{b}$ or not two $\mathfrak{b}$ in Borel realm?

Question: Two $\mathfrak{b}$ or not two $\mathfrak{b}$?

$$b_S(\mathcal{I},\mathcal{I},\mathcal{I}) \stackrel{?}{=} \mathfrak{b}_R(\mathcal{I}) \quad \text{for Borel ideals}$$

Answer

We do not know, but examples suggest: not two $\mathfrak{b}$.

Theorem (Brendle-Mejía, 2014)

If $\mathcal{I}$ is an analytic P-ideal, then $\mathfrak{b}_R(\mathcal{I}) = \mathfrak{b}$.

Consequently, $\mathfrak{b}_S(\mathcal{I},\mathcal{I},\mathcal{I}) = \mathfrak{b}_R(\mathcal{I})$ for analytic P-ideals.

Theorem (Brendle-Mejía, 2014)

If $\mathcal{I} = \mathcal{ED}_{\mathrm{Fin}} = \{A \subseteq \Delta : \exists M\, \forall n\, (|\{k : (n,k) \in A\}| \leq M)\}$, then $\mathfrak{b}_R(\mathcal{I}) = \aleph_1$.

Consequently, $\mathfrak{b}_S(\mathcal{ED}_{\mathrm{Fin}}, \mathcal{ED}_{\mathrm{Fin}}, \mathcal{ED}_{\mathrm{Fin}}) = \mathfrak{b}_R(\mathcal{ED}_{\mathrm{Fin}})$.

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