Spaces not distinguishing various kinds of convergence of sequences of real functions

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New results in my talk are from a join work with Adam Kwela:

"*Spaces not distinguishing ideal pointwise and σ-uniform convergence*"

which is available at arXiv (2308.09557).

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- Topological space = nonempty normal (a.k.a. T_4) space
- Function = continuous real-valued function defined on X ($f : X \to \mathbb{R}$)

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Four kinds of convergence of sequences of functions

Pointwise convergence

$(f_n) \xrightarrow{P} 0$ if $\forall_{\varepsilon>0} \forall_{x\in X} \exists_k \forall_{n>k} (|f_n(x)| < \varepsilon)$

Equal convergence (1979 – Császár-Laczkovich)
Quasi-normal convergence (1991 – Bukovská)

$(f_n) \xrightarrow{QN} 0$ if $\exists_{(\varepsilon_n)\to 0} \forall_{x\in X} \exists_k \forall_{n>k} (|f_n(x)| < \varepsilon_n)$

Uniform convergence

$(f_n) \xrightarrow{U} 0$ if $\forall_{\varepsilon>0} \exists_k \forall_{n>k} \forall_{x\in X} (|f_n(x)| < \varepsilon)$

σ-uniform convergence

$(f_n) \xrightarrow{\sigma-U} 0$ if there is a partition $X_1, X_2, \ldots$ of X such that

$$\forall_{k\in\mathbb{N}} \quad (f_n \restriction X_k) \xrightarrow{U} 0$$

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Theorem (Easy)

$(f_n) \xrightarrow{U} 0 \implies (f_n) \xrightarrow{\sigma-U} 0$ and $(f_n) \xrightarrow{QN} 0 \implies (f_n) \xrightarrow{P} 0$

Theorem (Császár-Laczkovich, 1979)

$$(f_n) \xrightarrow{\sigma-U} 0 \iff (f_n) \xrightarrow{QN} 0$$

Question

$(f_n) \xrightarrow{U} 0 \overset{?}{\Longleftarrow} (f_n) \xrightarrow{\sigma-U} 0$ and $(f_n) \xrightarrow{QN} 0 \overset{?}{\Longleftarrow} (f_n) \xrightarrow{P} 0$

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$\forall_{(f_n:X\to\mathbb{R})} ((f_n) \xrightarrow{U} 0 \Longleftarrow (f_n) \xrightarrow{\sigma-U} 0) \iff X$ is finite

Definition (Bukovský-Recław-Repický, 1991)

X is a QN-space $\iff \forall_{(f_n:X\to\mathbb{R})} ((f_n) \xrightarrow{QN} 0 \Longleftarrow (f_n) \xrightarrow{P} 0)$

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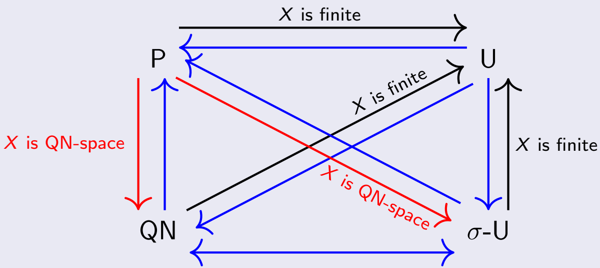

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Ideals on $\mathbb{N}$

Definition

A family $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is an ideal on $\mathbb{N}$ if

1. $A \subseteq B \in \mathcal{I} \implies A \in \mathcal{I}$,
2. $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$,
3. $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$ and $\mathbb{N} \notin \mathcal{I}$.

Example

1. $\mathrm{Fin} = \{A \subseteq \mathbb{N} : A \text{ is finite}\}$
2. $\mathcal{I}_{1/n} = \left\{A \subseteq \mathbb{N} : \sum_{n \in A} \frac{1}{n} < \infty\right\}$
3. $\mathcal{I}_d = \left\{A \subseteq \mathbb{N} : \lim_{n \to \infty} \frac{|A \cap \{1, \ldots, n\}|}{n} = 0\right\}$

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Ideal convergence of sequences of reals

Definition

A sequence (x_n) of reals is convergent to zero if

$$\forall_{\varepsilon>0} \quad \exists_k \ \forall_{n>k} \quad (|x_n| < \varepsilon)$$

equivalently:

$$\forall_{\varepsilon>0} \quad \exists_{A\in \mathrm{Fin}} \ \forall_{n\in\mathbb{N}\setminus A} \quad (|x_n| < \varepsilon)$$

Definition

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$.
A sequence (x_n) of reals is $\mathcal{I}$-convergent to zero if

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A sequence (x_n) of reals is $\mathcal{I}$-convergent to zero ($(x_n) \xrightarrow{\mathcal{I}} 0$) if

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Example

Since $\mathrm{Fin} \subseteq \mathcal{I}$,

$$\text{if } (x_n) \to 0, \text{ then } (x_n) \xrightarrow{\mathcal{I}} 0.$$

Example

Let $x_n = \begin{cases} 1 & \text{if } n = 1, 4, 9, \ldots, k^2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$

Then (x_n) is a divergent sequence, but

$$(x_n) \xrightarrow{\mathcal{I}_{1/n}} 0,$$

because $\sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$, so $A = \{1, 4, 9, \ldots, k^2, \ldots\} \in \mathcal{I}_{1/n}$.

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Ideal convergence of sequences of functions

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Let $f_n : X \to \mathbb{R}$, $n = 1, 2, \ldots$

Ideal pointwise convergence

$(f_n) \xrightarrow{P} 0$ if $\forall_{\varepsilon>0} \forall_{x\in X} \quad \exists_k \forall_{n>k} \quad (|f_n(x)| < \varepsilon)$

$(f_n) \xrightarrow{\mathcal{I}-P} 0$ if $\forall_{\varepsilon>0} \forall_{x\in X} \ \exists_{A\in\mathcal{I}} \forall_{n\in\mathbb{N}\setminus A} \ (|f_n(x)| < \varepsilon)$

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$(f_n) \xrightarrow{\mathcal{I}-QN} f$ if $\exists_{(\varepsilon_n)\xrightarrow{\mathcal{I}} 0} \forall_{x\in X} \ \exists_{A\in\mathcal{I}} \forall_{n\in\mathbb{N}\setminus A} \ (|f_n(x)| < \varepsilon_n)$

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Ideal uniform convergence

$(f_n) \xrightarrow{\mathcal{I}-U} 0$ if $\forall_{\varepsilon>0} \ \exists_{A\in\mathcal{I}} \forall_{n\in\mathbb{N}\setminus A} \ \forall_{x\in X}(|f_n(x)| < \varepsilon)$

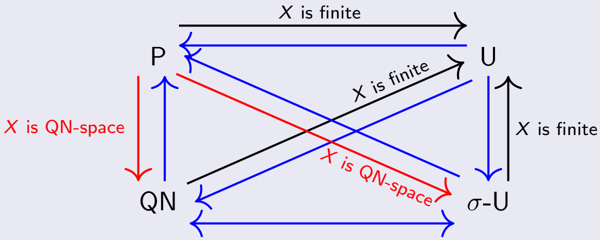
Ideal σ-uniform convergence

$(f_n) \xrightarrow{\mathcal{I}-\sigma-U} 0$ if there is a partition $X_1, X_2, \ldots$ of X such that

$$\forall_{k\in\mathbb{N}} \quad (f_n \restriction X_k) \xrightarrow{\mathcal{I}-U} 0$$

Ideal convergence of sequences of functions

Diagram of relationships



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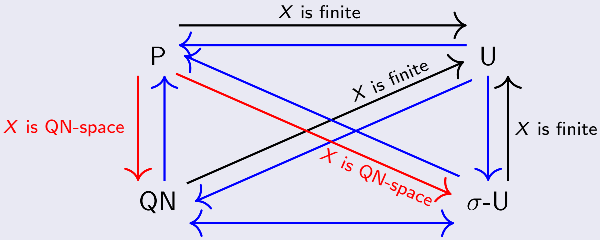
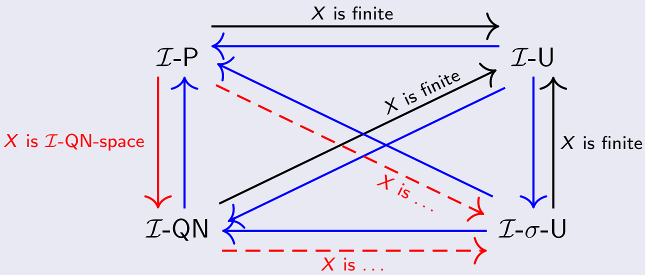
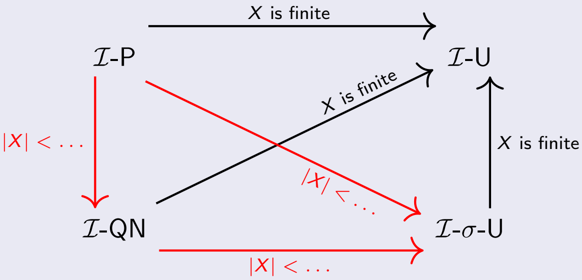




Minimal size of a space which distinguishes convergence



Definition

- non($\mathcal{I}$-P, $\mathcal{I}$-QN) $= \min\{|X| : \exists_{f_n : X \to \mathbb{R}} (f_n \xrightarrow{\mathcal{I}-P} 0 \wedge f_n \overset{\mathcal{I}-QN}{\not\to} 0)\}$
- non($\mathcal{I}$-QN, $\mathcal{I}$-σ-U) $= \min\{|X| : \exists_{f_n : X \to \mathbb{R}} (f_n \xrightarrow{\mathcal{I}-QN} 0 \wedge f_n \overset{\mathcal{I}-\sigma-U}{\not\to} 0)\}$
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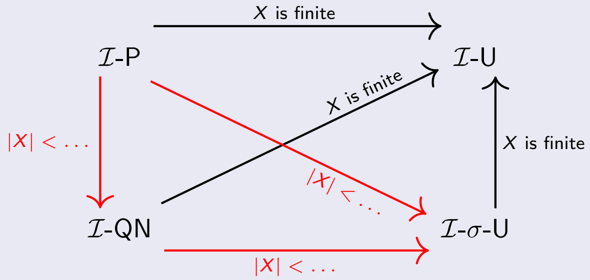

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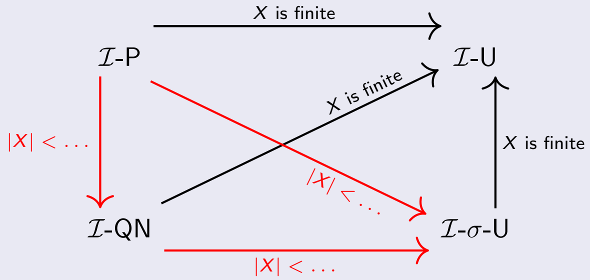

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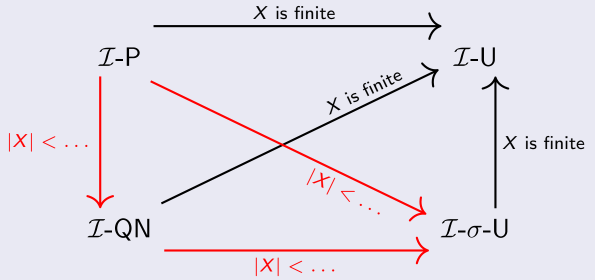
Minimal size of a space which distinguishes convergence



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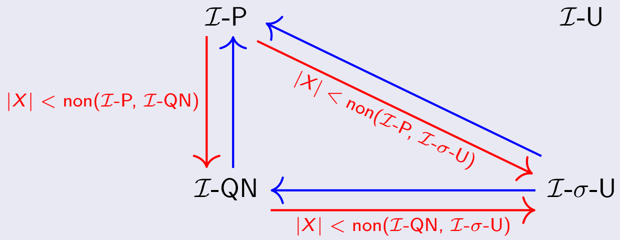
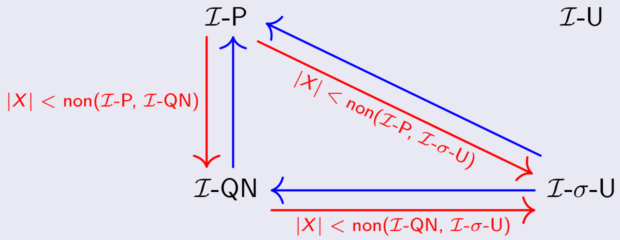


Theorem (Easy)

$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \min\{\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-QN}),\ \ \text{non}(\mathcal{I}\text{-QN}, \mathcal{I}\text{-}\sigma\text{-U})\}$$

Proof

- If $|X| < \text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U})$, then ..., so $|X| < \min\{\dots,\ \dots\}$
- If $|X| < \min\{\dots,\ \dots\}$, then ..., so $|X| < \text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U})$



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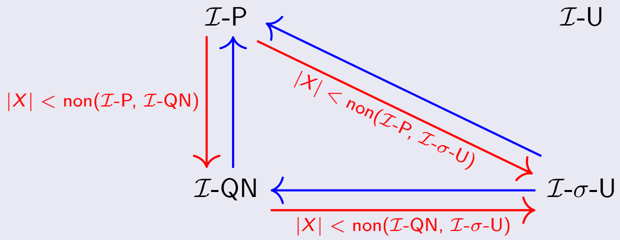

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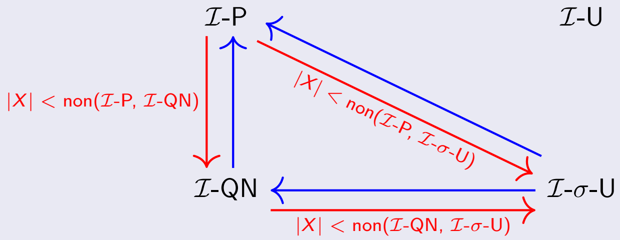
Minimal size of a space which distinguishes convergence

Relationship



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- If $|X| < \min\{\ldots, \ldots\}$, then ..., so $|X| < \text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U})$

Combinatorial characterizations of non(P, QN)

Theorem (Bukovský-Recław-Repický, 1991)

$$\text{non}(\text{P}, \text{QN}) = \text{non}(\text{Fin-P}, \text{Fin-QN}) = \mathfrak{b},$$

where $\mathfrak{b}$ is the bounding number i.e. the minimal size of an unbounded family of sequences of natural numbers ordered by the relation $\leq^*$ defined by

$$(a_n) \leq^* (b_n) \iff \exists_k \forall_{n>k} (a_n \leq b_n).$$

Definition

Let $\mathfrak{b}_\mathcal{I}$ is the minimal size of an unbounded family of sequences of natural numbers ordered by the relation $\leq^\mathcal{I}$ defined by

$$(a_n) \leq^\mathcal{I} (b_n) \iff \exists_{A\in\mathcal{I}} \forall_{n\in\mathbb{N}\setminus A} (a_n \leq b_n).$$

Remark

In general,

$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-QN}) \neq \mathfrak{b}_\mathcal{I}!$$

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$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-QN}) \neq \mathfrak{b}_\mathcal{I}!$$

Combinatorial characterization of non($\mathcal{I}$-P, $\mathcal{I}$-QN)

Theorem (F.-Staniszewski, 2015)

$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-QN}) = \mathfrak{b}_S(\mathcal{I}),$$

where

$$\mathfrak{b}_S(\mathcal{I}) = \min\left\{|\mathcal{E}| : \mathcal{E} \subseteq \mathcal{P}_\mathcal{I} \wedge \forall_{\{A_n\}\in\mathcal{P}_\mathcal{I}} \exists_{\{E_n\}\in\mathcal{E}} \bigcup_{n\in\mathbb{N}} (A_{n+1} \cap \bigcup_{i\leq n} E_i) \notin \mathcal{I}\right\}$$

and $\mathcal{P}_\mathcal{I}$ is the family of all partitions $\{A_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ such that $A_n \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Theorem (F.-Kwela, 2023)

$$\text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \mathfrak{b}_\sigma(\mathcal{I}),$$

where

$$\mathfrak{b}_\sigma(\mathcal{I}) = \min\{|\mathcal{E}| : \mathcal{E} \subseteq \mathcal{M}_\mathcal{I} \land \forall_{(A_n)\in\mathcal{M}_\mathcal{I}} \exists_{(E_n)\in\mathcal{E}} \forall_k \exists_{n>k} (E_n \not\subseteq A_n)\}$$

and $\mathcal{M}_\mathcal{I}$ is the family of all increasing sequences $(A_n : n \in \mathbb{N})$ such that $A_n \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Combinatorial characterization of non($\mathcal{I}$-QN, $\mathcal{I}$-σ-U)

Theorem (F.-Kwela, 2023)

$$\text{non}(\mathcal{I}\text{-QN}, \mathcal{I}\text{-}\sigma\text{-U}) = \text{add}_\omega(\mathcal{I}),$$

where

$$\text{add}_\omega(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \land \forall_{(B_n)\in\mathcal{I}^\omega} \exists_{A\in\mathcal{A}} \forall_n (A \not\subseteq B_n))\}$$

and $\mathcal{I}^\omega$ is the family of all sequences $(A_n : n \in \mathbb{N})$ such that $A_n \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Remark

- $\text{add}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \land \forall_{B\in\mathcal{I}} \exists_{A\in\mathcal{A}} (A \not\subseteq^* B))\}$
- If $\mathcal{I}$ is a P-ideal, then $\text{add}_\omega(\mathcal{I}) = \text{add}^*(\mathcal{I})$.
- There are non-P-ideals $\mathcal{I}$ with $\text{add}_\omega(\mathcal{I}) \neq \text{add}^*(\mathcal{I}) = \omega$.

Combinatorial characterization of non($\mathcal{I}$-QN, $\mathcal{I}$-σ-U)

Theorem (F.-Kwela, 2023)

$$\text{non}(\mathcal{I}\text{-QN}, \mathcal{I}\text{-}\sigma\text{-U}) = \text{add}_\omega(\mathcal{I}),$$

where

$$\text{add}_\omega(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \land \forall_{(B_n)\in\mathcal{I}^\omega} \exists_{A\in\mathcal{A}} \forall_n (A \not\subseteq B_n))\}$$

and $\mathcal{I}^\omega$ is the family of all sequences $(A_n : n \in \mathbb{N})$ such that $A_n \in \mathcal{I}$ for each $n \in \mathbb{N}$.

Remark

- $\text{add}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \land \forall_{B\in\mathcal{I}} \exists_{A\in\mathcal{A}} (A \not\subseteq^* B))\}$
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Examples

Corollary

$$\mathfrak{b}_\sigma(\mathcal{I}) = \min\{\mathfrak{b}_S(\mathcal{I}), \text{add}_\omega(\mathcal{I})\}$$

Proof

non($\mathcal{I}$-P, $\mathcal{I}$-σ-U) = min{non($\mathcal{I}$-P, $\mathcal{I}$-QN), non($\mathcal{I}$-QN, $\mathcal{I}$-σ-U)}

Examples

- $\mathfrak{b}_\sigma(\mathrm{Fin}) = \mathfrak{b}_S(\mathrm{Fin}) = \mathfrak{b} < \infty = \text{add}_\omega(\mathrm{Fin})$

- $\mathfrak{b}_\sigma(\mathcal{I}_d) = \text{add}_\omega(\mathcal{I}_d) = \text{add}(\mathcal{N}) \leq \mathfrak{b} = \mathfrak{b}_s(\mathcal{I}_d)$

- $\mathfrak{b}_\sigma(\mathrm{Fin} \otimes \mathrm{Fin}) = \mathfrak{b}_s(\mathrm{Fin} \otimes \mathrm{Fin}) = \text{add}_\omega(\mathrm{Fin} \otimes \mathrm{Fin}) = \mathfrak{b}$
- $\mathfrak{b}_\sigma(\mathcal{I}_{AD}) = \mathfrak{b}_s(\mathcal{I}_{AD}) = \text{add}_\omega(\mathcal{I}_{AD}) = \omega_1$

Examples

Corollary

$$\mathfrak{b}_\sigma(\mathcal{I}) = \min\{\mathfrak{b}_S(\mathcal{I}), \mathrm{add}_\omega(\mathcal{I})\}$$

Proof

non($\mathcal{I}$-P, $\mathcal{I}$-σ-U) $=$ min{non($\mathcal{I}$-P, $\mathcal{I}$-QN), non($\mathcal{I}$-QN, $\mathcal{I}$-σ-U)}

Examples

- $\mathfrak{b}_\sigma(\mathrm{Fin}) = \mathfrak{b}_S(\mathrm{Fin}) = \mathfrak{b} < \infty = \mathrm{add}_\omega(\mathrm{Fin})$

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- $\mathfrak{b}_\sigma(\mathrm{Fin}\otimes\mathrm{Fin}) = \mathfrak{b}_s(\mathrm{Fin}\otimes\mathrm{Fin}) = \mathrm{add}_\omega(\mathrm{Fin}\otimes\mathrm{Fin}) = \mathfrak{b}$
- $\mathfrak{b}_\sigma(\mathcal{I}_{AD}) = \mathfrak{b}_s(\mathcal{I}_{AD}) = \mathrm{add}_\omega(\mathcal{I}_{AD}) = \omega_1$

Examples

Corollary

$$\mathfrak{b}_\sigma(\mathcal{I}) = \min\{\mathfrak{b}_S(\mathcal{I}), \mathrm{add}_\omega(\mathcal{I})\}$$

Proof

non($\mathcal{I}$-P, $\mathcal{I}$-σ-U) = min{non($\mathcal{I}$-P, $\mathcal{I}$-QN), non($\mathcal{I}$-QN, $\mathcal{I}$-σ-U)}

Examples

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Examples

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Proof

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Examples

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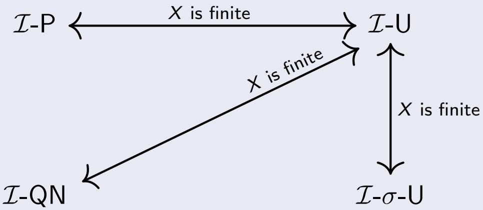
- $\mathfrak{b}_\sigma(\mathcal{I}_{AD}) = \mathfrak{b}_s(\mathcal{I}_{AD}) = \mathrm{add}_\omega(\mathcal{I}_{AD}) = \omega_1$

Is distinguishing convergence a topological notion?

Sometimes no!



Theorem

- $\forall_{(f_n:X\to\mathbb{R})}\ (f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-U}} 0) \iff |X| < \omega$
- $\forall_{(f_n:X\to\mathbb{R})}\ (f_n \xrightarrow{\mathcal{I}\text{-QN}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-U}} 0) \iff |X| < \omega$
- $\forall_{(f_n:X\to\mathbb{R})}\ (f_n \xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-U}} 0) \iff |X| < \omega$

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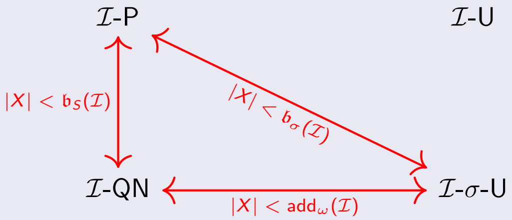

Theorem (F.-Kwela, 2023)

- $|X| < \mathfrak{b}_S(\mathcal{I}) \quad \overset{\Longrightarrow}{\not\Longleftarrow} \quad \forall_{(f_n : X \to \mathbb{R})} \left(f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-QN}} 0\right)$
- $|X| < \mathfrak{b}_\sigma(\mathcal{I}) \quad \overset{\Longrightarrow}{\not\Longleftarrow} \quad \forall_{(f_n : X \to \mathbb{R})} \left(f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0\right)$
- $|X| < \text{add}_\omega(\mathcal{I}) \quad \overset{\Longrightarrow}{\not\Longleftarrow} \quad \forall_{(f_n : X \to \mathbb{R})} \left(f_n \xrightarrow{\mathcal{I}\text{-QN}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0\right)$

In fact, there exists a Hausdorff compact (hence normal) space X of arbitrary cardinality such that

$$\forall_{(f_n : X \to \mathbb{R})} \left(f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \iff f_n \xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0\right)$$

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Remark

The above X is a one-point compactification of a discrete topological space of a given cardinality. This space is not separable (unless it is countable) and all but one points are isolated.

Theorem (F.-Kwela, 2023)

There exists a Hausdorff separable, sequentially compact, compact (hence normal) space X of arbitrary cardinality up to $\mathfrak{c}$ such that only countably many points of X are isolated and

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Subsets of reals distinguishing convergence

Remark

- Since
$$\mathrm{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \mathfrak{b}_\sigma(\mathcal{I}),$$
there exists a topological space X such that
$$|X| = \mathfrak{b}_\sigma(\mathcal{I}) \quad \text{and} \quad \exists_{(f_n:X\to\mathbb{R})} \left(f_n \xrightarrow{\mathcal{I}\text{-P}} 0 \not\Leftrightarrow f_n \xrightarrow{\mathcal{I}\text{-}\sigma\text{-U}} 0\right)$$
- The space X that is constructed in the proof of the equality
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Subsets of reals distinguishing convergence

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Definition

- The set $\mathcal{D}_{\mathcal{I}} \subseteq \mathbb{N}^{\mathbb{N}}$ is defined by

$$x \in \mathcal{D}_{\mathcal{I}} \iff x^{-1}[\{n\}] \in \mathcal{I} \text{ for every } n \in \mathbb{N}$$

- The binary relation $\preceq_{\mathcal{I}}$ on $\mathcal{D}_{\mathcal{I}}$ is defined by

$$x \preceq_{\mathcal{I}} y \iff \{m \in \omega : \exists k \in \omega\, (x(k) \leq m < y(k))\} \quad \text{is finite}$$

- The bounding number of the relation $\preceq$:

$$\mathfrak{b}(\preceq_{\mathcal{I}}) = \min\{|B| : B \text{ is } \preceq_{\mathcal{I}}\text{-unbounded}\}$$

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$$\mathfrak{b}_\sigma(\mathcal{I}) = \text{non}(\mathcal{I}\text{-P}, \mathcal{I}\text{-}\sigma\text{-U}) = \mathfrak{b}(\preceq_\mathcal{I})$$

Theorem (F.-Kwela, 2023)

If $X \subseteq \mathbb{N}^\mathbb{N}$ is $\preceq_\mathcal{I}$-unbounded, then

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There exists a subset X of reals such that $|X| = \mathfrak{b}_\sigma(\mathcal{I})$ and

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Distinguishing spaces not distinguishing convergence

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Proof

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- Consistently: $\text{add}(\mathcal{N}) < \mathfrak{b}$
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Do there exist a space X and an ideal $\mathcal{I}$ such that

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Remark

- For every ideal $\mathcal{I}$,

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- Hence, the cardinality argument does not work in this case.

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