The Borel complexity of sets of ideal limit points

properties of ideals *inspired* by limit points of sequences
(Part I)

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The talk is based on a join work with Adam Kwela and Paolo Leonetti

Redefinitions

- $\omega = \mathbb{N}$ is the set of all natural numbers

- X will stand for an uncountable Polish space (i.e. separable completely metrizable topological space)

Ideals on ω

Definition

A family $\mathcal{I} \subseteq \mathcal{P}(\omega)$ is an ideal on ω if

1. $\emptyset \in \mathcal{I}$ and $\omega \notin \mathcal{I}$,
2. $A \subseteq B \in \mathcal{I} \implies A \in \mathcal{I}$,
3. $A, B \in \mathcal{I} \implies A \cup B \in \mathcal{I}$,
4. $\mathcal{I}$ contains all finite subsets of ω.

Example

1. $\mathrm{Fin} = \{A \subseteq \omega : A \text{ is finite}\}$
2. $\mathcal{I}_{1/n} = \left\{A \subseteq \omega : \sum_{n \in A} \frac{1}{n} < \infty\right\}$ — the summable ideal
3. $\mathcal{I}_d = \left\{A \subseteq \omega : \lim_{n \to \infty} \frac{|A \cap \{1,\ldots,n\}|}{n} = 0\right\}$ — the density zero ideal

Convergent subsequences

Theorem (Bolzano-Weierstrass)

For every sequence $(x_n)_{n\in\omega}$ in $[0,1]$ there is an $A \notin \mathrm{Fin}$ such that the subsequence $(x_n)_{n\in A}$ is convergent.

Theorem (Folklore)

For every sequence $(x_n)_{n\in\omega}$ in $[0,1]$ there is an $A \notin \mathcal{I}_{1/n}$ such that the subsequence $(x_n)_{n\in A}$ is convergent.

Theorem (Fridy, 1993)

There exists a sequence $(x_n)_{n\in\omega}$ in $[0,1]$ such that for every $A \notin \mathcal{I}_d$ the subsequence $(x_n)_{n\in A}$ is not convergent.

Finite Bolzano-Weierstrass property

Definition (F.-Mrożek-Recław-Szuca, 2007)

An ideal $\mathcal{I}$ has finite Bolzano-Weierstrass property (FinBW property) if for every sequence $(x_n)_{n\in\omega}$ in $[0,1]$ there is $A \notin \mathcal{I}$ such that the subsequence $(x_n)_{n\in A}$ is convergent.

- Fin and $\mathcal{I}_{1/n}$ have the FinBW property
- $\mathcal{I}_d$ does not have the FinBW property

Theorem (F.-Mrożek-Recław-Szuca, 2007)

Every F_σ ideal has finite Bolzano-Weierstrass property.

Definition

An ideal $\mathcal{I}$ is F_σ if the set $\{\mathbf{1}_A : A \in \mathcal{I}\}$ is an F_σ subset of the Cantor space $2^\omega = \{0,1\}^\omega$.

The same for $F_{\sigma\delta}$, Borel, analytic, and other topological properties.

Topologised finite Bolzano-Weierstrass property

Definition (Kwela, 2023)

For a fixed ideal $\mathcal{I}$,

$$\mathit{FinBW}(\mathcal{I})$$

denote the class of all topological spaces X such that for every sequence $(x_n)_{n\in\omega}$ in X there is $A \notin \mathcal{I}$ such that the subsequence $(x_n)_{n\in A}$ is convergent in X.

- $[0,1] \in \mathit{FinBW}(\mathrm{Fin})$ and $[0,1] \in \mathit{FinBW}(\mathcal{I}_{1/n})$
- $[0,1] \notin \mathit{FinBW}(\mathcal{I}_d)$

Corollary

$$X \in \mathit{FinBW}(\mathcal{I}) \underset{\not\Longleftarrow}{\Longrightarrow} \text{ is sequentially compact } (\iff X \text{ is compact})$$

Topologised finite Bolzano-Weierstrass property

Corollary

If X is not compact, then $X \notin FinBW(\mathcal{I})$.

Theorem (Meza-Alcántara, 2009)

If X is compact, then $X \notin FinBW(\mathcal{I}) \iff conv \leq_K \mathcal{I}$.

Katětov order (Katětov, 1968)

$\mathcal{I} \leq_K \mathcal{J} \iff$ there exists $f : \omega \to \omega$ such that

$$\forall A \subseteq \omega \, (A \in \mathcal{I} \implies f^{-1}[A] \in \mathcal{J}).$$

The ideal *conv*

$conv = \{A \subseteq \mathbb{Q} : A \text{ has at most finitely many limit points in } \mathbb{R}\}$

Sets of limit and cluster points

Set of limit point of a sequence

$$\Lambda((x_n)_{n\in\omega}) = \{p \in X : \exists\, A \notin \mathrm{Fin}\ ((x_n)_{n\in A} \to p)\}$$
$$= \{p \in X : \exists\, A \notin \mathrm{Fin}\ \forall \underset{\text{open}}{U \ni p}\ \forall^\infty n \in A\ (x_n \in U)\}$$

Set of cluster points of a sequence

$$\Gamma((x_n)_{n\in\omega}) = \{p \in X : \forall \underset{\text{open}}{U \ni p}\ \exists A \notin \mathrm{Fin}\ \forall n \in A\ (x_n \in U)\}.$$

Theorem (Folklore)

- $\Lambda(x_n) = \Gamma(x_n)$.
- $\Lambda(x_n)$ and $\Gamma(x_n)$ are closed.
- For every nonempty closed set F there is a sequence $(x_n)_{n\in\omega}$ such that
$$F = \Gamma(x_n).$$

Ideal sets of limit and cluster points

Ideal set of limit point of a sequence

$$\Lambda_{\mathcal{I}}((x_n)_{n\in\omega}) = \{p \in X : \exists A \notin \mathcal{I}\ ((x_n)_{n\in A} \to p)\}$$
$$= \{p \in X : \exists A \notin \mathcal{I}\ \underbrace{\forall U \ni p}_{\text{open}}\ \forall^\infty n \in A\ (x_n \in U)\}$$

Set of cluster points of a sequence

$$\Gamma_{\mathcal{I}}((x_n)_{n\in\omega}) = \{p \in X : \underbrace{\forall U \ni p}_{\text{open}}\ \exists A \notin \mathcal{I}\ \forall n \in A\ (x_n \in U)\}.$$

Theorem (Kostyrko-Šalát-Wilczyński, 2001)

- $\Lambda_{\mathcal{I}}(x_n) \subseteq \Gamma_{\mathcal{I}}(x_n)$.
- ~~$\Lambda_{\mathcal{I}}(x_n)$, and~~ $\Gamma_{\mathcal{I}}(x_n)$ ~~are~~ is closed.
- T.F.A.E.
  - For every nonempty closed set F, there is a sequence $(x_n)_{n\in\omega}$ such that $F = \Gamma(x_n)$.
  - There exists an infinite partition of ω into sets which are not in $\mathcal{I}$.

ideal limit set $\neq$ ideal cluster set

- $\Lambda_{\mathcal{I}}(x_n) \subseteq \Gamma_{\mathcal{I}}(x_n)$.

Theorem (Fridy, 1993)

There exists a sequence $(x_n)_{n\in\omega}$ in $[0,1]$ such that

$$\Lambda_{\mathcal{I}_d}(x_n) \neq \Gamma_{\mathcal{I}_d}(x_n).$$

Theorem (He-Zang-Zang, 2022)

T.F.A.E.

- $\Lambda_{\mathcal{I}}(x_n) = \Gamma_{\mathcal{I}}(x_n)$ for every sequence $(x_n)_{n\in\omega}$.
- $\mathcal{I}$ is a P^+ ideal.

P-like properties of ideals

$\mathcal{I} \in P^{+}$ if for every sequence $A_1 \supseteq A_2 \supseteq \ldots$ such that $A_n \notin \mathcal{I}$

$$\exists A \notin \mathcal{I}\ \forall n\ (A \setminus A_n \text{ is finite}).$$

$\mathcal{I} \in P^{-}$ if for every sequence $A_1 \supseteq A_2 \supseteq \ldots$ such that $A_n \notin \mathcal{I}$ and $A_n \setminus A_{n+1} \in \mathcal{I}$ for every n

$$\exists A \notin \mathcal{I}\ \forall n\ (A \setminus A_n \text{ is finite}).$$

$\mathcal{I} \in P^{|}$ if for every sequence $A_1 \supseteq A_2 \supseteq \ldots$ such that $A_n \notin \mathcal{I}$ and $A_n \setminus A_{n+1} \notin \mathcal{I}$ for every n

$$\exists A \notin \mathcal{I}\ \forall n\ (A \setminus A_n \text{ is finite}).$$

Fact

$$\mathcal{I} \in P^{+} \iff \mathcal{I} \in P^{-} \text{ and } \mathcal{I} \in P^{|}.$$

ideal limit set versus ideal cluster set

- $\Lambda_{\mathcal{I}}(x_n) \subseteq \Gamma_{\mathcal{I}}(x_n)$.
- $\Lambda_{\mathcal{I}}(x_n) = \Gamma_{\mathcal{I}}(x_n)$ for each (x_n) $\iff$ $\mathcal{I}$ is a P^+ ideal.

For $A \subseteq X$, we write
$A^{|}$ to denote the derived set of A i.e. the set of all limit points of A
A^{-} to denote the set of all isolated points of A

Theorem (F.-Kwela-Leonetti, 2023)

T.F.A.E.

- $\Lambda_{\mathcal{I}}(x_n) \supseteq (\Gamma_{\mathcal{I}}(x_n))^{|}$ for every sequence $(x_n)_{n\in\omega}$.
- $\mathcal{I}$ is $P^{|}$.

Theorem (F.-Kwela-Leonetti, 2023)

T.F.A.E.

- $\Lambda_{\mathcal{I}}(x_n) \supseteq (\Gamma_{\mathcal{I}}(x_n))^{-}$ for every sequence $(x_n)_{n\in\omega}$.
- $\mathcal{I}$ is P^{-}.

Ideal limit set is not closed

Theorem (Kostyrko-Šalát-Wilczyński, 2001)

- ~~$\Lambda_{\mathcal{I}}(x_n)$ and~~ $\Gamma_{\mathcal{I}}(x_n)$ ~~are~~ is closed.

Theorem (Balcerzak-Leonetti, 2019)

If an ideal $\mathcal{I}$ is F_σ, then $\Lambda_{\mathcal{I}}(x_n)$ is closed for every sequence $(x_n)_{n\in\omega}$.

Theorem (Kostyrko-Mačaj-Šalát-Strauch, 2001)

- For every nonempty F_σ set $F \subseteq [0,1]$ there exists a sequence $(x_n)_{n\in\omega}$ in $[0,1]$ such that $F = \Lambda_{\mathcal{I}_d}(x_n)$.
- There exists a sequence (x_n) such that $\Lambda_{\mathcal{I}_d}(x_n)$ is not closed.

Question

What properties of ideals characterize Borel complexity of $\Lambda_{\mathcal{I}}$ sets?

Answer

You have to wait for Part II.

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