In search of higher forcing axioms

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Outline

1. The Gödel program
2. Forcing axioms
3. Higher forcing axioms
4. Guessing models

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The Gödel program

ZFC (Zermelo Fraenkel axioms with Choice) serves as a good axiomatization for mathematics. It describes the self evident properties of the cumulative hierarchy of sets and the set theoretic universe $\mathbb{V}$.

Cumulative hierarchy

- $V_0 = \emptyset$
- $V_{\alpha+1} = \mathcal{P}(V_\alpha)$
- $V_\alpha = \bigcup_{\xi<\alpha} V_\xi$, for α limit
- $\mathbb{V} = \bigcup_{\alpha \in \mathrm{ORD}} V_\alpha$.

However, ZFC does not decide some basic questions such as:

- Cantor's **Continuum Hypothesis** (CH), in fact says very little about cardinal arithmetic in general
- **Souslin's Hypothesis** (SH) and related problems in general topology
- **regularity properties** (i.e. Lebesgue measurability, the property of Baire, etc) of projective sets of reals
- **Whitehead's problem** in Homological Algebra
- **Kaplansky's conjecture** in Banach algebras
- many many more....

Gödel [1947]

... if the meaning of the primitive terms of set theory... are accepted as sound, it follows that the set-theoretical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality.

Gödel's program

Search for new axioms and rules of inference for set theory which would decide the value of the continuum and other problems undecidable in ZFC alone.

New axioms typically assert the richness of the set theoretic universe.

- **Large cardinal axioms** - assert that the set theoretic universe is 'tall'. Provide a linear hierarchy of consistency strength. Have impact on the low levels of the cumulative hierarchy. Do not decide cardinal arithmetic, e.g. the Continuum Hypothesis.
- **Forcing axioms** - Assert a kind of 'saturation' of the universe of sets, i.e. if a set satisfying certain properties can be found in a suitable generic extension of the universe then such a set already exists. Decide combinatorial questions about uncountable cardinals left open by ZFC. Have strong influence on cardinal arithmetic.

These two types of axioms are very closely intertwined. Typically one needs large cardinals to prove the consistency of strong forcing axioms.

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Forcing axioms

General form of forcing axioms. Let $\mathcal{K}$ be a class of forcing notions and κ an uncountable cardinal.

$\mathrm{FA}_\kappa(\mathscr{K})$

For every $\mathbb{P} \in \mathscr{K}$ and a family $\mathscr{D}$ of κ dense subsets of $\mathbb{P}$ there is a filter G in $\mathbb{P}$ such that $G \cap D \neq \varnothing$, for all $D \in \mathscr{D}$.

For $\kappa = \aleph_0$ this is just the Baire category theorem and is provable in ZFC.

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Martin's Axiom

Definition

A partial order $\mathbb{P}$ has the **countable chain condition (ccc)** if every antichain (pairwise incompatible set) in $\mathbb{P}$ is at most countable.

Martin's Axiom (MA_κ) is the axiom $\mathrm{FA}_\kappa(\mathrm{ccc})$.

Theorem (Martin, Solovay 1970)

Suppose λ is uncountable and $2^{<\lambda} = \lambda$. Then there is a generic extension $V[G]$ in which $\mathrm{MA}_{<\lambda}$ holds and $2^{\aleph_0} = \lambda$.

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Some consequences of Martin's Axiom

Theorem

Let κ be an uncountable cardinal. MA_κ *implies that:*

1. $2^{\aleph_0} = 2^\kappa$
2. *the union of κ Lebesgue null sets is Lebesgue null*
3. *the union of κ meager sets is meager*
4. *No non principal ultrafilter on ω has a base of cardinality κ.*

Theorem

$\mathrm{MA}_{\aleph_1}$ *implies that:*

1. *there is no $\aleph_1$-Souslin tree*
2. *the product of ccc topological spaces is ccc*
3. *(Shelah) There is a Whitehead group that is not free.*

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Strong forcing axioms

Martin's Axiom does not have much impact on cardinal arithmetic, but there are stronger axioms such as the **Proper Forcing Axiom (PFA)** and **Martin's Maximum (MM)** that do.

- PFA $\equiv \mathrm{FA}_{\aleph_1}$(proper)
- SPFA $\equiv \mathrm{FA}_{\aleph_1}$(semi-proper)
- MM $\equiv \mathrm{FA}_{\aleph_1}$(stationary set preserving) $\equiv$ SPFA.

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Definition (Proper forcing)

A forcing notion $\mathbb{P}$ is **proper** if for every θ large enough and every countable $N \prec H_\theta$, $\mathbb{P} \in N$, $p \in \mathbb{P} \cap N$, there is $q \leq p$ such that

$$q \Vdash_{\mathbb{P}} N[\dot{G}] \cap \mathrm{ORD} = \mathrm{N} \cap \mathrm{ORD}.$$

Such q is called **$(N, \mathbb{P})$-generic**.

Definition (Semi proper forcing)

Same as above, but require only

$$q \Vdash_{\mathbb{P}} N[\dot{G}] \cap \omega_1 = N \cap \omega_1.$$

Such q is called **$(N, \mathbb{P})$-semi-generic**.

Theorem (Baumgartner, Shelah 1983)

Assume there is a supercompact cardinal. Then there is a generic extension $V[G]$ of V in which PFA *holds.*

Theorem (Foreman, Magidor, Shelah 1988)

Assume there is a supercompact cardinal. Then there is a generic extension $V[G]$ of V in which MM *holds.*

Remark

$\mathrm{FA}_{\aleph_1}(\mathscr{K})$ for the class $\mathscr{K}$ of all posets or even posets preserving $\aleph_1$ is false in ZFC.

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Consequences of PFA

PFA has many important consequences that do not follow from $\mathrm{MA}_{\aleph_1}$.

Theorem (Baumgartner)

PFA *implies that:*

1. *every two* $\aleph_1$*-dense sets of reals without endpoints are isomorphic*
2. *there are no Kurepa trees on* ω_1
3. *there are no* $\aleph_2$*-Aronszajn trees.*

Theorem (Woodin)

PFA *implies Kaplansky's Conjecture, i.e. for every compact Hausdorff space* X*, every homomorphism of* $C(X)$ *to a Banach algebra is continuous.*

Structural consequences of PFA

Theorem (V. 1987)

PFA *implies that* $2^{\aleph_0} = \aleph_2$.

Theorem (Viale 2005)

PFA *implies the Singular Cardinal Hypothesis.*

Theorem (Todorcevic 1983)

PFA *implies the failure of Jensen's principle* $\square_\kappa$, *for all* $\kappa \geq \aleph_1$.

This implies that the consistency of PFA requires some very large cardinal assumptions.

Structural consequences of MM

Theorem (Foreman, Magidor, Shelah 1988)

MM *implies that:*

1. NS_{ω_1} *is* $\aleph_2$*-saturated, i.e. there is no family of* $\aleph_2$ *pairwise almost disjoint stationary subsets of* ω_1.
2. *Chang's conjecture, i.e.* $(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$.

Theorem (Woodin)

Asssume MM. *Then* $\underset{\sim}{\delta}^1_2 = \omega_2$.

$\underset{\sim}{\delta}^1_2$ is the supremum of the lengths of $\underset{\sim}{\delta}^1_2$-prewellorderings of $\mathbb{R}$. This is an **effective** failure of CH, i.e. at the level of descriptive set theory.

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Motivating question

MA_κ is consistent for arbitrary large cardinals κ, but is too weak to decide many basic questions, such as the value of the continuum.

Question

Are there versions of strong forcing axioms for cardinals $\kappa > \aleph_1$?

The main technical issue involves iterations of forcing notions while preserving cardinals. Another problem is that we have a proper class of forcing notions to worry about and we are doing an iteration of length κ, for a fixed cardinal κ. This is handled as in the classical approach using a supercompact cardinals and a Laver diamond sequence to capture all forcing notions in the given class.

Standard iteration techniques:

1. finite support iteration - preserves all cardinals for ccc posets
2. countable support iteration - preserves ω_1 for proper posets
3. revised countable support - preserves ω_1 for semiproper posets

In 2. and 3. if we do an iteration of length κ inaccessible, all iterands are of size $< \kappa$, and we take direct limits at α, for stationary many α, then the whole iteration has the κ-cc, so cardinals $\geq \kappa$ are also preserved.

Limitations

- finite support iteration of non ccc forcing even of countable length collapses ω_1
- the proofs that c.s. (r.c.s.) iterations of proper (semiproper) posets preserve ω_1 depend on **diagonalization**. This can work for only one cardinal, e.g. ω_1.

There is work by Shelah and Roslanowski on **uncountable support** iterations, but this requires the individual posets to be highly closed.

Our ability to iterate forcing while doing something interesting at **both** ω_1 and ω_2 remained severely limited. Forcing axioms for $\aleph_2$ many dense sets, such as Generalized Martin's Axiom (Baumgartner, Shelah) typically apply only to σ-closed posets.

Recent work

- **Neeman** introduced finite support iterations with finite $\in$-chains of elementary submodels as **side conditions**. This allowed him to decouple the size of the support with the preservation of cardinals. He gave a new proof of the consistency of PFA using models of two types: countable and transitive, and proposed several higher versions of PFA (i.e. for $\aleph_2$-dense sets) using models of three types, countable, size $\aleph_1$ and transitive.

- **Aspero and Mota** defined a class of forcing notions $\mathscr{K}_{1.5}$, they call $\aleph_{1.5}$-cc, and use symmetric systems of countable elementary submodels of H_θ (for suitable θ) to iterate such forcings and get the consistency of the forcing axiom $\mathrm{FA}_\kappa(\mathscr{K}_{1.5})$, for κ arbitrarily large.

- **Krueger** also developed a framework for forcing with adequate finite systems of countable elementary submodels as side conditions. He used them to add various objects of size ω_2, but no general iteration theorem.

Recent work

In 2014 I adapted Neeman's method in order to iterate semiproper forcings. This lead to the notion of **virtual models**.

Definition

Fix an inaccessible κ. Let E be the set of all α such that $V_\alpha \prec V_\kappa$.

- *For $\alpha \in E$ we say that a model M is a* ***virtual α-model*** *if $\mathrm{Hull}(M, V_\alpha)$ is transitive and $V_\alpha \prec \mathrm{Hull}(M, V_\alpha)$.*
- *$M \simeq_\alpha N$ if there is an isomorphism $\sigma : \mathrm{Hull}(M, V_\alpha) \to \mathrm{Hull}(N, V_\alpha)$ such that $\sigma[M] = N$.*
- *We define $M \in_\alpha N$ if there is $M' \in N$ such that $M \simeq_\alpha M'$.*

We think of $\simeq_\alpha$ and $\in_\alpha$ as versions of equality and membership relations for virtual α-models. If $\alpha < \beta$ and both are in E, there is a natural projection $M \mapsto M \restriction \alpha$ mapping virtual β-models to virtual α-models.

We can use finite systems of countable virtual models as the side conditions in our iteration. The advantages over Neeman's method:

- There is no need for transitive models, the method is simpler
- We can iterate semiproper posets, in fact even more general classes including Namba like forcings. We do not use all countable virtual models, but only some special ones that we call **full**.

If we want to iterate forcing while preserving two cardinals we should use two types of models. We need to define a virtual version of **intersection**, we call it the meet and denote it by $\wedge$. It should behave well with respect to projections, $\in_\alpha$ and $\simeq_\alpha$, for $\alpha \in E$.

It turns out that if the models of the second type have some second order properties the theory is smoother. Moreover, just the pure side condition forcing with models of the right type gives some strong consequences. The key notion is that of a **guessing model**.

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The notion of a **guessing model** emerged from the work of Viale and Weiss in an attempt to find the combinatorial essence of some large cardinal axioms that can hold at small cardinals.

Definition (Viale)

Let R be a model of a fragment of set theory and $M \prec R$. Let γ be a cardinal. Let $Z \in M$ and $f : Z \to 2$ be a function.

1. *f is γ-**approximated** in M if $f \restriction C \in M$, for all $C \in \mathcal{P}_\gamma(Z) \cap M$.*
2. *f is **guessed** in M if there is $\overline{f} \in M$ such that $f \restriction M = \overline{f} \restriction M$.*

*We say that M is a γ-**guessing model** if every $f \in R$ which is γ-approximated in M is guessed in M.*

Write $\mathcal{P}^*_\kappa(R)$ for the set of all $M \prec R$ such that $M \cap \kappa \in \kappa$. For $\gamma \le \kappa$ we let

$$\mathfrak{G}_{\kappa,\gamma}(R) = \{M \in \mathcal{P}^*_\kappa(R) : M \text{ is } \gamma\text{-guessing}\}.$$

Definition (Viale)

$\mathrm{GM}(\kappa,\gamma)$ *is the statement that* $\mathfrak{G}_{\kappa,\gamma}(H_\theta)$ *is stationary, for all sufficiently large* θ.

We are primarily interested in $\gamma = \omega_1$ and $\kappa = \omega_2$, i.e. ω_1-guessing models of size ω_1.

Lemma (Viale)

1. *If M is $\aleph_0$-guessing then $\kappa_M = M \cap \kappa$ and κ are inaccessible.*
2. *$M \prec V_\delta$ is $\aleph_0$-guessing iff $\bar{M} = V_{\bar{\delta}}$, for some $\bar{\delta}$, where $\bar{M}$ is the transitive collapse of M.*

The following is a reformulation of Magidor's characterization of supercompactness in terms of $\aleph_0$-guessing models.

Theorem (Magidor)

κ is supercompact iff $\mathrm{GM}(\kappa, \aleph_0)$ holds.

Remark

For this reason we use the term **Magidor models** for $\aleph_0$-guessing models.

Theorem (Viale, Weiss)

PFA *implies* $\mathrm{GM}(\omega_2, \omega_1)$.

Theorem (Weiss)

$\mathrm{GM}(\omega_2, \omega_1)$ *implies*

1. *the failure of* $\square(\lambda)$, *for all regular* $\lambda \geq \omega_2$
2. *the tree property holds at* $\aleph_2$.

Theorem (Krueger, Viale)

$\mathrm{GM}(\omega_2, \omega_1)$ *implies the Singular Cardinal Hypothesis.*

Theorem (Cox, Krueger)

$\mathrm{GM}(\omega_2, \omega_1)$ *is consistent with* $2^{\aleph_0}$ *arbitrarily large.*

Approachability ideal

Guessing models are closely related to the approachability ideal $I[\lambda]$.

Definition

Let λ be a regular cardinal and $\bar{a} = (a_\xi : \xi < \lambda)$ a sequence of bounded subsets of λ. We let $B(\bar{a})$ denote the set of all $\delta < \lambda$ such that there is a cofinal $c \subseteq \delta$ such that:

1. $\mathrm{otp}(c) < \delta$, *in particular δ is singular,*
2. *for all $\gamma < \delta$, there is $\eta < \delta$ such that $c \cap \gamma = a_\eta$.*

Definition (Shelah)

Suppose λ is regular. $I[\lambda]$ is the ideal generated by the sets $B(\bar{a})$, for sequences $\bar{a}$ as above, and the non stationary ideal NS_λ.

Approachability ideal

This ideal was defined by Shelah in the late 1970s. $I[\lambda]$ and its variations have been extensively studied over the past 40 years.

For regular $\kappa < \lambda$ we let $S^{\kappa}_{\lambda} = \{\alpha < \lambda : \mathrm{cof}(\alpha) = \kappa\}$.

Theorem (Shelah)

Suppose λ is a regular cardinal.

1. *Then $S^{<\lambda}_{\lambda^+} \in I[\lambda^+]$.*
2. *Suppose κ is regular and $\kappa^+ < \lambda$. Then there is a stationary subset of S^{κ}_{λ} which belongs to $I[\lambda]$.*

The **approachability property** AP_{κ^+} states that $\kappa^+ \in I[\kappa^+]$. For a regular cardinal κ the issue is to understand $I[\kappa^+] \upharpoonright S^{\kappa}_{\kappa^+}$.

Approachability ideal

We concentrate on the case $\kappa = \omega_1$.

Fact

Suppose $\bar{a} = (a_\xi : \xi < \omega_2)$ *is a sequence of bounded subsets of* ω_2. *Let* $M \prec H_\theta$ *be an* ω_1*-guessing model of size* ω_1 *such that* $\bar{a} \in M$. *Then* $M \cap \omega_2 \notin B(\bar{a})$.

Therefore, $\mathrm{GM}(\omega_2, \omega_1)$ implies that $S^{\omega_1}_{\omega_2} \notin I[\omega_2]$. However, one can ask a stronger question.

Question (Shelah)

Can $I[\omega_2] \restriction S^{\omega_1}_{\omega_2}$ consistently be the nonstationary ideal on $S^{\omega_1}_{\omega_2}$?

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Approachability ideal

Note that this cannot follow from $\mathrm{GM}(\omega_2, \omega_1)$ since it requires the continuum to be at least ω_3.

Theorem (Mitchell)

Suppose κ is κ^+-Mahlo. Then there is a generic extension in which $\kappa = \omega_2$ and $I[\omega_2] \upharpoonright S^{\omega_1}_{\omega_2}$ is the non stationary ideal on $S^{\omega_1}_{\omega_2}$.

Remark

In Mitchell's model $\omega_3 \in I[\omega_3]$. It is not known if one can have Mitchell's result for two consecutive cardinals, say ω_2 and ω_3.

Strong guessing models

Definition

Let R be a model of a fragment of ZFC. We say that $M \prec R$ is a ***strong ω_1-guessing model*** *if M can be written as the union of an increasing ω_1-continuous $\in$-chain $(M_\xi : \xi < \omega_2)$ of ω_1-guessing models of size ω_1.*

$$\mathfrak{G}^+_{\omega_3,\omega_1}(R) = \{M \in [R]^{\omega_2} : M \text{ is a strong } \omega_1\text{-guessing model}\}.$$

Definition

$\mathrm{GM}^+(\omega_3, \omega_1)$ *states that $\mathfrak{G}^+_{\omega_3,\omega_1}(H_\theta)$ is stationary, for all large enough θ.*

Remark

$\mathrm{GM}^+(\omega_3, \omega_1)$ obviously implies Mitchell's result.

Strong guessing models

Theorem

$\mathrm{GM}^+(\omega_3, \omega_1)$ *implies the following:*

1. $2^{\aleph_0} \geq \aleph_3$.
2. *there are no weak* ω_1*- Kurepa trees.*
3. *the tree property at* ω_2 *and* ω_3.
4. *the failure of* $\square(\lambda)$, *for all* $\lambda \geq \omega_2$.
5. *Singular Cardinal Hypothesis.*
6. $I[\omega_2] \restriction S^{\omega_1}_{\omega_2}$ *is the non stationary ideal on* $S^{\omega_1}_{\omega_2}$.

The point is that these are the consequences that we would expect to have from a higher version of PFA.

Theorem (Mohammadpour, V.)

Suppose $\kappa < \lambda$ are supercompact cardinals. There there is a generic extension $V[G]$ of V in which κ is ω_2, λ is ω_3, and $\mathrm{GM}^+(\omega_3, \omega_1)$ holds.

The forcing notion $\mathbb{P}_\kappa^\lambda$ we use consists of finite collections $p = \mathcal{M}_p$ of virtual models of one of two types: countable and κ-Magidor models. We also require that, for every $\alpha \in E$, the collection of models $M \in \mathcal{M}_p$ that are **active** at α forms an $\in_\alpha$-chains and is closed under meets.
We say that $q \leq p$ if, for every α-model $M \in \mathcal{M}_p$, there is $N \in \mathcal{M}_q$ such that $N \restriction \alpha = M$.

One can then start adding working parts and do an iteration, but this is a topic of another lecture...

Thank You!

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