Strong forcing axioms and the Continuum problem

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Topics explored in the talk

Baire Category theorem

$\Longleftrightarrow$

Forcing axioms

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Baire Category theorem

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Forcing axioms

Algebraic extensions of fields ($\mathbb{Q} \mapsto \mathbb{Q}[x]/x^2 + 1 = 0$)

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Bounded forcing axioms

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Baire Category theorem

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Algebraic extensions of fields ($\mathbb{Q} \mapsto \mathbb{Q}[x]/x^2 + 1 = 0$)

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Bounded forcing axioms

Duality theorem (Hilbert's nullstellensatz)

$\Longleftrightarrow$

Forcing axioms imply bounded forcing axioms ($\mathsf{MM}^{++} \to (*)$)

Topics explored in the talk

Model companionship

$\Longleftrightarrow$

Algebraic maximality (algebraically closed fields)

$\Longleftrightarrow$

Bounded forcing axioms — Determinacy

Section 1

Basics of Set Theory

Morse-Kelley Axioms of Set Theory MK

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- Uppercase variables $X, Y, Z, \ldots$ denote classes (which are possibly also sets).

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Universal axioms

- **Extensionality:** Two classes (or sets) are equal if they have exactly the same elements.
- **Comprehension (a):** Every class (or set) is a subset of V where

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*(a **set** is a class which belongs to some class).*

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- **Foundation:** There is no infinite sequence $\langle x_n : n \in \mathbb{N} \rangle$ such that $x_{n+1} \in x_n$ for all n.

  V is not a set, else $x_n = V$ for all n violates Foundation.

Morse-Kelley Axioms of Set Theory MK

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  - $1 = \{0\} = \{\emptyset\}$,
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Morse-Kelley Axioms of Set Theory MK

Basic construction principles:

- **Union, Pair, Product:** If X, Y are sets so are $X \cup Y$, $\{X, Y\}$, $X \times Y$.
- **Separation:** If P is a class and X is a set, $P \cap X$ is a set.

Morse-Kelley Axioms of Set Theory MK

Strong construction principles:

- **Comprehension (b):** For every property $\psi(x)$, $P_\psi = \{a \in V : \psi(a)\}$ is a class.
- **Replacement:** If F is a class function and $X \subseteq \mathrm{dom}(F)$ is a set, $F[X]$ is a set.
- **Powerset:** If X is a set so is $\mathcal{P}(X) = \{Y : Y \subseteq X\}$.
- **Global Choice:** For all classes $C = \{X_i : i \in I\}$ of non-empty sets X_i, $\prod_{i \in I} X_i$ is non-empty.

Given sets X, Y

Cardinality

- $|X|$ is the (proper) class $\{Y : \exists f : X \to Y \text{ bijection}\}$;
- $|X| \leq |Y|$ iff *there is* $f : X \to Y$ *injection* iff *there is* $g : Y \to X$ *surjection*;
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  - $|[0;1]| \leq |(0;1)|$ and $|[0;1]| \geq |(0;1)|$ witnessed by continuous functions.
  - $f : [0;1] \to (0;1)$ bijection, f is not continuous.

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  If $g : X \to \mathcal{P}(X)$, g is not a surjection as witnessed by

$$Y_g = \{x \in X : x \notin g(x)\}.$$

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Cardinals

- $\aleph_0 = |\mathbb{N}|$;
- $\aleph_1 = \aleph_0^+ = \min\{|Z| : |Z| > \aleph_0\}$;
- $2^{\aleph_0} = |\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.

Continuum Hypothesis CH (Cantor 1878, Hilbert 1900)

- $\aleph_1 = 2^{\aleph_0}$, or equivalently
- if $Z \subseteq \mathbb{R}$, either $|Z| = |\mathbb{R}|$ or $|Z| \leq |\mathbb{N}|$.

Counterexamples to CH?

- No *closed* subset of $\mathbb{R}$ is a counterexample to CH (Cantor 1883).
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Figure: Analytic and coanalytic sets

The projective susets of $\mathbb{R}^n$ are those subsets of $\mathbb{R}^n$ which are Σ^1_m (or Π^1_m) for some m.

Figure: Projective sets

Counterexamples to CH? continued

Assume there is a proper class of Woodin cardinals. Then:

- No *universally Baire* subset of $\mathbb{R}$ is a counterexample to CH (Feng-Magidor-Woodin 1992 + Steel-Martin 1989 + Davis 1964).
- Borel sets, analytic sets, projective sets, ... are all universally Baire (Feng-Magidor-Woodin 1992 + Steel-Martin 1989).

Definition

$U \subseteq \mathbb{R}$ is universally Baire if $f^{-1}[U]$ has the Baire property in X for any continuous $f : X \to \mathbb{R}$ with (X, τ) compact Hausdorff.

- Analytic and coanalytic sets are universally Baire provably in MK (without large cardinals).
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Independence of CH

CH is independent of the axioms of set theory:

- There is a model of the axioms of MK where CH holds (Gödel 1939).
- There is a model of the axioms of MK where CH fails (Cohen 1963).
- In the model of the axioms of MK where CH fails produced by Cohen, this failure can be witnessed by a Σ^1_2-set of reals.

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Section 2

Gödel's program

WHAT IS CANTOR'S CONTINUUM PROBLEM?

KURT GÖDEL, Institute for Advanced Study

The American Matematical Monthly, 54(9), 1947

WHAT IS CANTOR'S CONTINUUM PROBLEM?

KURT GÖDEL, Institute for Advanced Study

On the undecidability of CH:

> Only someone who (like the intuitionist) denies that the concepts and axioms of classical set theory have any meaning (or any well-defined meaning) could be satisfied with such a solution, not someone who believes them to describe some well-determined reality. For in this reality Cantor's conjecture must be either true or false, and its undecidability from the axioms as known today can only mean that these axioms do not contain a complete description of this reality;

WHAT IS CANTOR'S CONTINUUM PROBLEM?

KURT GÖDEL, Institute for Advanced Study

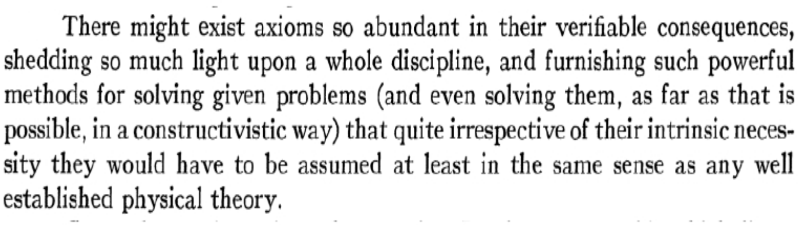
On Large Cardinals:

For first of all the axioms of set theory by no means form a system closed in itself, but, quite on the contrary, the very concept of set[17] on which they are based suggests their extension by new axioms which assert the existence of still further iterations of the operation "set of." These axioms can also be formulated as propositions asserting the existence of very great cardinal numbers or (which is the same) of sets having these cardinal numbers. The simplest of these strong "axioms of infinity" assert the existence of inaccessible numbers (and of numbers inaccessible in the stronger sense) $>\aleph_0$.


WHAT IS CANTOR'S CONTINUUM PROBLEM?

KURT GÖDEL, Institute for Advanced Study

On success as a criterion to detect new axioms:


There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even solving them, as far as that is possible, in a constructivistic way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established physical theory.

Section 3

Large cardinals

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- The existence of arbitrarily many Grothendieck universes is equivalent to:

There is a proper class of inaccessible cardinals.

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Adamek-Rosicky, *Locally presentable and accessible categories*, CUP, 1994.

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Fact

Assume Vopenka's principle. Then there is a proper class of Woodin cardinals.

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From nLab:

The implication of VP on homotopy theory, model categories and cohomology localization are discussed in the following articles

- Jiří Rosický, Walter Tholen, *Left-determined model categories and universal homotopy theories* Transactions of the American Mathematical Society Vol. 355, No. 9 (Sep., 2003), pp. 3611-3623 (JSTOR).
- Carles Casacuberta, Dirk Scevenels, Jeff Smith, *Implications of large-cardinal principles in homotopical localization* Advances in Mathematics Volume 197, Issue 1, 20 October 2005, Pages 120-139
- Joan Bagaria, Carles Casacuberta, Adrian Mathias, Jiří Rosicky *Definable orthogonality classes in accessible categories are small*, arXiv
- Giulio Lo Monaco, *Vopěnka's principle in ∞-categories*, arxiv:2105.04251

JOURNALS » JEMS » VOL. 17, NO. 3

Journal of the European Mathematical Society

Volume 17, No. 3 (2015)

Section 4

Forcing axioms

Forcing axioms relative to a cardinal κ:

The powerset of X is "as thick as possible" for given X of size κ,

Forcing axioms for κ can be divided in two categories:

- **topological maximality:** strong forms of Baire's category theorem, generic points, MM^{++}.
- **algebraic maximality:** closure of $\mathcal{P}(X)$ under a variety of set theoretic operations for any fixed X of size κ, algebraically closed structures, Woodin's axiom $(*)$.

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Forcing axioms for κ can be divided in two categories:

- **topological maximality:** strong forms of Baire's category theorem, generic points, MM^{++}.
- **algebraic maximality:** closure of $\mathcal{P}(X)$ under a variety of set theoretic operations for any fixed X of size κ, algebraically closed structures, Woodin's axiom $(*)$.

The rest of the talk is mainly aimed at formulating precisely these two concepts.

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Section 5

Topological maximality for set theory

Baire's category theorem

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Definition

Let κ be an infinite cardinal and (X, τ) a topological space. $\mathsf{FA}_\kappa(X, \tau)$ holds if $\bigcap_{i \in \kappa} D_i$ is *dense* in X for all $\{D_i : i \in \kappa\}$ family of dense open subsets of X.

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What about $\mathrm{FA}_{\aleph_1}(X,\tau)$?

Example

Let Y be an *uncountable set* and (X,τ) be the Stone-Čech compactification of the space $Y^{\mathbb{N}}$ with product topology induced by the discrete topology on Y.
Then $\mathrm{FA}_{\aleph_1}(X,\tau)$ fails.

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Assume (X, τ) is a compact Hausdorff space which is not SSP*.*
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Martin's maximum MM $\equiv$ $\mathrm{FA}_{\aleph_1}(X, \tau)$ *holds for all compact Hausdorff spaces* (X, τ) *which are* SSP.

MK decides $\mathrm{FA}_{\aleph_1}(X,\tau)$ fails if $(X,\tau) \notin$ SSP, but does not decide whether all $(X,\tau) \in$ SSP satisfy $\mathrm{FA}_{\aleph_1}(X,\tau)$.

$CH = \{(X,\tau) : (X,\tau) \text{ is compact Hausdorff}\}$

- $\mathrm{FA}_{\aleph_1} = \{(X,\tau) \in CH : \mathrm{FA}_{\aleph_1}(X,\tau) \text{ holds}\}$
- $\mathrm{FA}^c_{\aleph_1} = CH \setminus \mathrm{FA}_{\aleph_1} = \{(X,\tau) \in CH : \mathrm{FA}_{\aleph_1}(X,\tau) \text{ fails}\}$

MK + MM decides $\mathsf{FA}_{\aleph_1}(X, \tau)$ if and only if it is not impossible.

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Some applications of MM^{++}

Assume MM. Then:

- $2^{\aleph_0} = \aleph_2 = \aleph_1^+$.
  Foreman, Magidor, Shelah, 1988.
- Whitehead's conjecture on free groups is false, (i.e. there are uncountable Whitehead groups which are not free).
  Shelah, Israel Journal of Mathematics, 18(3), 1974.
- There are five uncountable linear orders such that any uncountable linear order contains an isomorphic copy of one of them.
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All the conclusions of these theorems are independent of MK+*Vopenka's principle*.

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Section 6

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We adjoin to the universe of sets V an ideal element G with some constraints, and we form $V[G]$.

Forcing

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CATEGORY THEORIST

Given the compact extremally disconnected Hausdorff space $\mathrm{St}(\mathrm{B})$,

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SET THEORIST	**CATEGORY THEORIST**
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the boolean valued model V^{B} by Cohen/Scott-Solovay-Vopenka;	the topos Sh$(\mathrm{St}(\mathsf{B})$, CompHaus$)$, given by topological sheaves on $\mathrm{St}(\mathsf{B})$ with target a compact Hausdorff space;

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	The properties of	
$V[G]$ depend mainly on B		Sh$(\mathrm{St}(\mathsf{B}),$ CompHaus$)/_G$ depend mainly on $\mathrm{St}(\mathsf{B})$
	and minimally on G.	

Section 7

Algebraic closure and model companionship

Algebraic closure of structures for $\{+, \cdot, 0, 1\}$

Structures	Axioms	Example
Commutative semirings with no zero divisors	$\forall x, y\, (x \cdot y = y \cdot x)$ $\forall x, y, z\, [(x \cdot y) \cdot z = x \cdot (y \cdot z)]$ $\forall x\, (x \cdot 1 = x \wedge 1 \cdot x = x)$ $\forall x, y\, (x + y = y + x)$ $\forall x, y, z\, [(x + y) + z = x + (y + z)]$ $\forall y\, (x + 0 = x \wedge 0 + x = x)$ $\forall x, y, z\, [(x + y) \cdot z = (x \cdot y) + (x \cdot z)]$ $\forall x, y\, [x \cdot y = 0 \rightarrow (x = 0 \vee y = 0)]$	$\mathbb{N}$
Integral domains	$\forall x \exists y\, (x + y = 0)$	$\mathbb{Z}$
Fields	$\forall x\, [x \neq 0 \rightarrow \exists y\, (x \cdot y = 1)]$	$\mathbb{Q}$
Algebraically closed fields	for all $n \geq 1$ $\forall x_0 \ldots x_n \exists y\, \sum x_i \cdot y^i = 0$	$\mathbb{C}$

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Existentially closed structures and model companionship

$$\langle \mathbb{Z}, +, \cdot, 0, 1 \rangle \quad \sqsubseteq \quad \langle \mathbb{C}, +, \cdot, 0, 1 \rangle \quad \sqsubseteq \quad \langle \mathbb{C}[X], +, \cdot, 0, 1 \rangle$$

Existentially closed structures and model companionship

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$\exists x\,(x^2 - 2 = 0)$? $\exists x\,(x^3 + 2x + i = 0)$?

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Given a vocabulary τ and τ-structures $\mathcal{M} \sqsubseteq \mathcal{N}$, $\mathcal{M} \prec_1 \mathcal{N}$ if every Σ_1-formula with parameters in $\mathcal{M}$ and true in $\mathcal{N}$ is true also in $\mathcal{M}$.

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Existentially closed structures and model companionship

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Example

In the vocabulary $\{+, \cdot, 0, 1\}$, the atomic formulae are diophantine equations and the quantifier free formulae with parameters in a ring $\mathcal{M}$ define the constructible sets (in the sense of algebraic geometry) of $\mathcal{M}$:

$$\bigvee_{j=1}^{l} \left[\bigwedge_{i=1}^{k_j} p_{ij}(a_1^{ij}, \ldots, a_{m_{ij}}^{ij}, x_1, \ldots, x_n) = 0 \wedge \bigwedge_{d=1}^{m_j} \neg q_{dj}(b_1^{dj}, \ldots, b_{k_{dj}}^{dj}, x_1, \ldots, x_n) = 0 \right]$$

with each a_k^{ij}, b_k^{dj} elements of $\mathcal{M}$ and $p_{ij}(y_1, \ldots, y_{m_{ij}}, x_1, \ldots, x_n) = 0$, $q_{dj}(z_1, \ldots, z_{k_dj}, x_1, \ldots, x_n) = 0$ diophantine equations (of degree 1 in the y_l, z_h-s).

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Given a vocabulary τ and τ-structures $\mathcal{M} \sqsubseteq \mathcal{N}$, $\mathcal{M} \prec_1 \mathcal{N}$ if every Σ_1-formula with parameters in $\mathcal{M}$ and true in $\mathcal{N}$ is true also in $\mathcal{M}$.

- A τ-formula $\phi(x_1, \ldots, x_n)$ is quantifier free if it is a boolean combination of atomic formulae.
- A τ-formula $\psi(x_0, \ldots, x_n)$ is a Σ_1-formula if it is of the form $\exists y_0, \ldots, y_k \phi(y_0, \ldots, y_k, x_0, \ldots, x_n)$ with $\phi(y_0, \ldots, y_k, x_0, \ldots, x_n)$ quantifier free.

Existentially closed structures and model companionship

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Given a τ-theory S, a τ-structure $\mathcal{M}$ is S-ec if:

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For S the $\{+, \cdot, 0, 1\}$-theory of integral domains the algebraically closed fields are the S-ec models.

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Example

The $\{+, \cdot, 0, 1\}$-theory of integral domains has the $\{+, \cdot, 0, 1\}$-theory of algebraically closed fields as its model companion.

The right vocabulary for a mathematical theory

Every mathematical theory can be axiomatized in first order logic by suitably choosing the vocabulary for its basic concepts.

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Consider Group Theory

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Axioms of groups in $\{\cdot, e\}$

$$\forall x, y, z\, [(x \cdot y) \cdot z = x \cdot (y \cdot z)],$$
$$\forall y\, (e \cdot y = y \wedge y \cdot e = y),$$
$$\forall x \exists y\, [x \cdot y = e \wedge y \cdot x = e].$$

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Axioms of groups in $\{R, e\}$ with R a ternary relation symbol

$\forall x, y \exists! z\, R(x, y, z)$,

$\forall x, y, z, w, t\,[((R(x, y, w) \wedge R(y, z, t)) \to \exists u\,(R(x, t, u) \wedge R(w, z, u))]$,

$\forall y\,[R(e, y, y) \wedge R(y, e, y)]$,

$\forall x \exists y\,[R(x, y, e) \wedge R(y, x, e)]$.

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$\forall y\, [R(e, y, y) \wedge R(y, e, y)],$

$\forall x \exists y\, [R(x, y, e) \wedge R(y, x, e)].$

The two axiomatizions are equivalent in the vocabulary $\{R, \cdot, e\}$, modulo the axiom

$$\forall x, y, z\, (R(x, y, z) \leftrightarrow x \cdot y = z)$$

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Formalizing in the $\{\in\}$-vocabulary the notion of ordered pair:

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Kuratowski's trick: $\langle y, z\rangle$ is coded in set theory by the set $\{\{y\}, \{y, z\}\}$.
In set theory the standard $\in$-formula expressing $x = \langle y, z\rangle$ is

$$\exists t \exists u\, [\forall w\, (w \in x \leftrightarrow w = t \vee w = u) \wedge \forall v\, (v \in t \leftrightarrow v = y) \wedge \forall v\, (v \in u \leftrightarrow v = y \vee v = z)].$$

The right vocabulary for set theory

The vocabulary $\in_{\Delta_0}$ for set theory

- constants for $\emptyset, \mathbb{N}$,
- relation symbols R_ϕ for any lightface Δ_0-property $\phi(x_1, \ldots, x_n)$,
- function symbols for a finite list of basic set theoretic constructors.

The right vocabulary for set theory

Lightface Δ_0-properties

- $\{R \in V : R$ is an n-ary relation$\}$,
- $\{f \in V : f$ is a function$\}$,
- $\left\{\langle a, b\rangle \in V^2 : a \subseteq b\right\}$,
- ...
- $\{\langle a_1, \ldots, a_n\rangle \in V^n : (V, \in) \models \phi(a_1, \ldots, a_n)\}$ for any $\in$-formula $\phi(x_1, \ldots, x_n)$ where quantified variables are bounded to range in a set.

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The *lightface Δ_0-properties* are those described in the last item above and include all those listed in some of the above items.

The right vocabulary for set theory

Lightface Δ_0-properties

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- $\{f \in V : f$ is a function$\}$,
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Complicated set theoretic relations

- $\left\{\langle X, Y\rangle \in V^2 : |X| = |Y|\right\}$,
- $\left\{\langle X, Y\rangle \in V^2 : X = \mathcal{P}(Y)\right\}$,
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- any relation which is not a Δ_1-property

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- ...
- any relation which is not a Δ_1-property ($\Delta_0 \subseteq \Delta_1$).

The right vocabulary for set theory

Basic set theoretic operations

- $\pi^n_j : \langle a_1, \ldots, a_n \rangle \mapsto a_j$,
- $\langle X, Y \rangle \mapsto X \times Y$,
- $\langle X, Y \rangle \mapsto \{X, Y\}$,
- $\ldots$
- Any provably total function whose graph is a lightface Δ_0-property.

The right vocabulary for set theory

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- Any provably total function whose graph is a lightface Δ_0-property.

Complicated set theoretic operations

- $X \mapsto \mathcal{P}(X)$,
- $X \mapsto \kappa$ where κ is the least ordinal in $|X|$,
- $\ldots$
- any operation whose graph is not expressible by a Δ_1-property.

The right vocabulary for set theory

The vocabulary $\in_{\Delta_0}$ for set theory

- constants for $\emptyset, \mathbb{N}$,
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Lightface Δ_0-properties

$$\{\langle a_1, \ldots, a_n\rangle \in V^n : (V, \in) \models \phi(a_1, \ldots, a_n)\}$$

for any $\in$-formula $\phi(x_1, \ldots, x_n)$ where quantified variables are bounded to range in a set.

Basic set theoretic operations

Any total function whose graph is a lightface Δ_0-property.

Section 8

Formalization of set theory

Axioms of Set Theory in $\in_{\Delta_0} \cup \{\text{Set}, V\}$

Notational convention: smallcase variables indicate sets, uppercase variables indicate classes.

Universal axioms

- **Extensionality:** $\forall X, Y\,[(X \subseteq Y \wedge Y \subseteq X) \leftrightarrow X = Y]$.
- **Comprehension:** $\forall X\,(\text{Set}(X) \leftrightarrow X \in V) \wedge \forall X\,(X \subseteq V)$.
- **Foundation:**

$$\forall F\,[(F \text{ is a function } \wedge \text{dom}(F) = \mathbb{N}) \rightarrow \exists n \in \mathbb{N}\, F(n+1) \notin F(n)].$$

Axioms of Set Theory in $\in_{\Delta_0} \cup \{\mathrm{Set}, V\}$

Existence Axioms:

- **Emptyset:** $(\forall x\, x \notin \emptyset) \wedge (\emptyset \in V)$,
- **Infinity:**
  $\mathrm{Set}(\mathbb{N}) \wedge \forall x\, [x \in \mathbb{N} \leftrightarrow (x$ is a finite Von Neumann ordinal$)]$.

Axioms of Set Theory in $\in_{\Delta_0} \cup \{\text{Set}, V\}$

Basic construction principles:

- **Union and Pair:** $\forall X, Y, w\, [w \in X \cup Y \leftrightarrow (w \in X \vee w \in Y)], \ldots$
- **Separation:** $\forall P, x\, [(x \in V) \rightarrow (P \cap x) \in V]$.

Axioms of Set Theory in $\in_{\Delta_0} \cup \{\text{Set}, V\}$

Strong construction principles:

- **Comprehension (b):** For every $\in_{\Delta_0}$-formula $\psi(\vec{x}, \vec{Y})$

$$\forall \vec{Y}\, \exists Z\, \forall x\, [x \in Z \leftrightarrow (x \in V \wedge \exists x_0, \ldots, x_n (x = \langle x_0, \ldots, x_n \rangle \wedge \psi(x_0, \ldots, x_n, \vec{Y})))].$$

- **Replacement:**

$$\forall F, x\, [(F \text{ is a function } \wedge (x \in V) \wedge (x \subseteq \mathrm{dom}(F))) \rightarrow (F[x] \in V)].$$

- **Powerset:**

$$\forall x\, [(x \in V) \rightarrow [\forall z\, (z \in \mathcal{P}(X) \leftrightarrow z \subseteq x) \wedge \mathcal{P}(x) \in V]].$$

- **Choice:**

$$\forall F[\; F \text{ is a function} \wedge \forall x\, (x \in \mathrm{dom}(F) \rightarrow F(x) \neq \emptyset) \;\rightarrow\; \exists G\, (G \text{ is a function} \wedge \mathrm{dom}(G) = \mathrm{dom}(F) \wedge \forall x\, (x \in \mathrm{dom}(G) \rightarrow G(x) \in F(x))\;].$$

Section 9

Algebraic maximality for set theory

The H_κs

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i.e. if letting

- $\bigcup^0 X = X$,
- $\bigcup^{n+1} X = \bigcup(\bigcup^n X)$,
- $\mathrm{trcl}(X) = \bigcup_{n\in\mathbb{N}}(\bigcup^n X)$,

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Example

- $\{\mathbb{R}\}$ is not hereditarily finite;
- each $n \in \mathbb{N}$ is hereditarily finite (recall that $n = \{0, \ldots, n-1\}$ for all $n \in \mathbb{N}$);

The H_κs

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Remark

- $\{\mathbb{R}\}$ *is not hereditarily countable;*
- *Any subset of* $\mathbb{N}$ *is hereditarily countable;*
- $\mathbb{Q}$ *and* $\mathbb{Z}$ *as defined in any textbook are hereditarily countable;*
- $\mathbb{R}$ *and* $\mathcal{P}(\mathbb{N})$ *are subsets of* $H_{\aleph_1}$ *(but not elements!);*
- $\mathcal{P}(\mathbb{N})$ *is definable by the atomic* $\in_{\Delta_0}$*-formula* $(x \subseteq \mathbb{N})$ *in the structure* $\langle H_{\aleph_1}, \in_{\Delta_0}\rangle$*;*
- *similarly for* $\mathbb{R}$ *or for any Polish space.*

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Definition

Given a cardinal κ, a set X is *hereditarily of size at most κ* if $\mathtt{trcl}(X)$ has size at most κ;
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Remark

- $\mathcal{P}(\aleph_1)$ *is definable by the atomic* $\in_{\Delta_0}$*-formula* $(x \subseteq \aleph_1)$ *in parameter* $\aleph_1$ *(the first uncountable ordinal) in the structure* $\langle H_{\aleph_2}, \in_{\Delta_0}\rangle$,
- NS, *the non-stationary ideal on* $\aleph_1$, *is* Σ_1*-definable in parameter* $\aleph_1$ *in the same structure.*

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$H_{\aleph_1^+} = H_{\aleph_2}$ is the set of all sets which are hereditarily of size at most $\aleph_1$.

$$H_{\aleph_0} \subseteq H_{\aleph_1} \subseteq H_{\aleph_2} \subseteq \cdots \subseteq H_{\kappa^+} \subseteq \ldots$$

$$V = \bigcup \{H_{\kappa^+} : \kappa \text{ an infinite cardinal}\}$$

Existentially closed structures for set theory

Theorem (Levy)

Let κ be an infinite cardinal.
Then

$$\langle H_{\kappa^+}, \in_{\Delta_0}, A : A \subseteq \mathcal{P}(\kappa)\rangle \prec_1 \langle V, \in_{\Delta_0}, A : A \subseteq \mathcal{P}(\kappa)\rangle$$

Algebraic maximality for $\mathcal{P}(\mathbb{N})$

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Theorem (Shoenfield, 1961)

Let $V[G]$ be a forcing extension of V. Then

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Algebraic maximality for $\mathcal{P}(\mathbb{N})$

- UB^V denotes the family of universally Baire subsets of $\mathbb{R}$ existing in V.
- (modulo a Borel isomorphism) $\mathbb{R} \approx \mathcal{P}(\mathbb{N}) \approx 2^{\mathbb{N}}$ and UB is a family of subsets of $\mathcal{P}(\mathbb{N})$.
- Every univ. Baire set A of V can be naturally lifted to a univ. Baire set $A^{V[G]}$ of $V[G]$ for any forcing extension $V[G]$ of V.

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Theorem (Feng-Magidor-Woodin, 1992)

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Theorem (Woodin, 1985+Martin-Steel, 1989+ V.-Venturi, 2020)

Assume there is a proper class of Woodin's cardinals. Then the theory of

$$\langle H_{\aleph_1}, \in_{\Delta_0}, A : A \in \mathrm{UB}^V \rangle$$

*is the **model companion** of the theory of*

$$\langle V[G], \in_{\Delta_0}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$$

for any forcing extension $V[G]$ of V.

Algebraic maximality for $\mathcal{P}(\mathbb{N})$

Theory	**degree of algebraic closure**
MK	$\langle H_{\aleph_1}, \in_{\Delta_0}, A : A \in \mathrm{UB}^V \rangle$ is Σ_1-elementary in $\langle V[G], \in_{\Delta_0}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ for all generic extension $V[G]$ of V
MK+ large cardinal axioms	The theory of $\langle H_{\aleph_1}, \in_{\Delta_0}, A : A \in \mathrm{UB}^V \rangle$ is the **model companion** of the theory of $\langle V[G], \in_{\Delta_0}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ for all generic extension $V[G]$ of V

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MK+ large cardinal axioms	The theory of $\langle H_{\aleph_1}, \in_{\Delta_0}, A : A \in \mathrm{UB}^V \rangle$ is the **model companion** of the theory of $\langle V[G], \in_{\Delta_0}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ for all generic extension $V[G]$ of V

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part I

- NS $\subseteq \mathcal{P}(\aleph_1)$ is the ideal of non-stationary subsets of $\aleph_1$.

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Definition

Strong Bounded Martin's maximum BMM^{++} holds if whenever B is an SSP cba and $V[G]$ is a forcing extension of V by B

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS} \rangle \prec_1 \langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]} \rangle.$$

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Theorem (Bagaria, Woodin)

MM^{++} *implies* BMM^{++}.

Applications of BMM^{++}

Assume BMM^{++}. Then:

- $2^{\aleph_0} = \aleph_2 = \aleph_1^{+}$.
  Todorčević, Mathematical Research Letters, 9(2), 2006.
- Whitehead's conjecture on free groups is false,
  (i.e. there are uncountable Whitehead groups which are not free).
  Shelah, Israel Journal of Mathematics, 18(3), 1974.
- THIS IS NOT KNOWN TO FOLLOW FROM BMM^{++}:
  There are five uncountable linear orders such that any uncountable linear order contains an isomorphic copy of one of them.
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Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *II*

- UB^V denotes the family of universally Baire subsets of $\mathbb{R}$ existing in V.
- $\mathsf{NS} \subseteq \mathcal{P}(\aleph_1)$ is the ideal of non-stationary subsets of $\aleph_1$.

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Definition (Woodin-Schindler?)

UB-BMM^{++} holds if whenever B is an SSP cba and $V[G]$ is a forcing extension of V by B

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Theorem (Woodin)

MM^{++} *implies* UB-BMM^{++}*.*

$(*)_{\mathrm{UB}}$ is a natural strengthening of Woodin's axiom $(*)$.

Theorem (Asperó-Schindler)

Assume there is a proper class of Woodin cardinals. Then $(*)_{\mathrm{UB}}$ *if and only if* UB-BMM^{++}*.*

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is Σ_1*-elementary in*

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Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

Recall that ψ is a Π_2-sentence if it is of the form $\forall\vec{x}\,\exists\vec{y}\phi(\vec{x},\vec{y})$ with $\phi(\vec{x},\vec{y})$ quantifier free.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

Recall that ψ is a Π_2-sentence if it is of the form $\forall\vec{x}\,\exists\vec{y}\phi(\vec{x},\vec{y})$ with $\phi(\vec{x},\vec{y})$ quantifier free.

In signature $\in_{\Delta_0}$ $\neg$CH can be formalized by the Π_2-sentence in parameter $\aleph_1$ (the first uncountable ordinal/cardinal):

$$\forall f\,[\underbrace{f \text{ is a function}}_{\Delta_0(f)} \wedge \underbrace{\mathrm{dom}(f)=\aleph_1}_{\Delta_0(f,\aleph_1)}) \rightarrow \exists r\,(\underbrace{r\subseteq\mathbb{N}}_{\Delta_0(r,\mathbb{N})} \wedge \underbrace{r\notin \mathrm{ran}(f)}_{\Delta_0(r,f)})]$$

Note that $\aleph_1 \in H_{\aleph_2}$.

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Theorem (Woodin)

Assume there are class many supercompacts, Sealing, and NS *is precipitous. TFAE:*

- $(*)_{\mathrm{UB}}$ *(or* $\mathrm{UB}\text{-}\mathrm{BMM}^{++}$*).*
- *For any* Π_2*-sentences* ψ *for* $\in_{\Delta_0} \cup \{\aleph_1, \mathrm{NS}\} \cup \left\{A : A \in \mathrm{UB}^V\right\}$

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \aleph_1, \mathrm{NS}, A : A \in \mathrm{UB}^V\rangle \models \psi$$

if and only if

ψ is true in $H_{\aleph_2}^{V[G]}$ for some forcing extension $V[G]$ of V.

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Sealing can be removed if one replaces UB^V with $\mathcal{P}(\mathbb{R})^{L(\mathrm{Ord}^{\mathbb{N}})}$ in the formulation of BMM^{*++} and in the relevant spots.

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

Theorem (V.)

Assume there are class many supercompacts, Sealing, and NS *is precipitous. TFAE:*

- $(*)_{\mathrm{UB}}$ *(or* UB-BMM^{++}*).*
- *The theory T of the structure*

$$\mathcal{M} = \langle H_{\aleph_2}, \in_{\Delta_0}, \aleph_1, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle$$

*is the **model companion** of the theory S of the structure*

$$\langle V, \in_{\Delta_0}, \aleph_1, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle.$$

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- *Letting $S_{\forall\vee\exists}$ be the boolean combination of existential sentences which are in S, and ψ be a Π_2-sentence,*
  $\mathcal{M}$ ***models*** ψ ***if and only*** $\psi + S_{\forall\vee\exists}$ ***is consistent****.*

Algebraic maximality for $\mathcal{P}(\aleph_1)$ part *III*

Theorem (V.)

Assume there are class many supercompacts, Sealing, and NS *is precipitous. TFAE:*

- $(*)_{\mathrm{UB}}$ *(or* UB-BMM^{++}*).*
- *For any* Π_2*-sentences* ψ

$$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle \models \psi$$

if and only if

ψ is true in $H_{\aleph_2}^{V[G]}$ for some forcing extension $V[G]$ of V.

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where S is the theory of the structure

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Algebraic maximality for $\mathcal{P}(\aleph_1)$

Theory	degree of algebraic closure
MK	$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle$ is a *substructure* of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ for all generic extension $V[G]$ of V by an SSP-forcing
MK+ forcing axioms	$\langle H_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}, A : A \in \mathrm{UB}^V \rangle$ is a Σ_1-*substructure* of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ for all generic extension $V[G]$ of V by an SSP-forcing
MK+ large cardinal axioms	for all generic extension $V[G]$ of V the theories of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ have the same **model companion** theory
MK+ large cardinals + forcing axioms	for all generic extension $V[G]$ of V the theories of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ have as **model companion** the theory of $\langle H^V_{\aleph_2}, \in_{\Delta_0}, \mathrm{NS}^V, A^V : A \in \mathrm{UB}^V \rangle$

Algebraic maximality for $\mathcal{P}(\aleph_1)$

Theory	degree of algebraic closure
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MK+ large cardinal axioms	for all generic extension $V[G]$ of V the theories of $\langle V[G], \in_{\Delta_0}, \mathrm{NS}^{V[G]}, A^{V[G]} : A \in \mathrm{UB}^V \rangle$ have the same **model companion** theory
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Section 10

Appendixes

Appendix 1: Universally Baire sets

Definition

Let (X, τ) be a locally compact Polish space. $A \subseteq X$ is *universally Baire* if for all continuous $f : Y \to X$ with (Y, σ) compact Hausdorff, $f^{-1}[A]$ has the Baire property in (Y, σ).

Universal Baireness describes the **absolutely** regular sets of reals:

Consider $2^{\mathbb{N}}$ as a closed subspace of $[0; 1]$. It is meager.

Now take a subset P of $2^{\mathbb{N}}$ which does not have the Baire property in $2^{\mathbb{N}}$.

Seen as a subset of $[0; 1]$, P is meager, hence it has the Baire property, but P is also the preimage under the inclusion map of $2^{\mathbb{N}}$ inside $[0; 1]$.

This map is continuous, and the preimage of P does not have the Baire property in $2^{\mathbb{N}}$.

Hence $P \subseteq [0; 1]$ is not universally Baire, even if it has the Baire property.

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Let (X, τ) be a locally compact Polish space. $A \subseteq X$ is *universally Baire* if for all continuous $f : Y \to X$ with (Y, σ) compact Hausdorff, $f^{-1}[A]$ has the Baire property in (Y, σ).

Universal Baireness describes the **absolutely** regular sets of reals:

Consider $2^{\mathbb{N}}$ as a closed subspace of $[0; 1]$. It is meager.
Now take a subset P of $2^{\mathbb{N}}$ which does not have the Baire property in $2^{\mathbb{N}}$.

Seen as a subset of $[0; 1]$, P is meager, hence it has the Baire property, but P is also the preimage under the inclusion map of $2^{\mathbb{N}}$ inside $[0; 1]$.

This map is continuous, and the preimage of P does not have the Baire property in $2^{\mathbb{N}}$.

Hence $P \subseteq [0; 1]$ is not universally Baire, even if it has the Baire property.

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Appendix 2: Stationary sets and the non-stationary ideal

Definition

- C is a club subset of $\aleph_1$ if $\sup(C) = \aleph_1$ and for all $\beta \notin C$ there is $\alpha < \beta$ such that $[\alpha, \beta] \cap C$ is empty.
- $S \subseteq \aleph_1$ is stationary if for all C club subset of $\aleph_1$ $S \cap C$ is non-empty.
- NS $\subseteq \mathcal{P}(\aleph_1)$ is the ideal of non-stationary subsets of $\aleph_1$ (i.e. subsets disjoint from some club).
- NS is saturated if the boolean algebra $\mathcal{P}(\aleph_1)/_{\mathrm{NS}}$ has only partitions of size at most $\aleph_1$.

Theorem

- *Assume* NS *is saturated. Then it is precipitous.*
- *Assume* MM. *Then* NS *is saturated.*
- NS *is precipitous is consistent with* CH.

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Appendix 3: Sealing

Definition (Woodin)

Given $(\mathcal{D}, W, \in_{\Delta_0})$ transitive model of MK, let N^W be the set $\mathcal{P}(H_{\aleph_1})^{L(\mathrm{UB})^W}$, where $L(\mathrm{UB})^W$ is the smallest transitive model of ZF containing UB^W.

Sealing holds in a model $(\mathcal{C}, V, \in_{\Delta_0})$ of MK if whenever $V[G]$ is a forcing extension of V and $V[H]$ a forcing extension of $V[G]$ we have that

$$(N^{V[G]}, H_{\aleph_1}^{V[G]}, \in_{\Delta_0}) \prec (N^{V[H]}, H_{\aleph_1}^{V[H]}, \in_{\Delta_0}).$$

Theorem (Woodin)

Assume V models κ is supercompact and there are class many Woodin cardinals. Let $V[H]$ be a generic extension of V where 2^κ is countable. Then sealing holds in $V[H]$.

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Appendix 4: Some references

- J. Bagaria, *Natural axioms on set theory and the continuum problem*, CRM Preprint, 591, 2004.
- P. Koellner, *On the question of absolute undecidability*, in *Kurt Gödel: essays for his centennial*, Lect. Notes Log. 33, 2010.
- G. Venturi and M. Viale, *What model companionship can say about the Continuum problem*, The Review of Symbolic Logic, pp. 1–40, eprint: 2204.13756.
- M. Viale *STRONG FORCING AXIOMS AND THE CONTINUUM PROBLEM [after Asperó's and Schindler's proof that MM++ implies Woodin's Axiom* $(*)$*]*, Séminaire BOURBAKI Avril 2023, 75e année, 2022–2023, no 1207, p. 1 à 34
- W. H. Woodin, *The Continuum hypothesis Part I*, Notices of AMS, 48(6), 2001.
- W. H. Woodin, *The Continuum hypothesis Part II*, Notices of AMS, 48(7), 2001.