Inner models from extended logics

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Gdańsk, December 2023

European Research Council
Established by the European Commission

Gödel

$$
\begin{aligned}
L_0 &= \emptyset \\
L_{\alpha+1} &= \mathrm{Def}(L_\alpha) \\
L_\nu &= \bigcup_{\alpha<\nu} L_\alpha \text{ for limit } \nu \\
L &= \bigcup_\alpha L_\alpha
\end{aligned}
$$

Jensen

$$
\begin{aligned}
J_0 &= \emptyset \\
J_{\alpha+\omega} &= \text{rud}(J_\alpha \cup \{J_\alpha\}) \\
J_{\omega\nu} &= \bigcup_{\alpha<\nu} J_{\omega\alpha} \text{ for limit } \nu \\
L &= \bigcup_\alpha J_{\omega\alpha}
\end{aligned}
$$

Suppose $\mathcal{L}^*$ is a logic

$$
\begin{aligned}
L'_0 &= \emptyset \\
L'_{\alpha+1} &= \mathrm{Def}_{\mathcal{L}^*}(L'_\alpha) \\
L'_\nu &= \bigcup_{\alpha<\nu} L'_\alpha \text{ for limit } \nu \\
C(\mathcal{L}^*) &= \bigcup_\alpha L'_\alpha
\end{aligned}
$$

A typical set in $L'_{\alpha+1}$ has the form

$$X = \{a \in L'_\alpha : (L'_\alpha, \in) \models \varphi(a, \vec{b})\},$$

where $\varphi(x, \vec{y}) \in \mathcal{L}^*$ and $\vec{b} \in L'_\alpha$.

Theorem

For any $\mathcal{L}^*$ the class $C(\mathcal{L}^*)$ is a transitive model of ZF containing all the ordinals.

Proof.

As in the usual proof of ZF in L. Let us prove the Comprehension Schema as an example. Suppose $A, \vec{b}$ are in $C(\mathcal{L}^*)$, $\varphi(x, \vec{y})$ is a first order formula of set theory and

$$X = \{a \in A : C(\mathcal{L}^*) \models \varphi(a, \vec{b})\}.$$

Let α be an ordinal such that $A \in L'_\alpha$ and $\varphi(x, y)$ is absolute for $L'_\alpha, C(\mathcal{L}^*)$. Now

$$X = \{a \in L'_\alpha : L'_\alpha \models a \in A \wedge \varphi(a, \vec{b})\}.$$

Hence $X \in C(\mathcal{L}^*)$. □

Definition

A logic $\mathcal{L}^*$ is *adequate to truth in itself* if for all finite vocabularies K there is function $\varphi \mapsto \ulcorner\varphi\urcorner$ from all formulas $\varphi(x_1, \ldots, x_n) \in \mathcal{L}^*$ in the vocabulary K into ω, and a formula $\mathrm{Sat}_{\mathcal{L}^*}(x, y, z)$ in $\mathcal{L}^*$ such that:

1. The function $\varphi \mapsto \ulcorner\varphi\urcorner$ is one to one and has a recursive range.
2. For all admissible sets M, formulas φ of $\mathcal{L}^*$ in the vocabulary K, structures $\mathcal{N} \in M$ in the vocabulary K, and $a_1, \ldots, a_n \in N$ the following conditions are equivalent:
   2.1 $M \models \mathrm{Sat}_{\mathcal{L}^*}(\mathcal{N}, \ulcorner\varphi\urcorner, \langle a_1, \ldots, a_n\rangle)$
   2.2 $\mathcal{N} \models \varphi(a_1, \ldots, a_n)$.

   We may admit ordinal parameters in this definition.

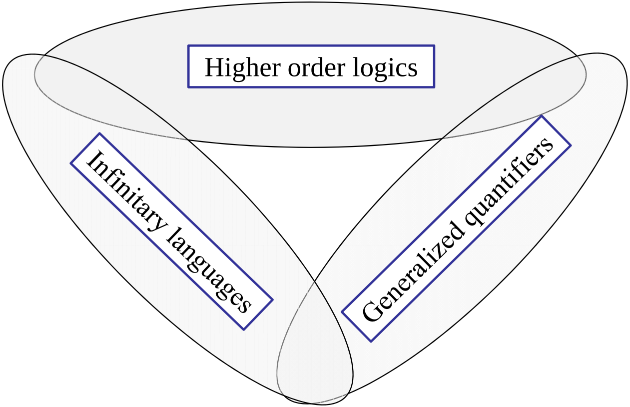
Lemma

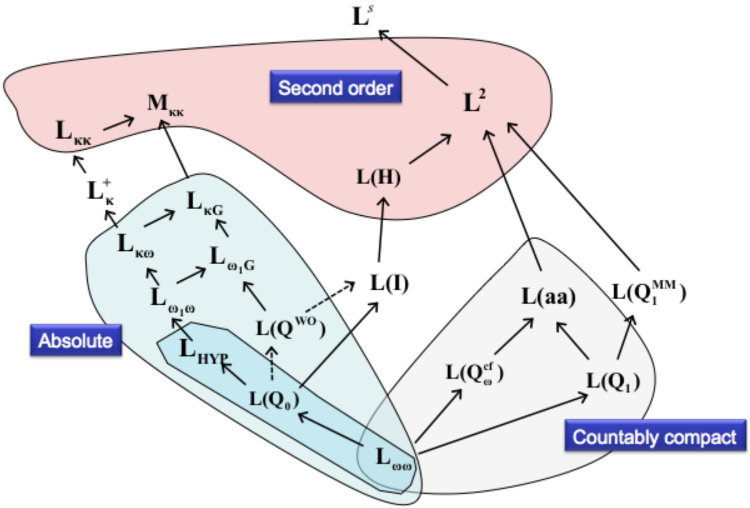
If $\mathcal{L}^*$ is adequate to truth in itself, there are formulas $\Phi_{\mathcal{L}^*}(x)$ and $\Psi_{\mathcal{L}^*}(x, y)$ of $\mathcal{L}^*$ in the vocabulary $\{\in\}$ such that if M is an admissible set and $\alpha = M \cap \mathrm{On}$, then:

1. $\{a \in M : (M, \in) \models \Phi_{\mathcal{L}^*}(a)\} = L'_\alpha \cap M$.
2. $\{(a, b) \in M \times M : (M, \in) \models \Psi_{\mathcal{L}^*}(a, b)\}$ *is a well-order* $<'_\alpha$ *the field of which is* $L'_\alpha \cap M$.

Some history

- Chang in Mostowski's seminar in Warsaw 1967: $\mathcal{L}^* = L_{\kappa\kappa}$.
- Chang, PSPM 1971: $\mathcal{L}^* = L_{\kappa\kappa}$.
- Myhill-Scott, PSPM 1971: $\mathcal{L}^* = L^2$.
- Gloede, “Higher Set Theory" 1977: $\mathcal{L}^* = L_{\kappa\lambda}$
- Kennedy-Magidor-V, JML 2021: $\mathcal{L}^* = L(Q)$
- Welch, JSL 2022: $\mathcal{L}^* = L(I)$
- Friedman-Gitman-Müller, APAL 2023.
- Ur Ya'ar, APAL 2024: $\mathcal{L}^* = L(Q^1, \ldots, Q^n)$
- SQuaRE group in the AIM 2021-2024.


Higher order logics
Infinitary languages
Generalized quantifiers


L^s
Second order
$M_{\kappa\kappa}$
L^2
$L_{\kappa\kappa}$
L(H)
L^+_κ
$L_{\kappa G}$
$L_{\kappa\omega}$
$L_{\omega_1 G}$
L(I)
L(aa)
$L(Q_1^{MM})$
$L_{\omega_1\omega}$
$L(Q^{WO})$
Absolute
L_{HYP}
$L(Q^{cf}_\omega)$
$L(Q_1)$
$L(Q_0)$
Countably compact
$L_{\omega\omega}$

- $C(\mathcal{L}_{\omega\omega}) = L$
- $C(\mathcal{L}_{\omega_1\omega}) = L(\mathbb{R})$
- $C(\mathcal{L}_{\omega_1\omega_1}) =$ Chang model
- $C(\mathcal{L}^2) = \mathrm{HOD}$

Possible attributes of inner models

- Forcing absolute.
- Support large cardinals.
- Satisfy Axiom of Choice.
- Arise “naturally".
- Decide questions such as CH.

- L: Forcing-absolute but no large cardinals (above WC)
- HOD: Has large cardinals but forcing-fragile
- $L(\mathbb{R})$: Forcing-absolute, has large cardinals, but no AC
- Extender models: Tailor made to support given large cardinals

Theorem (Essentially Gloede 1978)

Suppose $\mathcal{L}^*$ (and its syntax) are ZFC-absolute with parameters from L. Then $C(\mathcal{L}^*) = L$.

Corollary

$C(\mathcal{L}(Q_\alpha)) = L$ for all α.

Definition

Magidor-Malitz quantifier of dimension n:

$$\mathcal{M} \models Q_\alpha^{\mathrm{MM},n} x_1, \ldots, x_n \varphi(x_1, \ldots, x_n) \iff$$

$$\exists X \subseteq M (|X| \geq \aleph_\alpha \wedge \forall a_1, \ldots, a_n \in X : \mathcal{M} \models \varphi(a_1, \ldots, a_n)).$$

Can express Souslinity of a tree.

Consistently, $C(Q_1^{\mathrm{MM},2}) \neq L$, but:

Theorem

If $0^\sharp$ exists, then $C(Q_\alpha^{MM,<\omega}) = L$.

Lemma

Suppose $0^\sharp$ exists and $A \in L$, $A \subseteq [\alpha]^2$. If there is (in V) an uncountable B such that $[B]^2 \subseteq A$, then there is such a set B in L.

The inner model C^*.

Definition

The cofinality quantifier Q^{cf}_{ω} is defined as follows:

$$\mathcal{M} \models Q^{\mathrm{cf}}_{\omega} xy \varphi(x, y, \vec{a}) \iff \{(c, d) : \mathcal{M} \models \varphi(c, d, \vec{a})\}$$
is a linear order of cofinality ω.

- Axiomatizable
- Fully compact
- Downward Löwenheim-Skolem down to $\aleph_1$

Definition

$$C^* =_{def} C(Q^{\mathrm{cf}}_\omega)$$

Note:

$$\{\alpha < \beta : \mathrm{cf}^V(\alpha) = \omega\} \in C^*$$

Theorem

If $0^\sharp$ exists, then $0^\sharp \in C^*$.

Proof.

Let

$$X = \{\xi < \aleph_\omega : \xi \text{ is a regular cardinal in } L \text{ and } \mathrm{cf}(\xi) > \omega\}$$

Now $X \in C^*$ and

$$0^\sharp = \{\ulcorner \varphi(x_1, ..., x_n) \urcorner : L_{\aleph_\omega} \models \varphi(\gamma_1, ..., \gamma_n) \text{ for some } \gamma_1 < ... < \gamma_n \text{ in } X\}.$$

□

Welch JSL 2022 proves the stronger result $0^k \in C^*$, where 0^k is a sharp for a proper class of measurable cardinals.

- More generally, $x^\sharp \in C^*(x)$ for any $x \in C^*$ such that $x^\sharp$ exists.
- Hence $C^* \neq L(x)$ whenever x is a set of ordinals such that $x^\sharp$ exists in V.

Theorem

The Dodd-Jensen Core model is contained in C^*.

Theorem

Suppose L^μ exists. Then C^* contains some L^ν.

Theorem

If there is a measurable cardinal κ, then $V \neq C^*$.

Proof.

Suppose $V = C^*$ but κ is a measurable cardinal. Let $i : V \to M$ with critical point κ and $M^\kappa \subseteq M$. Now $(C^*)^M = (C^*)^V = V$, whence $M = V$. This contradicts Kunen's result that there cannot be a non-trivial $i : V \to V$. □

Theorem

If E is an infinite set of measurable cardinals (in V), then $E \notin C^*$. Moreover, then $C^* \neq \mathrm{HOD}$.

Proof.

As Kunen's result that if there are uncountably many measurable cardinals, then AC is false in the Chang model. □

Stationary Tower Forcing

Suppose λ is Woodin.

- There is a forcing $\mathbb{Q}$ such that in $V[G]$ there is $j : V \to M$ with $V[G] \models M^\omega \subseteq M$ and $j(\omega_1) = \lambda$.
- For all regular $\omega_1 < \kappa < \lambda$ there is a cofinality ω preserving forcing $\mathbb{P}$ such that in $V[G]$ there is $j : V \to M$ with $V[G] \models M^\omega \subseteq M$ and $j(\kappa) = \lambda$.

Theorem

If there is a Woodin cardinal, then ω_1 is (strongly) Mahlo in C^*.

Proof.

To prove that ω_1 is strongly inaccessible in C^* suppose $\alpha < \aleph_1$ and $f : \omega_1 \to (2^\alpha)^{C^*}$ is one-one. Let $\mathbb{Q}$, G and $j : V \to M$ with $M^\omega \subset M$ and $j(\omega_1) = \lambda$ (= Woodin) be as above. Thus $j(f) : \lambda \to ((2^\alpha)^{C^*})^M$. Let $a = j(f)(\omega_1^V)$. If $a \in V$, then $j(a) = a$, whence $a = f(\delta)$ for some $\delta < \omega_1$. Then $a = j(a) = j(f)(j(\delta)) = j(f)(\delta)$ contradicting the fact that $a = j(f)(\omega_1)$. Hence $a \notin V$.
Now, $(C^*)^M = C^*_{<\lambda} \subseteq V$. Hence $a \in C^*_{<\lambda} \subseteq V$, a contradiction.

□

Theorem

Suppose there is a Woodin cardinal λ. Then every regular cardinal κ such that $\omega_1 < \kappa < \lambda$ is weakly compact in C^*.

Proof.

Suppose λ is a Woodin cardinal, $\kappa > \omega_1$ is regular and $< \lambda$. To prove that κ is strongly inaccessible in C^* we can use the "second" stationary tower forcing $\mathbb{P}$ above. With this forcing, cofinality ω is not changed, whence $(C^*)^M = C^*$. □

Theorem

If there is a proper class of Woodin cardinals, then the regular cardinals $\geq \aleph_2$ are indiscernible in C^*.

Proof.

We use the “second" stationary tower forcing $\mathbb{P}$ to show first that the Woodin cardinals are indiscernible, and after that the regular cardinals $\geq \aleph_2$ are indiscernible. Remember that here $\mathbb{P}$ and j preserve C^*. □

Theorem

If $V = L^{\mu}$, then C^* is the inner model $M_{\omega^2}[E]$, where $E = \{\kappa_{\omega \cdot n} : n < \omega\}$.

Theorem

Suppose there is a proper class of Woodin cardinals. Suppose $\mathcal{P}$ is a forcing notion and $G \subseteq \mathcal{P}$ is generic. Then

$$Th((C^*)^V) = Th((C^*)^{V[G]}).$$

Moreover, the theory $Th(C^*)$ is independent of the cofinality used, and forcing does not change the reals of these models.

Proof.

Let H_1 be generic for $\mathbb{Q}$. Now

$$j_1 : (C^*)^V \to (C^*)^{M_1} = (C^*)^{V[H_1]} = (C^*_{<\lambda})^V.$$

Let H_2 be generic for $\mathbb{Q}$ over $V[G]$. Then

$$j_2 : (C^*)^{V[G]} \to (C^*)^{M_2} = (C^*)^{V[H_2]} = (C^*_{<\lambda})^{V[G]} = (C^*_{<\lambda})^V.$$

□

Theorem

$|\mathcal{P}(\omega) \cap C^*| \leq \aleph_2$.

Theorem

If there are three Woodin cardinals and a measurable cardinal above them, then there is a cone of reals x such that $C^*(x)$ satisfies the Continuum Hypothesis.

If two reals x and y are Turing-equivalent, then $C^*(x) = C^*(y)$. Hence the set

$$\{y \subseteq \omega : C^*(y) \models CH\} \tag{1}$$

is closed under Turing-equivalence. Need to show that

(I) The set (1) is projective.

(II) For every real x there is a real y such that $x \leq_T y$ and y is in the set (1).

Lemma

Suppose there is a Woodin cardinal and a measurable cardinal above it. The following conditions are equivalent:

(i) $C^*(y) \models CH$.

(ii) *There is a countable iterable structure M with a Woodin cardinal such that $y \in M$, $M \models \exists\alpha(\text{“}L'_\alpha(y) \models CH\text{”})$ and for all countable iterable structures N with a Woodin cardinal such that $y \in N$: $\mathcal{P}(\omega)^{(C^*)^N} \subseteq \mathcal{P}(\omega)^{(C^*)^M}$.*

Consistency results about C^*

Suppose $V = L$. Let us add a Cohen real r. We can code this real with a modified Namba forcing so that in the end for all $n < \omega$: $\mathrm{cf}^V(\aleph^L_{n+2}) = \omega \iff n \in r$.

Theorem

Suppose $V = L$ and κ is a cardinal of cofinality $> \omega$. There is a forcing notion $\mathbb{P}$ which forces $C^* \models 2^\omega = \kappa$ and preserves cardinals between L and C^*.

Theorem

It is consistent, relative to the consistency of an inaccessible cardinal, that $V = C^*$ and $2^{\aleph_0} = \aleph_2$.

The inner model $C(aa)$.

Definition

$\mathcal{M} \models \mathtt{aa}\, s \varphi(s) \iff \{A \in [M]^{\leq\omega} : (\mathcal{M}, A) \models \varphi(s)\}$ contains a club of countable subsets of M. (i.e. almost all countable subsets A of M satisfy $\varphi(A)$.) We denote $\neg\mathtt{aa}\, s \neg\varphi$ by $\mathtt{stat}\, s \varphi$.

$$C(aa) =_{\text{def}} C(\mathcal{L}(\mathtt{aa}))$$

$$C^* \subseteq C(aa)$$

Suppose $\mathcal{L}^*$ is a logic the sentences of which are (coded by) natural numbers. We define the hierarchy (J'_α), $\alpha \in \mathrm{Lim}$ as follows:

$$\mathrm{Tr} = \{(\alpha, \varphi(\bar\alpha)) : (J'_\alpha, \in, \mathrm{Tr}\restriction\alpha) \models \varphi(\bar\alpha), \varphi(\bar x) \in \mathcal{L}^*, \bar\alpha \in J'_\alpha, \alpha \in \mathrm{Lim}\},$$

where

$$\mathrm{Tr}\restriction\alpha = \{(\beta, \psi(\bar\alpha)) \in \mathrm{Tr} : \beta \in \alpha \cap \mathrm{Lim}\},$$

and

$$\begin{aligned} J'_0 &= \emptyset \\ J'_{\alpha+\omega} &= \mathrm{rud}_{\mathrm{Tr}}(J'_\alpha \cup \{J'_\alpha\}) \\ J'_{\omega\nu} &= \textstyle\bigcup_{\alpha<\nu} J'_{\omega\alpha}, \text{ for } \nu \in \mathrm{Lim}\,. \end{aligned}$$

Definition

1. A first order structure $\mathcal{M}$ is *club-determined* if

$$\mathcal{M} \models \forall \vec{s} \forall \vec{x} [\mathtt{aa} \vec{t} \varphi(\vec{x}, \vec{s}, \vec{t}) \vee \mathtt{aa} \vec{t} \neg \varphi(\vec{x}, \vec{s}, \vec{t})],$$

   where $\varphi(\vec{x}, \vec{s}, \vec{t})$ is any formula in $\mathcal{L}(\mathtt{aa})$.

2. We say that the inner model $C(aa)$ is *club-determined* if every level L'_α is.

Theorem

If there are a proper class of Woodin cardinals or PFA holds, then $C(aa)$ is club-determined.

Proof.

Suppose L'_α is the least counter-example. W.l.o.g $\alpha < \omega_2^V$. Let δ be Woodin. The hierarchies

$$C(aa)^M, C(aa)^{V[G]}, C(aa_{<\delta})^V$$

are all the same and the (potential) failure of club-determinateness occurs in all at the same level. □

Some ingredients

Lemma

If δ is Woodin, $S \subseteq \delta$ is in M and M thinks that S is stationary, then $V[G]$ thinks that S is stationary.

Lemma

Suppose $C(aa)$ is club-determined, δ is Woodin, $\mathbb{P}$ is the countable stationary tower, $G \subseteq \mathbb{P}$ is generic and M is the associated generic ultrapower. Then $C(aa)^M = C(aa_{<\delta})^V$.

Theorem

Suppose there are a proper class of Woodin cardinals. Then the theory of $C(aa)$ is (set) forcing absolute.

Proof.

Suppose $\mathbb{P}$ is a forcing notion and δ is a Woodin cardinal $> |\mathbb{P}|$. Let $j : V \to M$ be the associated elementary embedding. Now

$$C(aa) \equiv (C(aa))^M = (C(aa_{<\delta}))^V.$$

On the other hand, let $H \subseteq \mathbb{P}$ be generic over V. Then δ is still Woodin, so we have the associated elementary embedding $j' : V[H] \to M'$. Again

$$(C(aa))^{V[H]} \equiv (C(aa))^{M'} = (C(aa_{<\delta}))^{V[H]}.$$

Finally, we may observe that $(C(aa_{<\delta}))^{V[H]} = (C(aa_{<\delta}))^V$. Hence

$$(C(aa))^{V[H]} \equiv (C(aa))^V.$$

Theorem

Suppose there are a proper class of Woodin cardinals or PFA holds. Then every regular $\kappa \geq \aleph_1$ is measurable in $C(aa)$.

Proof.

For α big enough for L'_α to contain all subsets of κ in $C(aa)$, consider the normal filter:

$$\mathcal{F} = \{X \subseteq \kappa : X \in L'_\alpha, L'_\alpha \models aas(\sup(s \cap \kappa) \in X)\}.$$

Suppose $X \subseteq \kappa$ is in $C(aa)$. Since L'_α is club determined,

$$L'_\alpha \models aas(\sup(s \cap \kappa) \in X) \text{ or}$$

$$L'_\alpha \models aas(\sup(s \cap \kappa) \notin X).$$

In the first case $X \in \mathcal{F}$. In the second case $\kappa \setminus X \in \mathcal{F}$. □

Theorem

Suppose there is a supercompact cardinal. Then every regular $\kappa \geq \aleph_1$ is measurable in $C(aa)$.

Theorem

If Club Determinacy holds, then $C(aa)$ satisfies CH.

The proof is based on the concept of an aa-premouse.

The inner model HOD_1.

Recall:

$$\mathrm{HOD} = C(L^2).$$

Let

$$\mathrm{HOD}_1 =_{\mathrm{df}} C(\Sigma^1_1).$$

Note:

- $\{\alpha < \beta : \mathrm{cf}^V(\alpha) = \omega\} \in \mathrm{HOD}_1$
- $\{(\alpha, \beta) \in \gamma^2 : |\alpha|^V \leq |\beta|^V\} \in \mathrm{HOD}_1$
- $\{\alpha < \beta : \alpha \text{ cardinal in } V\} \in \mathrm{HOD}_1$
- $\{(\alpha_0, \alpha_1) \in \beta^2 : |\alpha_0|^V \leq (2^{|\alpha_1|})^V\} \in \mathrm{HOD}_1$
- $\{\alpha < \beta : (2^{|\alpha|})^V = (|\alpha|^+)^V\} \in \mathrm{HOD}_1$

Lemma

1. $C^* \subseteq \mathrm{HOD}_1$.
2. $C(Q_1^{MM,<\omega}) \subseteq \mathrm{HOD}_1$
3. $C(I) \subseteq \mathrm{HOD}_1$.
4. *If* $0^\sharp$ *exists, then* $0^\sharp \in \mathrm{HOD}_1$

Theorem

It is consistent, relative to the consistency of infinitely many weakly compact cardinals that for some λ:

$$\{\kappa < \lambda : \kappa \text{ weakly compact (in } V)\} \notin \mathrm{HOD}_1,$$

and, moreover, $\mathrm{HOD}_1 = L \neq \mathrm{HOD}$.

Further work

- Further work has focused on closer investigation of the relationship between C^* and $C(aa)$, on inner models of $C(aa)$ with large cardinals, on GCH in these inner models, and on further extensions of $C(aa)$.
- Goldberg, Kennedy, Larson, Magidor, Rajala, Schindler, Steel, Väänänen, Welch, Wilson, Ya'ar.
- The reals of $C(aa)$ are in M_1 (Magidor-Schindler).

Thank you!