A NOTE ON NONREGULAR MATRICES AND IDEALS ASSOCIATED WITH THEM

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Abstract. We consider summability methods and ideals on $\mathbb{N}$ generated by some nonregular matrices.

1. Introduction

The paper consists of two parts. In the first part (Section 3) we consider summability methods for nonregular matrices (see Subsection 1.1 for an outline of the results). In the second part (Section 4) we examine ideals on $\mathbb{N}$ defined with the aid of nonregular matrices (see Subsection 1.2 for an outline of the results). Both parts are independent and can be read separately.

All the notions and notations used in the introduction are defined in Section 2.

1.1. **Matrix summability.** In [13, Theorem 3.4], Osikiewicz proved the formula to calculate the limit of certain kinds of A-summable sequences, namely, so called *spliced sequences*. In [2] the authors proved the following theorem which is, in a sense, a generalization of Osikiewicz Theorem [13, Theorem 3.4] to a larger class of sequences.

Theorem 1.1 (Bartoszewicz-Das-Głąb [2, Theorem 3])**.** *Let A be a nonnegative regular matrix. Let x be a bounded sequence such that $\delta^A(x,r)$ exists for every $r \in \mathbb{R}$, the sum $\sum_{r\in\mathbb{R}} r\delta^A(x,R)$ is well defined and x has only countably many cluster points. Then*

$$\lim{}^A x = \sum_{r\in\mathbb{R}} r\delta^A(x,r).$$

In the above mentioned papers the authors were interested in summability methods for regular matrices. In Section 3 we extend Theorem 1.1 to a class of nonregular matrices (see Theorem 3.2).

1.2. **Matrix ideals.** For every nonnegative regular matrix A we define the family $\mathcal{I}(A) = \{B \subseteq \mathbb{N} : d^A(B) = 0\}$, where d^A is A-density. It turns out that $\mathcal{I}(A)$ is an ideal of subsets of $\mathbb{N}$ and we call it *the matrix ideal*. It is known (see e.g. [9, Section 4]) that every Erdős-Ulam ideal on $\mathbb{N}$ is a matrix ideal (precisely it is a matrix ideal defined with the aid of Nörlund matrix). For instance, if A is the Cesàro matrix, $\mathcal{I}(A) = \mathcal{I}_d$ is the ideal of all sets of the asymptotic density zero.

In Section 4 we examine ideals on $\mathbb{N}$ which are generated by nonregular matrices.

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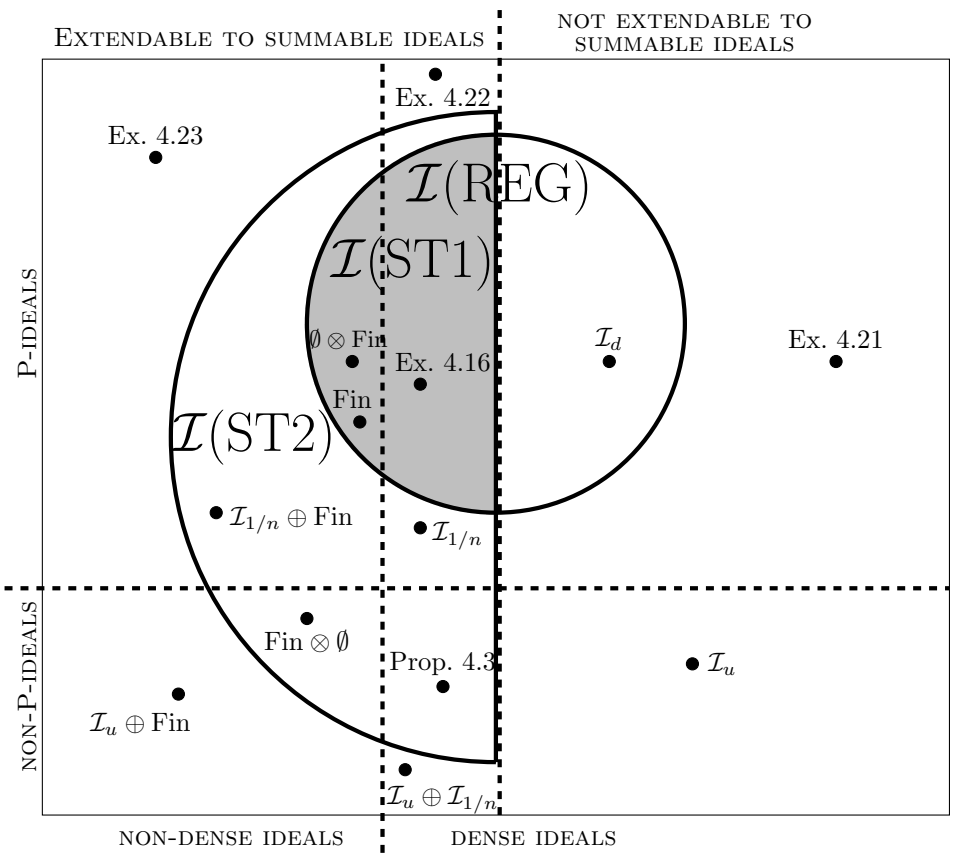
It is known that the matrix ideals generated by regular matrices are $F_{\sigma\delta}$ P-ideals (see e.g. [2, Proposition 13]). It turns out that the matrix ideals generated by nonregular matrices are also $F_{\sigma\delta}$ (Proposition 4.1(1)) and a large subclass of matrix ideals generated by nonregular matrices consists of P-ideals (Proposition 4.1(2)). However, there are examples of matrix ideals generated by nonregular matrices which are not P-ideals.

It is known (see [9, Proposition 4.11]) that no dense summable ideal is a matrix ideal generated by any regular matrix. We show (Proposition 4.6) that every summable ideal is a matrix ideal generated by some nonregular matrix.

We divide the family of nonregular matrices into two, quite natural, subclasses (Definition 2.3) and show some relations between ideals generated by matrices from these subclasses.

The following diagram summarizes all the relationships between considered subclasses of matrix ideals that hold in general.



Explanation of the above diagram:

- In the whole diagram (the largest rectangle) we have only $F_{\sigma\delta}$ ideals.
- In the circle we have all matrix ideals generated by regular matrices.
- In the large semicircle we have all matrix ideals generated by semiregular matrices of type 2.

- In the shadowed small semicircle (which is the intersection of the circle and the large semicircle – see Theorem 4.9) we have all matrix ideals generated by semiregular matrices of type 1.
- Matrix ideals generated by semiregular matrices (of either type) can be extended to summable ideals (see Theorem 4.7).
- Matrix ideals generated by regular matrices and matrices of type 1 are P-ideals (see Proposition 4.1(2)).
- There is no more inclusions between considered classes of ideals as pointed out by the appropriate examples.

2. Preliminaries

By $\mathbb{N}$ we mean the set of positive natural numbers.

By $A^{<\omega}$ we mean the set of all finite sequences of elements of A. If $s \in A^{<\omega}$ and $a \in A$, then $s\hat{\ }a$ denotes the concatenation of sequences s and (a).

We use the convention that $0 \cdot \infty = 0$.

The sum $\sum_{r\in\mathbb{R}} a_r$ is *well defined* if $\sum_{r\in\mathbb{R}} a_r^+ < \infty$ or $\sum_{r\in\mathbb{R}} a_r^- > -\infty$, where $a_r^+ = \max\{a_r, 0\}$ and $a_r^- = \min\{a_r, 0\}$ for every $r \in \mathbb{R}$.

2.1. Matrices.

Definition 2.1. We say that a matrix $A = (a_{i,k})$ is

(1) *nonnegative* if $a_{i,k} \geq 0$ for every $i, k \in \mathbb{N}$,
(2) *admissible* if $\lim_{i\to\infty} a_{i,k} = 0$ for every $k \in \mathbb{N}$.

Definition 2.2. We say that a matrix $A = (a_{i,k})$ is *regular* if

(1) A is admissible,
(2) $\sup_i \sum_{k\in\mathbb{N}} |a_{i,k}| < \infty$,
(3) $\lim_{i\to\infty} \sum_{k\in\mathbb{N}} a_{i,k} = 1$.

Below we introduce the notion of semiregular matrices which is crucial to the contents of the paper. It seems that this notion was not consider in the literature so far.

Definition 2.3. A matrix $A = (a_{i,k})$ is *semiregular* if

(1) A is admissible,
(2) $\lim_{i\to\infty} \sum_{k\in\mathbb{N}} a_{i,k} = \infty$.

A semiregular matrix $A = (a_{i,k})$ is of

- *type 1* if $\sum_{k\in\mathbb{N}} |a_{i,k}| < \infty$ for all but finitely many i;
- *type 2* if $\sum_{k\in\mathbb{N}} |a_{i,k}| = \infty$ for infinitely many i.

2.2. Matrix summability.

Below we recall the notion of matrix summability (for more on the subject we refer the readers to the classical monograph [5] or the recent book [3]).

Notation. For an infinite matrix of reals $A = (a_{i,k})_{i,k\in\mathbb{N}}$, a sequence of reals $x = (x_k)$ and $i \in \mathbb{N}$ we write

$$A_i(x) = \sum_{k=1}^{\infty} a_{i,k} x_k.$$

Definition 2.4. Let $A = (a_{i,k})_{i,k\in\mathbb{N}}$ be an infinite matrix of reals. We say that a sequence $x = (x_k)$ is *A-summable* if

(1) the series $A_i(x)$ is convergent for all but finitely many $i \in \mathbb{N}$,
(2) the sequence $(A_i(x))$ is convergent.

The real $\lim_{i\to\infty} A_i(x)$ is called the *A-limit of the sequence* x and is denoted by $\lim^A x$.

Remark. The well-known Silverman-Toeplitz theorem (see [14] and [15]) says that a matrix A is regular if and only if $\lim_{i\to\infty} A_i(x) = \lim_{i\to\infty} x$ for every ordinary convergent sequence x.

Example 2.5. For the identity matrix $I = (a_{i,k})$ where $a_{i,i} = 1$ and $a_{i,k} = 0$ for $i \neq k$, the I-limit is equal to the ordinary limit (i.e. $\lim^I = \lim$).

Example 2.6. For the Cesàro matrix $C = (a_{i,k})$ where $a_{i,k} = 1/i$ for $k \leq i$ and $a_{i,k} = 0$ for $k > i$, the C-summability is called the *Cesáro summability*, and $\lim^C x = \lim \frac{x_1+\cdots+x_n}{n}$.

2.3. **Matrix densities.**

Definition 2.7 (Freedman-Sember [10], see also Drewnowski-Paúl [6]; for regular matrices see Buck [4])**.** Let A be a nonnegative matrix. For a set $B \subseteq \mathbb{N}$ we define *upper A-density of B* by

$$\overline{d^A}(B) = \limsup_{i\to\infty} d_i^A(B)$$

and *lower A-density of B* by

$$\underline{d^A}(B) = \liminf_{i\to\infty} d_i^A(B),$$

where

$$d_i^A(B) = \sum_{k\in B} a_{i,k}.$$

Moreover we define *A-density of B* by

$$d^A(B) = \lim_{i\to\infty} d_i^A(B)$$

provided that the considered limit exists. Note that $d_i^A(B) = A_i(\mathbf{1}_B)$ and $d^A(B) = \lim^A \mathbf{1}_B$, where $\mathbf{1}_B$ is the characteristic function of the set B.

Definition 2.8. For a sequence $x = (x_n)$, a real $r \in \mathbb{R}$ and a matrix A we define *upper and lower A-densities of x at r* by $\overline{\delta^A}(x,r) = \lim_{\varepsilon\to 0^+} \overline{d^A}(\{n : |x_n - r| \leq \varepsilon\})$ and $\underline{\delta^A}(x,r) = \lim_{\varepsilon\to 0^+} \underline{d^A}(\{n : |x_n - r| \leq \varepsilon\})$. If $\overline{\delta^A}(x,r) = \underline{\delta^A}(x,r)$, we write $\delta^A(x,r) = \overline{\delta^A}(x,r)$.

2.4. **Ideals on $\mathbb{N}$.**

Definition 2.9. A family $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is called an *ideal on* $\mathbb{N}$ if

(1) $\mathbb{N} \notin \mathcal{I}$,
(2) $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$,
(3) $A \subseteq B \wedge B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$,
(4) $\mathcal{I}$ contains all finite subsets of $\mathbb{N}$.

Remark. We can talk about ideals on any other infinite countable set by identifying this set with $\mathbb{N}$ via a fixed bijection.

Definition 2.10. Ideals $\mathcal{I}$ and $\mathcal{J}$ are *isomorphic* (in short $\mathcal{I} \approx \mathcal{J}$) if there exists a bijection $\phi : \mathbb{N} \to \mathbb{N}$ such that $A \in \mathcal{I} \iff \phi[A] \in \mathcal{J}$ for every $A \subseteq \mathbb{N}$.

Definition 2.11. An ideal $\mathcal{I}$ is

(1) *dense* if for every infinite $A \subseteq \mathbb{N}$ there is an infinite $B \in \mathcal{I}$ such that $B \subseteq A$,
(2) a *P-ideal* if for every countable family $\mathcal{A} \subseteq \mathcal{I}$ there is $B \in \mathcal{I}$ such that $A \setminus B$ is finite for every $A \in \mathcal{A}$.

Definition 2.12. For ideals $\mathcal{I}, \mathcal{J}$ and $A \notin \mathcal{I}$ we define the following new ideals

(1) $\mathcal{I} \restriction A = \{B \cap A : B \in \mathcal{I}\}$,
(2) $\mathcal{I} \oplus \mathcal{J} = \{A \subseteq \mathbb{N} \times \{0,1\} : \{n : (n,0) \in A\} \in \mathcal{I} \wedge \{n : (n,1) \in A\} \in \mathcal{J}\}\}$,
(3) $\mathcal{I} \oplus \mathcal{P}(\mathbb{N}) = \{A \subseteq \mathbb{N} \times \{0,1\} : \{n : (n,0) \in A\} \in \mathcal{I}\}$,
(4) $\mathcal{I} \otimes \mathcal{J} = \{A \subseteq \mathbb{N} \times \mathbb{N} : \{n : \{k : (n,k) \in A\} \notin \mathcal{J}\} \in \mathcal{I}\}$,
(5) $\mathcal{I} \otimes \emptyset = \{A \subseteq \mathbb{N} \times \mathbb{N} : \{n : \{k : (n,k) \in A\} \neq \emptyset\} \in \mathcal{I}\}$.

Remark. By identifying sets of natural numbers with their characteristic functions, we equip $\mathcal{P}(\mathbb{N})$ with the topology of the Cantor space $\{0,1\}^{\mathbb{N}}$ and therefore we can assign topological complexity to ideals. In particular, an ideal $\mathcal{I}$ is F_σ ($F_{\sigma\delta}$ resp.) if it is an F_σ ($F_{\sigma\delta}$ resp.) subset of the Cantor space.

Definition 2.13. For every $f : \mathbb{N} \to [0,\infty)$ such that $\sum_{n=1}^{\infty} f(n) = \infty$ we define a *summable ideal generated by a function* f by $\mathcal{I}_f = \{B \subseteq \mathbb{N} : \sum_{n \in B} f(n) < \infty\}$. In particular, if $f(n) = 1/n$ we obtain the ideal $\mathcal{I}_{1/n} = \{B \subseteq \mathbb{N} : \sum_{n \in B} \frac{1}{n} < \infty\}$. It is known that summable ideals are F_σ P-ideals (see e.g. [7, Example 1.2.3]).

Notation. The family of all

- P-ideals is denoted by $\mathcal{I}(\mathrm{P})$,
- dense ideals is denoted by $\mathcal{I}(\mathrm{DENSE})$,
- $F_{\sigma\delta}$ ideals is denoted by $\mathcal{I}(\mathrm{F}_{\sigma\delta})$,
- summable ideals is denoted by $\mathcal{I}(\mathrm{SUM})$,
- ideals that can be extended to a summable ideal is denoted by $\mathcal{I}(\mathrm{SUM\text{-}EXT})$.

2.5. **Matrix ideals.**

Proposition 2.14. *If A is a nonnegative, admissible matrix such that $\overline{d^A}(\mathbb{N}) \neq 0$, then the family*

$$\mathcal{I}(A) = \{B \subseteq \mathbb{N} : d^A(B) = 0\}$$

is an ideal on $\mathbb{N}$. We call it a matrix ideal generated by the matrix A.

Proof. (1) Since $\overline{d^A}(\mathbb{N}) \neq 0$, so $\mathbb{N} \notin \mathcal{I}(A)$.

(2) If $B, C \in \mathcal{I}(A)$, then $0 \leq \overline{d^A}(B \cup C) \leq \overline{d^A}(B) + \overline{d^A}(C) = 0$, hence $B \cup C \in \mathcal{I}(A)$.

(3) If $B \subseteq C$ and $C \in \mathcal{I}(A)$, then $0 \leq \overline{d^A}(B) \leq \overline{d^A}(C) = 0$, hence $B \in \mathcal{I}(A)$.

(4) Let B be a finite subset of $\mathbb{N}$. Let $\varepsilon > 0$. Since A is admissible, so there is N such that $a_{i,k} < \varepsilon/|B|$ for every $i > N$ and $k \leq \max(B)$. Then $0 \leq \overline{d^A}(B) = \limsup_{i\to\infty} d_i^A(B) = \limsup_{i\to\infty} \sum_{k \in B} a_{i,k} \leq |B| \cdot \frac{\varepsilon}{|B|} = \varepsilon$. Since ε was arbitrary, we obtain $d^A(B) = 0$, hence $B \in \mathcal{I}(A)$. □

Remark. In the rest of the paper, once we consider $\mathcal{I}(A)$ we tacitly assume that A is a nonnegative, admissible matrix such that $\overline{d^A}(\mathbb{N}) \neq 0$. Note that nonnegative regular and semiregular matrices satisfy all these requirements.

Notation. The family of all matrix ideals generated by *nonnegative*

- regular matrices is denoted by $\mathcal{I}(\mathrm{REG})$,
- semiregular matrices of type 1 is denoted by $\mathcal{I}(\mathrm{ST1})$,
- semiregular matrices of type 2 is denoted by $\mathcal{I}(\mathrm{ST2})$.

3. Osikiewicz-like theorem

Proposition 3.1. *Let A be a nonnegative semiregular matrix. Let x be a bounded sequence such that $\delta^A(x,r)$ exists for every $r \in \mathbb{R}$. Then there is $r \in \mathbb{R}$ such that $\delta^A(x,r) = \infty$.*

Proof. Since x is bounded, there is $M > 0$ such that $d^A(\{n \in \mathbb{N} : x_n \in [-M, M]\}) = \infty$. Denote the set $[-M, M]$ by $B_\emptyset$. Now for every established B_s, $s \in \{0,1\}^{<\omega}$, define the closed intervals

$$B_{s\hat{}0} = \left[\min B_s, \frac{\min B_s + \max B_s}{2}\right] \text{ and } B_{s\hat{}1} = \left[\frac{\min B_s + \max B_s}{2}, \max B_s\right].$$

It is easy to see that for every B_s such that $\overline{d^A}(\{n \in \mathbb{N} : x_n \in B_s\}) = \infty$ we either have $\overline{d^A}(\{n \in \mathbb{N} : x_n \in B_{s\hat{}0}\}) = \infty$ or $\overline{d^A}(\{n \in \mathbb{N} : x_n \in B_{s\hat{}1}\}) = \infty$. Since $d^A\{n \in \mathbb{N} : x_n \in B_\emptyset\} = \infty$, we can find a sequence $s \in \{0,1\}^\omega$ such that $\overline{d^A}(\{n \in \mathbb{N} : x_n \in B_{s\restriction n}\}) = \infty$ for every $n \in \mathbb{N}$. For such $s \in \{0,1\}^\omega$ we pick $r \in \mathbb{R}$ such that $\bigcap_{n\in\mathbb{N}} B_{s\restriction n} = \{r\}$.

Clearly, every neighborhood of r contains $B_{s\restriction n}$ for some $n \in \mathbb{N}$, so $\overline{\delta^A}(x,r) = \infty$. Since $\delta^A(x,t)$ exists for every $t \in \mathbb{R}$, $\delta^A(x,r) = \infty$. □

Remark. Notice that when $\limsup_{i\to\infty} \sum_{k\in\mathbb{N}} a_{i,k} = \infty$ and $\liminf_{i\to\infty} \sum_{k\in\mathbb{N}} a_{i,k} < \infty$ then the condition "$\delta^A(x,r)$ exists for every $r \in \mathbb{R}$" cannot hold for bounded sequences, because the above theorem shows that when $\overline{d^A}(\{n \in \mathbb{N} : x_n \in [-M, M]\}) = \infty$ for some $M > 0$ then there is some $r \in \mathbb{R}$ such that $\overline{\delta^A}(x,r) = \infty$, but for all $r \in \mathbb{R}$ we have $\underline{\delta^A}(x,r) \le \liminf_{i\to\infty} \sum_{k\in\mathbb{N}} a_{i,k} < \infty$.

Theorem 3.2. *Let A be a nonnegative semiregular matrix. Let x be a bounded sequence such that $\delta^A(x,r)$ exists for every $r \in \mathbb{R}$, the sum $\sum_{r\in\mathbb{R}} r\delta^A(x,r)$ is well defined and $\delta^A(x,0) < \infty$. Then*

$$\lim {}^A x = \sum_{r\in\mathbb{R}} r\delta^A(x,r) = \pm\infty.$$

Proof. By Proposition 3.1, there is some $R \in \mathbb{R} \setminus \{0\}$ such that $\delta^A(x,R) = \infty$. We may assume that $R > 0$. Now we know that $\sum_{r<0} r\delta^A(x,r) > -\infty$, thus $\delta^A(x,r) < \infty$ for all $r < 0$. Therefore, $\overline{d^A}(\{n \in \mathbb{N} : x_n \le 0\}) < \infty$. Otherwise, by Proposition 3.1 we would find $r \le 0$ such that $\delta^A(x,r) = \infty$, a contradiction with either $\sum_{r<0} r\delta^A(x,r) > -\infty$ or $\delta^A(x,0) < \infty$.

Now we know that $\sum_{r\in\mathbb{R}} r\delta^A(x,r) = \infty$. To finish the proof, we will show that $\lim^A x^+ = \infty$ and $\lim^A x^- > -\infty$. Let $M > 0$ be such that $|x_n| \le M$ for all $n \in \mathbb{N}$. It is easy to see that for every $\varepsilon > 0$ and almost all $n \in \mathbb{N}$ we have $\sum_{k\in\mathbb{N}} a_{n,k} x_k^- > -M(\overline{d^A}(\{n \in \mathbb{N} : x_n \le 0\}) + \varepsilon)$, thus $\lim^A x^- > -M \cdot \overline{d^A}(\{n \in \mathbb{N} : x_n \le 0\}) > -\infty$. It is also not difficult to see that $\lim^A x^+ > (r/2) \cdot \overline{d^A}(\{n \in \mathbb{N} : x_n > r/2\}) = \infty$. □

The following proposition shows that we cannot drop the assumption "$\delta^A(x,0) < \infty$" in Theorem 3.2 in general.

Proposition 3.3. *Let A be a nonnegative semiregular matrix. There is a bounded sequence x such that $\delta^A(x,r)$ exists for every $r \in \mathbb{R}$, the sum $\sum_{r\in\mathbb{R}} r\delta^A(x,R)$ is well defined, $\delta^A(x,0) = \infty$ and*

$$\lim {}^A x \neq \sum_{r\in\mathbb{R}} r\delta^A(x,r).$$

Proof. Since $\lim_{i\to\infty}\sum_{k\in\mathbb{N}} a_{i,k} = \infty$, so for each n there is i_n such that $\sum_{k\in\mathbb{N}} a_{i,k} > n^2$ for every $i \geq i_n$. Without any loss of generality we can assume that the sequence (i_n) is increasing. Now choose k_n such that $\sum_{k<k_n} a_{i,k} > n^2$ for each $i_n \leq i < i_{n+1}$. Again without any loss of generality we can assume that the sequence (k_n) is increasing.

Let us now define the bounded sequence x by $x_k = 1$ for $k < k_1$ and $x_k = \frac{1}{n}$ whenever $k_{n-1} \leq k < k_n$ and $n \geq 2$.

Since $A_i(x) = \sum_{k\in\mathbb{N}} a_{i,k}x_k \geq n$ for every $i_n \leq i < i_{n+1}$ then $\lim^A x = \lim_{i\to\infty} A_i(x) = \infty$.

On the other hand $\delta^A(x,0) = \infty$ and $\delta^A(x,r) = 0$ for every $r \in \mathbb{R}\setminus\{0\}$, so $\sum_{r\in\mathbb{R}} r\delta^A(x,r) = 0\cdot\infty = 0$. □

Remark. In the proof of Proposition 3.3 we have constructed a bounded sequence x such that $\lim^A x = \infty$. Naturally, for some matrices we can construct x with $\lim^A x$ being finite and nonzero (see Example 3.4), but in general it is not possible (see Example 3.5).

Example 3.4. Let $A = (a_{i,k})$ be given by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Let $x = (1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \dots)$. Then x is a bounded sequence such that $\delta^A(x,0) = \infty$ and $\delta^A(x,r) = 0$ for every $r \in \mathbb{R}\setminus\{0\}$ (hence the sum $\sum_{r\in\mathbb{R}} r\delta^A(x,R)$ is well defined). Moreover, $\lim^A x = 1 \neq 0 = \sum_{r\in\mathbb{R}} r\delta^A(x,R)$.

Example 3.5. Let $A = (a_{i,k})$ be given by

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1/2 & 1/2 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1/2 & 1/2 & 1/2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

Note that for any bounded sequence x, $A_{2i-1}(x) = 2A_{2i}(x)$ for every $i \in \mathbb{N}$ and so, either $\lim^A x = 0$ or $\lim^A x = \pm\infty$ if that limit exists.

4. Matrix ideals generated by nonregular matrices

4.1. **Basic properties of matrix ideals.** Freedman and Sember [10, Propositions 3.1 and 3.2] proved that $\mathcal{I}(A)$ is a P-ideal for every regular matrix A, whereas in [2, Proposition 13] the authors proved that $\mathcal{I}(A)$ is $F_{\sigma\delta}$ ideal in this case.

Below we show that the same holds for matrix ideals generated by semiregular matrices of type 1. In the case of matrices of type 2 we obtain $F_{\sigma\delta}$ ideals, but there are both P-ideals (see Propositions 4.4 and 4.5) and non-P-ideals (see Propositions 4.2 and 4.3).

Proposition 4.1.

(1) $\mathcal{I}(\mathrm{REG}) \cup \mathcal{I}(\mathrm{ST1}) \cup \mathcal{I}(\mathrm{ST2}) \subseteq \mathcal{I}(\mathrm{F}_{\sigma\delta})$.
(2) $\mathcal{I}(\mathrm{REG}) \cup \mathcal{I}(\mathrm{ST1}) \subseteq \mathcal{I}(\mathrm{P})$.

Proof. (1) Let A be a semiregular matrix of either type. It suffice to observe that

$$\mathcal{I}(A) = \bigcap_{k\in\mathbb{N}} \bigcup_{N\in\mathbb{N}} \bigcap_{n>N} G_{n,k},$$

where $G_{n,k} = \{C \subseteq \mathbb{N} : A_n(\mathbf{1}_C) < 1/k\}$ and notice that $G_{n,k}$ is a closed set.

(2) Let A be a semiregular matrix of type 1. Take $B_1, B_2, B_3, \ldots \in \mathcal{I}(A)$. Let n_1 be such that for every $n \geq n_1$ we have $d_n^A(B_1) < 1/10$ and let i_1 be such that for every $n \leq n_1$ we have $\sum_{j>i_1} a_{n,j} < 1/10$. For every $k \in \mathbb{N}$ let n_k be such that for every $n \geq n_k$ we have $d_n^A(\bigcup_{l\leq k} B_l) < 1/10^k$ and let i_k be such that for every $n \leq n_k$ we have $\sum_{j>i_k} a_{n,j} < 1/10^k$.

Define $B = \bigcup_{k\in\mathbb{N}} B_k \setminus \{1, \ldots, i_k\}$. Then $B \setminus B_k$ is finite for every $k \in \mathbb{N}$ and $B \in \mathcal{I}(A)$. □

Proposition 4.2. $\mathcal{I}(\mathrm{ST2}) \setminus (\mathcal{I}(\mathrm{DENSE}) \cup \mathcal{I}(\mathrm{P})) \neq \emptyset$.

Proof. Let $\mathcal{I} = \mathrm{Fin} \otimes \emptyset$. It is not difficult to show that $\mathcal{I}$ is neither dense nor P-ideal (see e.g. [7]). Let $\{P_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into infinitely many infinite sets. Let $A = (a_{i,k})$ be a matrix defined by $a_{i,k} = 1$ for $k \in P_i$ and $a_{i,k} = 0$ for $k \notin P_i$. Then it is not difficult to see that $\mathcal{I}(A) \approx \mathcal{I}$. □

Proposition 4.3. $(\mathcal{I}(\mathrm{ST2}) \cap \mathcal{I}(\mathrm{DENSE})) \setminus \mathcal{I}(\mathrm{P}) \neq \emptyset$.

Proof. Let $\{P_n : n \in \mathbb{N}\}$ be a partition of $\mathbb{N}$ into infinitely many infinite sets. Let $A = (a_{i,k})$ be a matrix defined by $a_{i,k} = 1/i$ for $k \in P_i$ and $a_{i,k} = 0$ for $k \notin P_i$. The A is a semiregular matrix of type 2. Suppose that $\mathcal{I}(A)$ is a P-ideal. Since $P_n \in \mathcal{I}(A)$ for every n then there is $P \in \mathcal{I}(A)$ such that $P_n \setminus P$ is finite for every n. Then $\overline{d^A}(P) = \lim_{i\to\infty} d_i^A(P) = \infty$, so $A \notin \mathcal{I}(A)$, a contradiction. Note that $\lim_{i,k\to\infty} a_{i,k} = 0$, so by [6, Proposition 7.2] $\mathcal{I}(A)$ is dense. □

Proposition 4.4. $\mathcal{I}(\mathrm{ST2}) \cap \mathcal{I}(\mathrm{P}) \cap \mathcal{I}(\mathrm{DENSE}) \neq \emptyset$.

Proof. In Proposition 4.6 we will show that every summable ideal is a matrix ideal generated by a matrix of type 2. Let $\mathcal{I} = \mathcal{I}_{1/n}$. Then $\mathcal{I}$ is a summable ideal and it is not difficult to show that $\mathcal{I}$ is a dense P-ideal (see e.g. [7]). □

Proposition 4.5. $\mathcal{I}(\mathrm{ST2}) \cap \mathcal{I}(\mathrm{P}) \setminus \mathcal{I}(\mathrm{DENSE}) \neq \emptyset$.

Proof. In Proposition 4.6 we will show that every summable ideal is a matrix ideal generated by a matrix of type 2. Let $\mathcal{I} = \mathcal{I}_{1/n} \oplus \mathrm{Fin}$. It is not difficult to show that $\mathcal{I}$ is a summable non-dense P-ideal. □

4.2. **Extendability to summable ideals.** In [9, Proposition 4.11] the authors proved that no dense ideal generated by a regular matrix is equal to a dense summable ideal. Below we show that every summable ideal is a matrix ideal generated by a semiregular matrix of type 2 (Proposition 4.6) and every matrix ideal generated by a semiregular matrix of either type can be extended to a summable ideal (Theorem 4.7). Moreover we show that if a matrix ideal generated by a regular matrix can be extended to a summable ideal, then it is a matrix ideal generated by a semiregular matrix of type 2 (Theorem 4.8(1)). Finally we show that a matrix ideal generated by a semiregular matrix of type 1 is always a matrix ideal generated by a semiregular matrix of type 2 (Theorem 4.8(2)).

Proposition 4.6.

(1) $\mathcal{I}(\mathrm{SUM}) \cap \mathcal{I}(\mathrm{DENSE}) \cap \mathcal{I}(\mathrm{REG}) = \emptyset$.
(2) $\mathcal{I}(\mathrm{SUM}) \subseteq \mathcal{I}(\mathrm{ST2})$.

Proof. (1) It is proved in [9, Proposition 4.11]. (2) Let $\mathcal{I} = \mathcal{I}_f$ be a summable ideal. Define a matrix $A = (a_{i,k})$ by $a_{i,k} = f(k)$ when $k \geq i$ and $a_{i,k} = 0$ otherwise. Then

$$B \in \mathcal{I}(A) \iff 0 = \lim_{i\to\infty} \sum_{k\in B} a_{i,k} = \lim_{i\to\infty} \sum_{k\in B\wedge k\geq i} f(k) = 0$$

$$\iff \sum_{k\in B} f(k) < \infty \iff B \in \mathcal{I}_f.$$

□

Theorem 4.7. $\mathcal{I}(\mathrm{ST1}) \cup \mathcal{I}(\mathrm{ST2}) \subseteq \mathcal{I}(\mathrm{SUM\text{-}EXT})$.

Proof. Let $A = (a_{i,k})$ be a semiregular matrix of either type. Since all non-dense ideals can be extended to a summable ideal, we may assume that $\mathcal{I}$ is dense. In [6] the authors proved that when $\mathcal{I}(A)$ is dense then $\lim_{i,k\to\infty} a_{i,k} = 0$. Therefore, we can assume that $a_{i,k} \leq 1$ for all $i, k \in \mathbb{N}$.

Since $\lim_{i\to\infty} \sum_{k\in\mathbb{N}} a_{i,k} = \infty$, we can find $i_1, k_1 \in \mathbb{N}$ such that $\sum_{k\in[1,k_1]} a_{i_1,k} \in [1, 2]$. Suppose we have defined i_n, k_n for some $n \in \mathbb{N}$. Since A is semiregular, $\lim_{i\to\infty} \sum_{k\in[1,k_n]} a_{i,k} = 0$, hence we can find such $i_{n+1}, k_{n+1} \in \mathbb{N}$ that $\sum_{k\in[k_n+1,k_{n+1}]} a_{i_{n+1},k} \in [n + 1, n + 2]$.

Define the function f as follows. When $i \in [k_{n-1} + 1, k_n]$ for some $n \in \mathbb{N}$ (taking $k_0 = 0$), take $f(i) = a_{i_n,i}/(n(\sum_{k\in[k_{n-1}+1,k_n]} a_{i_n,k}))$. Then $\sum_{i\in\mathbb{N}} f(i) = \sum_{n\in\mathbb{N}} \sum_{i\in[k_{n-1}+1,k_n]} f(i) = \sum_{n\in\mathbb{N}} 1/n = \infty$. To finish the proof, we will take $B \in \mathcal{I}(A)$ and show that $B \in \mathcal{I}_f$.

When $B \in \mathcal{I}(A)$, we get $\lim_{i\to\infty} \sum_{k\in B} a_{i,k} = 0$, thus $\sum_{i\in B\cap[k_{n-1}+1,k_n]} a_{i_n,k} < 1$ for almost all $n \in \mathbb{N}$. Since $\sum_{k\in[k_{n-1}+1,k_n]} a_{i_n,k} \in [n, n + 1]$, it follows that $\sum_{i\in B\cap[k_{n-1}+1,k_n]} f(i) < 1/n^2$. Thus, there is $N \in \mathbb{N}$ such that $\sum_{i\in B\cap(k_N,\infty)} f(i) < \sum_{n\geq N} 1/n^2 < \infty$, which means that $B \in \mathcal{I}_f$. □

Theorem 4.8.

(1) $\mathcal{I}(\mathrm{REG}) \cap \mathcal{I}(\mathrm{SUM\text{-}EXT}) \subseteq \mathcal{I}(\mathrm{ST2})$.
(2) $\mathcal{I}(\mathrm{ST1}) \subseteq \mathcal{I}(\mathrm{ST2})$.

Proof. Let $\mathcal{I} = \mathcal{I}(A)$ be a matrix ideal such that $\mathcal{I} \subseteq \mathcal{I}_f$ for some summable ideal $\mathcal{I}_f$ and $A = (a_{i,k})$ is either regular matrix or semiregular matrix of type 1 (for a matrix of type 1 use Theorem 4.7 to get an appropriate extension).

We define the matrix $B = (b_{i,k})$ by $b_{i,k} = a_{i,k}$ when $i > k$ and $b_{i,k} = a_{i,k} + f(k)$ otherwise. Since $\sum_{k\in\mathbb{N}} f(k) = \infty$, we get $\sum_{k\in\mathbb{N}} b_{i,k} = \infty$ for all $i \in \mathbb{N}$, hence B is semiregular of type 2.

Clearly, $\mathcal{I}(B) \subseteq \mathcal{I}(A)$ as $b_{i,k} \geq a_{i,k}$ for all $i, k \in \mathbb{N}$. On the other hand, it is easy to see that $\mathcal{I}(B) \supseteq \mathcal{I}(A) \cap \mathcal{I}_f$. Since $\mathcal{I}(A) \subseteq \mathcal{I}_f$, we get $\mathcal{I}(B) \supseteq \mathcal{I}(A)$, thus $\mathcal{I}(A) = \mathcal{I}(B)$. □

4.3. **Regular plus type 2 equals type 1.** Below we show that an ideal that is simultaneously a matrix ideal generated by a regular matrix and a matrix ideal generated by a semiregular matrix of type 2 is a matrix ideal generated by a semiregular matrix of type 1 and vice versa.

Theorem 4.9. $\mathcal{I}(\mathrm{REG}) \cap \mathcal{I}(\mathrm{ST2}) = \mathcal{I}(\mathrm{ST1})$.

Proof. ($\subseteq$) Let $\mathcal{I} = \mathcal{I}(A) = \mathcal{I}(B)$ for a regular matrix $A = (a_{i,k})$ and a semiregular matrix $B = (b_{i,k})$ of type 2. First, we find sequences (i_n) and (k_n) such that $\sum_{k \le k_n} b_{i_n,k} \ge k$. We can find such sequences, because $\lim_{i\to\infty} \sum_{k\in\mathbb{N}} b_{i,k} = \infty$. Then we construct the matrix $C = (c_{i,k})$ in the following way. Take $c_{i_n,k} = a_{i_n,k} + b_{i_n,k}$ for $k \le k_n$ and put $c_{i,k} = a_{i,k}$ otherwise. Since A is regular, $\sum_{k\in\mathbb{N}} c_{i,k} < \infty$ for all but finitely many $i \in \mathbb{N}$. On the other hand, $\lim_{n\to\infty} \sum_{k\in\mathbb{N}} c_{i_n,k} = \infty$. Therefore, C is semiregular of type 1.

Clearly, $\mathcal{I}(C) \subseteq \mathcal{I}(A)$ as $c_{i,k} \ge a_{i,k}$ for all $i, k \in \mathbb{N}$. On the other hand, it is easy to see that $\mathcal{I}(C) \supseteq \mathcal{I}(A) \cap \mathcal{I}(B)$. Since $\mathcal{I} = \mathcal{I}(A) = \mathcal{I}(B)$, we get $\mathcal{I}(C) \supseteq \mathcal{I}$, thus $\mathcal{I}(C) = \mathcal{I}$.

($\supseteq$) By Theorem 4.8(2) we have $\mathcal{I}(\mathrm{ST1}) \subseteq \mathcal{I}(\mathrm{ST2})$. Below we show that $\mathcal{I}(\mathrm{ST1}) \subseteq \mathcal{I}(\mathrm{REG})$.

Let A be a semiregular matrix of type 1. Take the matrix $Z = (z_{i,k})$ given by $z_{i,k} = \min\{a_{i,k}, 1\}$ for all $i, k \in \mathbb{N}$.

For every $i \in \mathbb{N}$ let $y_i \in \mathbb{N}$ be such that $\sum_{k>y_i} z_{i,k} < 1/i$. Now, for every $i \in \mathbb{N}$ we define the set $N_i = \{C \subseteq [1, y_i] : \sum_{k\in C} z_{i,k} \in [1, 2)\}$ and take $n_i = \sum_{j\le i} |N_j|$. Since $\sup_{k\in\mathbb{N}} z_{i,k} \le 1$ for all $i \in \mathbb{N}$ and $\lim_{i\to\infty} \sum_{k\in\mathbb{N}} z_{i,k} = \infty$, N_i is nonempty for almost all i. Enumerate the elements of every N_i in any order by $(C_{n_{i-1}+1}, \dots, C_{n_i})$ where $n_0 = 0$.

We construct the matrix $B = (b_{j,k})$ in the following way. When $j \in (n_i, n_{i+1}]$ we take $b_{j,k} = z_{i+1,k} / \sum_{k\in C_j} z_{i+1,k}$ when $k \in C_j$ and $b_{j,k} = 0$ otherwise. Such B is a regular matrix as $\sum_{k\in\mathbb{N}} b_{j,k} = 1$ for all $j \in \mathbb{N}$.

It is easy to see that $\mathcal{I}(A) \subseteq \mathcal{I}(B)$ since $\sum_{k\in D} b_{j,k} \le \sum_{k\in D} z_{i+1,k} \le \sum_{k\in D} a_{i+1,k}$ for every $D \subseteq \mathbb{N}$ and $j \in (n_i, n_{i+1}]$. To finish this proof, we need to show that $\mathcal{I}(B) \subseteq \mathcal{I}(A)$. In order to do that, take $D \notin \mathcal{I}(A)$. Then there is $\varepsilon > 0$ and infinitely many $i \in \mathbb{N}$ such that both $d_i^A(D) > \varepsilon$ and $D_i^Z(D) > \varepsilon$ (of course we can take $\varepsilon < 2$). Therefore, there are also infinitely many $i \in \mathbb{N}$ such that $d_i^Z(D \cap [1, y_i]) > \varepsilon/2$. It follows that for every such $i \in \mathbb{N}$ we can find such a set $K \subseteq D \cap [1, y_i]$ that $d_i^Z(K) \in (\varepsilon/2, 2)$, hence there is a set $K \subseteq L \subseteq [1, y_i]$ with $d_i^Z(L) \in [1, 2)$. This set L belongs to N_i, thus there is $j \in (n_{i-1}, n_i]$ such that $d_j^B(D) \ge d_j^B(K) > \varepsilon/2$. It follows that $D \notin \mathcal{I}(B)$. □

4.4. **Tools to construct (non)matrix ideals.** Below we present two propositions that are useful in creating new examples of ideals generated by matrices as well as ideals that cannot be generated by some types of matrices (see Subsection 4.5). In Proposition 4.10 we show that the direct sum of matrix ideals is also a matrix ideal which is generated by a matrix, in a sense, of the greater type. In Proposition 4.11 we show that the restriction of a matrix ideal to a positive set is also a matrix ideal which is generated by a matrix of the same type or, in a sense, a smaller type.

Proposition 4.10.

(1) *If* $\mathcal{I}, \mathcal{J} \in \mathcal{I}(\mathrm{REG})$, *then* $\mathcal{I} \oplus \mathcal{J} \in \mathcal{I}(\mathrm{REG})$.
(2) *If* $\mathcal{I} \in \mathcal{I}(\mathrm{ST1})$ *and* $\mathcal{J} \in \mathcal{I}(\mathrm{REG}) \cup \mathcal{I}(\mathrm{ST1})$, *then* $\mathcal{I} \oplus \mathcal{J} \in \mathcal{I}(\mathrm{ST1})$.
(3) *If* $\mathcal{I} \in \mathcal{I}(\mathrm{ST2})$ *and* $\mathcal{J} \in \mathcal{I}(\mathrm{REG}) \cup \mathcal{I}(\mathrm{ST1}) \cup \mathcal{I}(\mathrm{ST2})$, *then* $\mathcal{I} \oplus \mathcal{J} \in \mathcal{I}(\mathrm{ST2})$.

Proof. (1) Let $A = (a_{i,k})$ and $B = (b_{i,k})$ be regular matrices. Define $C = (c_{i,k})$ by $c_{i,2k-1} = a_{i,k}/2$ and $c_{i,2k} = b_{i,k}/2$ for all $i, k \in \mathbb{N}$. It gives us $\lim_{i\to\infty} \sum_{k\in\mathbb{N}} c_{i,k} = 1$, because $\lim_{i\to\infty} \sum_{k\in\mathbb{N}} a_{i,k} = 1$ and $\lim_{i\to\infty} \sum_{k\in\mathbb{N}} b_{i,k} = 1$, thus C is regular. It is

easy to see that $\mathcal{I}(C) \restriction 2\mathbb{N}-1 \approx \mathcal{I}(A)$ and $\mathcal{I}(C) \restriction 2\mathbb{N} \approx \mathcal{I}(B)$. Hence $\mathcal{I}(A) \oplus \mathcal{I}(B) \approx \mathcal{I}(C)$.

(2) Let $A = (a_{i,k})$ be a semiregular matrix of type 1 and $B = (b_{i,k})$ be a regular matrix or semiregular matrix of type 1. Define $C = (c_{i,k})$ by $c_{i,2k-1} = a_{i,k}$ and $c_{i,2k} = b_{i,k}$ for all $i, k \in \mathbb{N}$. Then $\lim_{i\to\infty} \sum_{k\in\mathbb{N}} c_{i,k} = \infty$ and $\sum_{k\in\mathbb{N}} c_{i,k} < \infty$ for all but finitely many $i \in \mathbb{N}$, because $\sum_{k\in\mathbb{N}} a_{i,k} < \infty$, $\sum_{k\in\mathbb{N}} b_{i,k} < \infty$ for almost all $i \in \mathbb{N}$ and $\lim_{i\to\infty} \sum_{k\in\mathbb{N}} a_{i,k} = \infty$, thus C is semiregular of type 1. It is easy to see that $\mathcal{I}(C) \restriction 2\mathbb{N} - 1 \approx \mathcal{I}(A)$ and $\mathcal{I}(C) \restriction 2\mathbb{N} \approx \mathcal{I}(B)$. Hence $\mathcal{I}(A) \oplus \mathcal{I}(B) \approx \mathcal{I}(C)$.

(3) Let $A = (a_{i,k})$ be a semiregular matrix of type 2 and $B = (b_{i,k})$ be a regular matrix or semiregular matrix of either type. Define $C = (c_{i,k})$ by $c_{i,2k-1} = a_{i,k}$ and $c_{i,2k} = b_{i,k}$ for all $i, k \in \mathbb{N}$. Then $\lim_{i\to\infty} \sum_{k\in\mathbb{N}} c_{i,k} = \infty$ and $\sum_{k\in\mathbb{N}} c_{i,k} = \infty$ for infinitely many $i \in \mathbb{N}$, because $\sum_{k\in\mathbb{N}} a_{i,k} = \infty$ for infinitely many $i \in \mathbb{N}$ and $\lim_{i\to\infty} \sum_{k\in\mathbb{N}} a_{i,k} = \infty$, thus C is semiregular of type 2. It is easy to see that $\mathcal{I}(C) \restriction 2\mathbb{N} - 1 \approx \mathcal{I}(A)$ and $\mathcal{I}(C) \restriction 2\mathbb{N} \approx \mathcal{I}(B)$. Hence $\mathcal{I}(A) \oplus \mathcal{I}(B) \approx \mathcal{I}(C)$. □

Proposition 4.11. *Let $B \notin \mathcal{I}$.*

(1) *If $\mathcal{I} \in \mathcal{I}(\mathrm{ST2})$, then $\mathcal{I} \restriction B \in \mathcal{I}(\mathrm{REG}) \cup \mathcal{I}(\mathrm{ST1}) \cup \mathcal{I}(\mathrm{ST2})$.*
(2) *If $\mathcal{I} \in \mathcal{I}(\mathrm{ST1})$, then $\mathcal{I} \restriction B \in \mathcal{I}(\mathrm{REG}) \cup \mathcal{I}(\mathrm{ST1})$.*
(3) *If $\mathcal{I} \in \mathcal{I}(\mathrm{REG})$, then $\mathcal{I} \restriction B \in \mathcal{I}(\mathrm{REG})$.*

Proof. (1) Let $A = (a_{i,k})$ be a semiregular matrix of type 2. Since $B \notin \mathcal{I}(A)$, we know that $\overline{d^A}(B) > 0$. We find two disjoint sets X, Y such that $X \cup Y = \mathbb{N}$, X is infinite, $\lim_{i\in X} d_i^A(B) = \overline{d^A}(B)$ and $d_i^A(B) < \overline{d^A}(B)$ for every $i \in Y$ (Y may be empty). Let $(x_1, x_2, \ldots)$ be an increasing enumeration of elements of X.

We will construct the matrix $C = (c_{i,k})$. We have three cases:

(1) $\overline{d^A}(B) = \infty$ and $d_i^A(B) = \infty$ for infinitely many $i \in \mathbb{N}$;
(2) $\overline{d^A}(B) = \infty$ and $d_i^A(B) < \infty$ for all but finitely many $i \in \mathbb{N}$;
(3) $\overline{d^A}(B) < \infty$.

In the first case, take $c_{i,k} = 0$ for $k \notin B$, for $k \in B$ take $c_{i,k} = a_{i,k}$ when $i \in X$ and when $i \in (x_j, x_{j+1})$ for some $j \in \mathbb{N}$, put $c_{i,k} = a_{i,k} + a_{x_j,k}$. Clearly, $\lim_{i\to\infty} \sum_{k\in\mathbb{N}} c_{i,k} = \infty$ and $\sum_{k\in\mathbb{N}} c_{i,k} = \infty$ for infinitely many $i \in \mathbb{N}$. Thus, C is a semiregular matrix of type 2. It is not difficult to see that $\mathcal{I}(A) \restriction B \approx \mathcal{I}(C)$.

In the second case, we construct the matrix C in the same way as in the first one. Once again, it is easy to see $\mathcal{I}(A) \restriction B \approx \mathcal{I}(C)$, $\lim_{i\to\infty} \sum_{k\in\mathbb{N}} c_{i,k} = \infty$ and $\sum_{k\in\mathbb{N}} c_{i,k} < \infty$ for all but finitely many $i \in \mathbb{N}$. Thus, C is a semiregular matrix of type 1.

In the last case, we know that $\overline{d^A}(B) = M < \infty$ for some $M > 0$. Denote $\sum_{k\in B} a_{i,k}$ by y_i for all $i \in Y$.

Define the matrix C by $c_{i,k} = 0$ for $k \notin B$, for $k \in B$ take $c_{i,k} = a_{i,k}/M$ when $i \in X$ and when $i \in (x_j, x_{j+1})$ for some $j \in \mathbb{N}$, put $c_{i,k} = (a_{i,k} + a_{x_j,k} \cdot \frac{M-y_i}{M})/M$. It is not difficult to see that $\mathcal{I}(A) \restriction B \approx \mathcal{I}(C)$ and $\lim_{i\to\infty} \sum_{k\in\mathbb{N}} c_{i,k} = 1$. Thus, C is a regular matrix.

(2) The proof is the same as the proof of (1), only this time the first case does not apply.

(3) The proof is the same as the proof of (1), only this time the first two cases do not apply. □

4.5. **More examples of (non)matrix ideals.** Below we present some examples of matrix ideals and non-matrix ideals. These examples show that there are no more inclusions between classes of ideals considered in the paper.

Example 4.12. $\emptyset \otimes \mathrm{Fin} \in \mathcal{I}(\mathrm{REG}) \cap \mathcal{I}(\mathrm{ST1}) \cap \mathcal{I}(\mathrm{ST2}) \setminus \mathcal{I}(\mathrm{DENSE})$.

It is not difficult to see that $\emptyset \otimes \mathrm{Fin}$ is not dense.

($\emptyset \otimes \mathrm{Fin} \in \mathcal{I}(\mathrm{REG})$). Let $C_1, C_2, \ldots$ be infinite pairwise disjoint subsets of $\mathbb{N}$ such that $\bigcup_{n\in\mathbb{N}} C_n = \mathbb{N}$ and $\emptyset \otimes \mathrm{Fin} \approx \{C \subseteq \mathbb{N} : \forall n |C \cap C_n| < \aleph_0\}$. Enumerate C_1 increasingly by $(c_1, c_2, c_3, \ldots)$. We construct the matrix $B = (b_{i,k})$ in the following way. If $i \in C_1$ then put $b_{i,i} = 1$ and $b_{i,k} = 0$ otherwise. If $i \in C_j$ for some $j > 1$ then put $b_{i,i} = 1/j$, $b_{i,c_i} = 1 - 1/j$ and $b_{i,k} = 0$ otherwise.

It is easy to see that $\sum_{k\in\mathbb{N}} b_{i,k} = 1$ for every $i \in \mathbb{N}$ and that for any $k \in \mathbb{N}$ we get $b_{i,k} \neq 0$ for at most two $i \in \mathbb{N}$, so the matrix B is regular. Furthermore, when $C \cap C_n$ is infinite for some $C \subseteq \mathbb{N}$ and $n \in \mathbb{N}$ then $d_i^B(C) \geq 1/n$ for all $i \in C \cap C_n$, hence $\mathcal{I}(B) \subseteq \emptyset \otimes \mathrm{Fin}$.

To finish the proof, we take $C \subseteq \mathbb{N}$ such that $C \cap C_n$ is finite for all $n \in \mathbb{N}$ and show that $C \in \mathcal{I}(B)$. To do that, we only need to notice that when $C \cap \bigcup_{j\leq n} C_j$ is finite for some $n \in \mathbb{N}$ then for all $i > \max \bigcup_{j\leq n} C_j$ we have $d_i^B(C) < 1/n$, thus $d^B(C) = 0$.

($\emptyset \otimes \mathrm{Fin} \in \mathcal{I}(\mathrm{ST2})$) First note that every non-dense ideal can be extended to a summable ideal, and then apply Theorem 4.8(1).

($\emptyset \otimes \mathrm{Fin} \in \mathcal{I}(\mathrm{ST1})$) Apply Theorem 4.9.

Definition 4.13. Let $p = (p_n)$ be a sequence of nonnegative reals with $s_i = \sum_{k=1}^{i} p_k \neq 0$ for every $i \in \mathbb{N}$ such that $\lim_{i\to\infty} s_i = \infty$ and $\lim_{i\to\infty} p_i/s_i = 0$. The family

$$\mathcal{EU}_p = \left\{ A \subseteq \mathbb{N} \ : \lim_{n\to\infty} \frac{\sum_{i\in A, i\leq n} p_i}{\sum_{i\leq n} p_i} = 0 \right\}$$

is called the *Erdős-Ulam ideal generated by* p. It is known that $\mathcal{EU}_p$ is an $F_{\sigma\delta}$ P-ideal (see e.g. [11] or [7]). The ideal

$$\mathcal{I}_d = \left\{ A : \lim_{n\to\infty} \frac{|A \cap \{1, \ldots, n\}|}{n} = 0 \right\}$$

of all sets of the *asymptotic density zero* is the Erdős-Ulam ideal $\mathcal{EU}_p$ with any constant sequence p.

Example 4.14. $\mathcal{EU}_p \in \mathcal{I}(\mathrm{REG}) \setminus (\mathcal{I}(\mathrm{ST1}) \cup \mathcal{I}(\mathrm{ST2}))$.

It is known that every Erdős-Ulam ideal $\mathcal{EU}_p$ is equal to $\mathcal{I}(A)$ for some regular matrix A (see e.g. [9, Section 4]). Moreover, in [8, Proposition 3.7] the authors proved that Erdős-Ulam ideals cannot be extended to a summable ideals. Therefore, by Theorem 4.7 they cannot be equal to $\mathcal{I}(A)$ for any semiregular matrix A.

Definition 4.15. Let (ϕ_n) be a sequence of measures on $\mathbb{N}$ concentrating on finite, pairwise disjoints sets. Then $\mathcal{I} = \{A \subseteq \mathbb{N} : \lim_{n\to\infty} \phi_n(A) = 0\}$ is called a *density ideal*. It is known that Erdős-Ulam ideals are density ideals and that a dense density ideal is an Erdős-Ulam ideal if and only if $\limsup_{n\to\infty} \phi_n(\mathbb{N}) = \infty$ (see e.g. [7, Theorem 1.13.3]).

Example 4.16. $\mathcal{I} \in \mathcal{I}(\mathrm{ST1}) \cap \mathcal{I}(\mathrm{DENSE})$ for every dense density ideal $\mathcal{I}$ that is not an Erdős-Ulam ideal.

The matrix A is given by $a_{i,k} = \phi_i(\{k\})$ for every $i, k \in \mathbb{N}$. Clearly, $\sum_{k\in\mathbb{N}} a_{i,k} = \phi_i(\mathbb{N}) < \infty$ for all $i \in \mathbb{N}$ and $\limsup_{i\to\infty} \sum_{k\in\mathbb{N}} a_{i,k} = \limsup_{i\to\infty} \phi_i(\mathbb{N}) = \infty$, hence A is semiregular of type 1. It is also easy to see that for every $B \subseteq \mathbb{N}$ we have $d^A(B) = 0$ if and only if $\lim_{i\to\infty} \phi_i(B) = 0$, thus $\mathcal{I} = \mathcal{I}(A)$.

Example 4.17. $\mathcal{I}_f \oplus \mathrm{Fin} \in \mathcal{I}(\mathrm{ST2}) \cap \mathcal{I}(\mathrm{P}) \setminus (\mathcal{I}(\mathrm{REG}) \cup \mathcal{I}(\mathrm{DENSE}))$ for every dense summable ideal $\mathcal{I}_f$.

It is easy to see that $\mathcal{I}_f \oplus \mathrm{Fin}$ is a non-dense P-ideal. It is also a summable ideal, so Proposition 4.6 finishes the proof.

Example 4.18. The ideal

$$\mathcal{I}_u = \left\{ A \subseteq \mathbb{N} : \lim_{h\to\infty} \limsup_{n\to\infty} \frac{|A \cap \{n, \dots, n+h\}|}{h+1} = 0 \right\}$$

of all sets of the *uniform density zero* is dense and $F_{\sigma\delta}$ (see e.g. [1, Theorem 2]). In [10, p.299], the authors proved that $\mathcal{I}_u$ is not a P-ideal. Thus, by Proposition 4.1, $\mathcal{I}_u \notin \mathcal{I}(\mathrm{REG})$. In [1, Corollary 1] the authors proved that $\mathcal{I}_u$ does not have BW property and in [8, Corollary 3.5 and Theorem 3.3] they proved that ideals without BW property cannot be extended to a summable ideals. Thus, by Theorem 4.7, $\mathcal{I}_u \notin \mathcal{I}(\mathrm{ST1}) \cup \mathcal{I}(\mathrm{ST2})$.

Example 4.19. $\mathcal{I}_u \oplus \mathcal{I}_f \in \mathcal{I}(\mathrm{DENSE}) \cap \mathcal{I}(\text{SUM-EXT}) \setminus (\mathcal{I}(\mathrm{P}) \cup \mathcal{I}(\mathrm{REG}) \cup \mathcal{I}(\mathrm{ST2}))$ for every dense summable ideal $\mathcal{I}_f$.

It is easy to see that $\mathcal{I}_u \oplus \mathcal{I}_f$ is a dense non-P-ideal ideal. Moreover $\mathcal{I}_u \oplus \mathcal{I}_f$ can be extended to a summable ideal (for instance to $\mathcal{P}(\mathbb{N}) \oplus \mathcal{I}_f$). Since $\mathcal{I}_u \notin \mathcal{I}(\mathrm{REG}) \cup \mathcal{I}(\mathrm{ST1}) \cup \mathcal{I}(\mathrm{ST2})$, so by Proposition 4.11, $\mathcal{I} \notin \mathcal{I}(\mathrm{REG}) \cup \mathcal{I}(\mathrm{ST1}) \cup \mathcal{I}(\mathrm{ST2})$.

Example 4.20. $\mathcal{I}_u \oplus \mathrm{Fin} \in \mathcal{I}(\text{SUM-EXT}) \setminus (\mathcal{I}(\mathrm{DENSE}) \cup \mathcal{I}(\mathrm{P}) \cup \mathcal{I}(\mathrm{REG}) \cup \mathcal{I}(\mathrm{ST2}))$.

Example 4.21. In [8, Example 3.6] the authors proved that there is an $F_{\sigma\delta}$ P-ideal $\mathcal{I}$ that cannot be extended to a summable ideal (it was a slight modification of the ideal constructed by Mazur in [12] – an example of a pathological F_σ ideal). Thus, by Theorem 4.7, $\mathcal{I} \notin \mathcal{I}(\mathrm{ST2})$. On the other hand, in [9, Theorem 4.12] the authors proved that $\mathcal{I} \notin \mathcal{I}(\mathrm{REG})$.

Example 4.22. Let $\mathcal{I}$ be the ideal from Example 4.21. Then

$$\mathcal{I} \oplus \mathcal{I}_f \in \mathcal{I}(\mathrm{DENSE}) \cap \mathcal{I}(\mathrm{P}) \cap \mathcal{I}(\text{SUM-EXT}) \setminus (\mathcal{I}(\mathrm{REG}) \cup \mathcal{I}(\mathrm{ST2}))$$

for every dense summable ideal $\mathcal{I}_f$.

It is not difficult to see that $\mathcal{I} \oplus \mathcal{I}_f$ is dense P-ideal. Moreover it can be extended to a summable ideal (for instance to $\mathcal{P}(\mathbb{N}) \oplus \mathcal{I}_f$). Since $\mathcal{I} \notin \mathcal{I}(\mathrm{REG}) \cup \mathcal{I}(\mathrm{ST2})$, so by Proposition 4.11, $\mathcal{I} \oplus \mathcal{I}_f \notin \mathcal{I}(\mathrm{REG}) \cup \mathcal{I}(\mathrm{ST2})$.

Example 4.23. If $\mathcal{I}$ is the ideal from Example 4.21, then

$$\mathcal{I} \oplus \mathrm{Fin} \in \mathcal{I}(\mathrm{P}) \cap \mathcal{I}(\text{SUM-EXT}) \setminus (\mathcal{I}(\mathrm{DENSE}) \cup \mathcal{I}(\mathrm{REG}) \cup \mathcal{I}(\mathrm{ST2})).$$

It is not difficult to see that $\mathcal{I} \oplus \mathrm{Fin}$ is non-dense P-ideal. Moreover it can be extended to a summable ideal (for instance to $\mathcal{P}(\mathbb{N}) \oplus \mathrm{Fin}$). Since $\mathcal{I} \notin \mathcal{I}(\mathrm{REG}) \cup \mathcal{I}(\mathrm{ST2})$, so by Proposition 4.11, $\mathcal{I} \oplus \mathrm{Fin} \notin \mathcal{I}(\mathrm{REG}) \cup \mathcal{I}(\mathrm{ST2})$.

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