KATĚTOV ORDER BETWEEN HINDMAN, RAMSEY, VAN DER WAERDEN AND SUMMABLE IDEALS

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Abstract. A family $\mathcal{I}$ of subsets of a set X is an *ideal on* X if it is closed under taking subsets and finite unions of its elements. An ideal $\mathcal{I}$ on X is below an ideal $\mathcal{J}$ on Y in the *Katětov order* if there is a function $f : Y \to X$ such that $f^{-1}[A] \in \mathcal{J}$ for every $A \in \mathcal{I}$. We show that the Hindman ideal, the Ramsey ideal and the summable ideal are pairwise incomparable in the Katětov order, where

- the *Ramsey ideal* consists of those sets of pairs of natural numbers which do not contain a set of all pairs of any infinite set (equivalently do not contain, in a sense, any infinite complete subgraph),
- the *Hindman ideal* consists of those sets of natural numbers which do not contain any infinite set together with all finite sums of its members (equivalently do not contain IP-sets that are considered in Ergodic Ramsey theory),
- the *summable ideal* consists of those sets of natural numbers such that the series of the reciprocals of its members is convergent.

Moreover, we show that in the Katětov order the above mentioned ideals are not below the *van der Waerden ideal* that consists of those sets of natural numbers which do not contain arithmetic progressions of arbitrary finite length.

Contents

1. Introduction

The Katětov order is an efficient tool for studying ideals over countable sets [19, 20, 21, 22, 35, 37]. Originally, the Katětov order (introduced by Katětov [24] in 1968) was used to study convergence in topological spaces, and our interest in Katětov order between the Hindman, Ramsey, van der Waerden and summable ideals stems from the study of sequentially compact spaces defined as, in a sense, topological counterparts of well-known combinatorial theorems: Ramsey's theorem

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for coloring graphs, Hindman's finite sums theorem and van der Waerden's arithmetical progressions theorem [3, 4, 25, 26, 27, 28]. It is known [11, 30] that an existence of a sequentially compact space which distinguishes the above mentioned classes of spaces is reducible to a question whether particular ideals are incomaparable in the Katětov order.

Beside our primary interest in the Katětov order described above we mention one more strength of this order. Using the Katětov order, we can classify non-definable objects (like ultrafilters or maximal almost disjoint families) using Borel ideals [20]. For instance, an ultrafilter $\mathcal{U}$ is a P-point if and only if the dual ideal $\mathcal{U}^*$ is not Katětov above Fin^2 (equivalently $\mathcal{U}$ is a Fin^2-ultrafilter as defined by Baumgartner [2]). It is known [10] that an existence of an ultrafilter which distinguishes between some classes of ultrafilters is reducible to a question whether particular ideals are incomaparable in the Katětov order.

Below we describe the results obtain in this paper and introduce a necessary notions and notations.

We write ω to denote the set of all natural numbers (with zero).

We write $[A]^2$ to denote the set of all unordered pairs of elements of A, $[A]^{<\omega}$ to denote the family of all finite subsets of A and $[A]^\omega$ to denote the family of all infinite countable subsets of A.

A family $\mathcal{I} \subseteq \mathcal{P}(X)$ of subsets of a set X is an *ideal on* X if it is closed under taking subsets and finite unions of its elements, $X \notin \mathcal{I}$ and $\mathcal{I}$ contains all finite subsets of X. By $\mathrm{Fin}(X)$ we denote the family of all finite subsets of X and we write Fin instead of $\mathrm{Fin}(\omega)$.

For an ideal $\mathcal{I}$ on X, we write $\mathcal{I}^+ = \{A \subseteq X : A \notin \mathcal{I}\}$ and call it the *coideal of* $\mathcal{I}$, and we write $\mathcal{I}^* = \{X \setminus A : A \in \mathcal{I}\}$ and call it the *filter dual to* $\mathcal{I}$. It is easy to see that $\mathcal{I} \restriction A = \{A \cap B : B \in \mathcal{I}\}$ is an ideal on A if and only if $A \in \mathcal{I}^+$.

For a set $B \subseteq \omega$, we write $FS(B)$ to denote the set of all finite (nonempty) sums of distinct elements of B i.e. $\mathrm{FS}(B) = \{\sum_{n\in F} n : F \in [B]^{<\omega} \setminus \{\emptyset\}\}$.

In this paper we are interested in the following four ideals:

- the *Ramsey ideal*
$$\mathcal{R} = \left\{A \subseteq [\omega]^2 : \forall B \in [\omega]^\omega \left([B]^2 \not\subseteq A\right)\right\},$$
- the *Hindman ideal*
$$\mathcal{H} = \{A \subseteq \omega : \forall B \in [\omega]^\omega \ \mathrm{FS}(B) \not\subseteq A\},$$
- the *van der Waerden ideal*
$$\mathcal{W} = \{A \subseteq \omega : A \text{ does not contain arithmetic progressions of arbitrary finite length}\},$$
- the *summable ideal*
$$\mathcal{I}_{1/n} = \left\{A \subseteq \omega : \sum_{n\in A} \frac{1}{n+1} < \infty\right\}.$$

The Ramsey ideal was introduced by Meza-Alcántara and Hrušák [22] (the authors noted that if we identify a set $A \subseteq [\omega]^2$ with a graph $G_A = (\omega, A)$, the ideal $\mathcal{R}$ can be seen as an ideal consisting of graphs without infinite complete subgraphs). Both the Hindman and van der Waerden ideals were introduced by Flaškova [14, p. 109]. The summable ideal is a particular instance of the so-called *summable ideals* which seem to be "ancient" compare to previously mentioned ideals as they were introduced in 1972 by Mathias [31, Example 3, p.206].

We say that an ideal $\mathcal{I}$ on X is below an ideal $\mathcal{J}$ on Y in the *Katětov order* [24] if there is a function $f : Y \to X$ such that $f^{-1}[A] \in \mathcal{J}$ for every $A \in \mathcal{I}$ (equivalently, $f[B] \notin \mathcal{I}$ for all $B \notin \mathcal{J}$). Note that the Katětov order has been extensively examined (even in its own right) for many years so far [1, 2, 5, 6, 7, 16, 17, 18, 19, 20, 21, 22, 34, 35, 36, 37, 39].

The aim of this paper is to prove the following

Theorem 1.1.

(1) The ideals $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{I}_{1/n}$ are pairwise incomparable in the Katětov order.
(2) The ideals $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{I}_{1/n}$ are not below the ideal $\mathcal{W}$ in the Katětov order.

As far as we are concerned, the remaining three questions about these ideals are still open.

Question 1.2. Is the ideal $\mathcal{W}$ below the ideal $\mathcal{R}$ ($\mathcal{H}$, $\mathcal{I}_{1/n}$, resp.) in the Katětov order?

Note that in the case of the summable ideal, Question 1.2 is a weakening of the famous Erdős-Turán conjecture which says that $\mathcal{W} \subseteq \mathcal{I}_{1/n}$.

2. Preliminaries

An ideal $\mathcal{I}$ on X is *tall* [32, Definition 0.6] if for every infinite set $A \subseteq X$ there exists an infinite set $B \subseteq A$ such that $B \in \mathcal{I}$. It is not difficult to see that $\mathcal{I}$ is not tall $\iff$ $\mathcal{I} \leq_K \mathcal{J}$ for every ideal $\mathcal{J}$ $\iff$ $\mathcal{I} \leq_K \mathrm{Fin}$ $\iff$ $\mathcal{I} \restriction A = \mathrm{Fin}(A)$ for some $A \in \mathcal{I}^+$. It is easy to show the following

Proposition 2.1. *The ideals $\mathcal{H}$, $\mathcal{R}$, $\mathcal{W}$ and $\mathcal{I}_{1/n}$ are tall.*

Ideals $\mathcal{I}$ and $\mathcal{J}$ on X and Y, respectively are *isomorphic* (in short: $\mathcal{I} \approx \mathcal{J}$) if there exists a bijection $\phi : X \to Y$ such that $A \in \mathcal{I} \iff \phi[A] \in \mathcal{J}$ for each $A \subseteq X$. An ideal $\mathcal{I}$ is *homogeneous* [29, Definition 1.3] if the ideals $\mathcal{I}$ and $\mathcal{I} \restriction A$ are isomorphic for every $A \in \mathcal{I}^+$.

Proposition 2.2 ([29, Examples 2.5 and 2.6])**.** *The ideals $\mathcal{H}$, $\mathcal{R}$ and $\mathcal{W}$ are homogeneous.*

By identifying subsets of X with their characteristic functions, we equip $\mathcal{P}(X)$ with the topology of the space 2^X (the product topology of countably many copies of the discrete topological space $\{0, 1\}$) and therefore we can assign topological notions to ideals on X. In particular, an ideal $\mathcal{I}$ is *Borel* (F_σ, resp.) if $\mathcal{I}$ is a Borel (F_σ, resp.) subset of 2^X.

If $A \subseteq \omega$ and $n \in \omega$, we write $A + n = \{a + n : a \in A\}$ and $A - n = \{a - n : a \in A, a \geq n\}$.

A set $D \subseteq \omega$ is *sparse* [25, p. 1598] if for each $x \in \mathrm{FS}(D)$ there exists the unique set $\alpha \subseteq D$ such that $x = \sum_{n \in \alpha} n$. This unique set will be denoted by $\alpha_D(x)$. For instance, the set $E = \{2^n : n \in \omega\}$ is sparse, and in the sequel, we write $\alpha(x)$ instead of $\alpha_E(x)$.

A set $D \subseteq \omega$ is *very sparse* [12, p. 894] if it is sparse and

$$\forall x, y \in \mathrm{FS}(D)\ (\alpha_D(x) \cap \alpha_D(y) \neq \emptyset \implies x + y \notin \mathrm{FS}(D)).$$

In the sequel, we will use the following

Lemma 2.3 ([12, Lemma 2.2])**.** *For every infinite set $D \subseteq \omega$ there is an infinite set $D' \subseteq D$ which is very sparse.*

3. Summable and van der Waerden ideals are not above Hindman and Ramsey ideals

To show that F_σ ideals are not above $\mathcal{H}$ nor $\mathcal{R}$ in the Katětov order one can use the following ideal on ω^2 introduced by Katětov [23, Definition 5.1]:

$$\mathrm{Fin}^2 = \left\{C \subseteq \omega^2 : \{n \in \omega : \{k \in \omega : (n, k) \in C\} \notin \mathrm{Fin}\} \in \mathrm{Fin}\right\}.$$

The following lemma and proposition can be found in [11], but we decided to include proofs here for the sake of completeness.

Lemma 3.1 ([11, Proposition 7.2])**.**

(a) $\mathrm{Fin}^2 \leq_K \mathcal{H}$.
(b) $\mathrm{Fin}^2 \leq_K \mathcal{R}$.

Proof. (a): Let $A_k = \{2^k(2n+1) : n \in \omega\}$ for each $k \in \omega$. Let $f : \omega \to \omega^2$ be any injective function such that $f[A_k] \subseteq \{k\} \times \omega$ for all $k \in \omega$. In [12, item (2) in the proof of Proposition 1.1], the authors showed that $A_k \in \mathcal{H}$ for every $k \in \omega$ (so $f^{-1}[\{k\} \times \omega] \in \mathcal{H}$ for all $k \in \omega$), whereas in [12, item (1) in the proof of Proposition 1.1] it is shown that for every $B \notin \mathcal{H}$ there is $k \in \omega$ such that $B \cap A_k$ is infinite (so $f^{-1}[C] \in \mathcal{H}$ whenever $C \subseteq \omega^2$ is such that $C \cap (\{k\} \times \omega)$ is finite for all $k \in \omega$). Thus, the function f witnesses the fact that $\mathrm{Fin}^2 \leq_K \mathcal{H}$.

(b): Let $A_n = \{\{k, i\} : i > k \geq n\}$ for every $n \in \omega$. Then $A_n \notin \mathcal{R}$, $A_0 = [\omega]^2$, $\bigcap_{n\in\omega} A_n = \emptyset$ and $A_n \setminus A_{n+1} = \{\{n, i\} : i > n\} \in \mathcal{R}$. Let $f : [\omega]^2 \to \omega^2$ be any injective function such that $f[A_n \setminus A_{n+1}] \subseteq \{n\} \times \omega$. Then $f^{-1}[\{n\} \times \omega] \in \mathcal{R}$ for all $n \in \omega$. Suppose, for sake of contradiction, that there is $C \subseteq \omega^2$ such that $C \cap (\{k\} \times \omega)$ is finite for all $k \in \omega$ (so $C \in \mathrm{Fin}^2$), but $B = f^{-1}[C] \notin \mathcal{R}$. Then $B \subseteq^* A_n$ for every $n \in \omega$. Let $H = \{h_n : n \in \omega\}$ be an infinite set such that $[H]^2 \subseteq B$ and $h_n < h_{n+1}$ for every $n \in \omega$. Since $[H]^2 \subseteq^* A_{h_1}$, there is a finite set F such that $[H]^2 \setminus F \subseteq A_{h_1}$. Since F is finite, there is $k > 0$ such that $\{h_0, h_n\} \notin F$ for every $n \geq k$. Then $\{\{h_0, h_n\} : n \geq k\} \subseteq [H]^2 \setminus F$ and $\{\{h_0, h_n\} : n \geq k\} \cap A_{h_1} = \emptyset$, a contradiction. □

Proposition 3.2 ([11, Theorem 7.7])**.**

(a) $\mathcal{H} \not\leq_K \mathcal{W}$.
(b) $\mathcal{R} \not\leq_K \mathcal{W}$.
(c) $\mathcal{H} \not\leq_K \mathcal{I}_{1/n}$.
(d) $\mathcal{R} \not\leq_K \mathcal{I}_{1/n}$.

Proof. (a): Suppose otherwise: $\mathcal{H} \leq_K \mathcal{W}$. Using Lemma 3.1 we get that $\mathrm{Fin}^2 \leq_K \mathcal{H}$, so $\mathrm{Fin}^2 \leq_K \mathcal{W}$. However, since $\mathcal{W}$ is F_σ (see [13, Example 4.12]), $\mathrm{Fin}^2 \not\leq_K \mathcal{W}$ (by [8, Theorems 7.5 and 9.1] and [1, Example 4.1]). A contradiction.

The proofs of items (b), (c) and (d) are similar to the proof of item (a), since $\mathrm{Fin}^2 \leq_K \mathcal{R}$ (by Lemma 3.1) and $\mathcal{I}_{1/n}$ is F_σ (see [33, Example 1.5]). □

4. Summable ideal is not below van der Waerden ideal

Proposition 4.1. $\mathcal{I}_{1/n} \not\leq_K \mathcal{W}$.

Proof. Suppose for sake of contradiction that there is a function $\phi : \omega \to \omega$ such that $\phi^{-1}[B] \in \mathcal{W}$ for every $B \in \mathcal{I}_{1/n}$. We construct a sequence $(F_n : n \in \omega)$ of finite subsets of ω such that for every $n \in \omega$ we have

(1) F_n is an arithmetic progression of length n,
(2) $\phi(x) \geq n2^n$ for every $x \in F_n$.

Suppose that F_i are constructed for $i < n$. Since $B = \{i \in \omega : i < n2^n\}$ is finite, $A = \phi^{-1}[B] \in \mathcal{W}$. Then $\omega \setminus A \notin \mathcal{W}$, so there is an arithmetic progression $F_n \subseteq \omega \setminus A$ of length n. This finishes the construction of F_n.

Let $A = \bigcup\{F_n : n \in \omega\}$. Then $A \notin \mathcal{W}$, but

$$\sum_{y\in\phi[A]} \frac{1}{y+1} \leq \sum_{n\in\omega} \left(\sum_{x\in F_n} \frac{1}{\phi(x)+1} \right) \leq \sum_{n\in\omega} \left(\sum_{x\in F_n} \frac{1}{n2^n+1} \right) = \sum_{n\in\omega} \frac{n}{n2^n+1} < \infty,$$

so $\phi[A] \in \mathcal{I}_{1/n}$, a contradiction. □

5. Summable ideal is not below Hindman ideal

Theorem 5.1. $\mathcal{I}_{1/n} \not\leq_K \mathcal{H}$.

Proof. This is proved in [12, Theorem 3.2], but below we provide a simpler proof.

Let $\phi : \omega \to \omega$ be an arbitrary function. We will show that ϕ is not a witness for $\mathcal{I}_{1/n} \leq_K \mathcal{H}$ i.e. we will find an infinite set $D \subseteq \omega$ such that $\phi[\mathrm{FS}(D)] \in \mathcal{I}_{1/n}$.

Using Canonical Hindman Theorem ([38, Theorem 2.1], see also [15, Theorem 5 at p. 133]), there is an infinite set $C = \{c_n : n \in \omega\} \subseteq \omega$ such that $\max \alpha(c_n) < \min \alpha(c_{n+1})$ for every $n \in \omega$ and one of the following five cases holds:

(1) $\forall x, y \in \mathrm{FS}(C)(\phi(x) = \phi(y))$,
(2) $\forall x, y \in \mathrm{FS}(C)(\phi(x) = \phi(y) \iff \min \alpha(x) = \min \alpha(y))$,
(3) $\forall x, y \in \mathrm{FS}(C)(\phi(x) = \phi(y) \iff \max \alpha(x) = \max \alpha(y))$,
(4) $\forall x, y \in \mathrm{FS}(C)(\phi(x) = \phi(y) \iff (\min \alpha(x) = \min \alpha(y) \text{ and } \max \alpha(x) = \max \alpha(y)))$,
(5) $\forall x, y \in \mathrm{FS}(C)(\phi(x) = \phi(y) \iff x = y)$.

Case 1. We take $D = C$ and see that the set $\phi[\mathrm{FS}(D)]$ has only one element, so it belongs to $\mathcal{I}_{1/n}$.

Case 2. We construct a strictly increasing sequence $\{k_n : n \in \omega\}$ such that $\phi(c_{k_n}) > 2^n$ for every $n \in \omega$.

Suppose that k_i are constructed for $i < n$. Since $\max \alpha(c_k) < \min \alpha(c_{k+1})$ for every $k \in \omega$, $\min \alpha(c_k) \neq \min \alpha(c_l)$ for distinct $k, l \in \omega$. Consequently, $\phi \restriction C$ is one-to-one, so we can find $k_n > k_{n-1}$ such that $\phi(c_{k_n}) > 2^n$. That finishes the inductive construction of k_n.

Let $D = \{c_{k_n} : n \in \omega\}$. If we show that $\phi[\mathrm{FS}(D)] \in \mathcal{I}_{1/n}$, the proof of this case will be finished. Using the properties of c_{k_n}'s we can see that $\phi[c_{k_n} + \mathrm{FS}(\{c_{k_i} : i > n\})] = \{\phi(c_{k_n})\}$, for every $n \in \omega$, so

$$\sum_{y \in \phi[\mathrm{FS}(D)]} \frac{1}{y+1} = \sum_{n \in \omega} \left(\sum_{y \in \{\phi(c_{k_n})\} \cup \phi[c_{k_n} + \mathrm{FS}(\{c_{k_i} : i > n\})]} \frac{1}{y+1} \right)$$
$$= \sum_{n \in \omega} \frac{1}{\phi(c_{k_n}) + 1} \leq \sum_{n \in \omega} \frac{1}{2^n + 1} < \infty.$$

Case 3. We construct a strictly increasing sequence $\{k_n : n \in \omega\}$ such that $\phi(c_{k_n}) > 2^n$ for every $n \in \omega$.

Suppose that k_i are constructed for $i < n$. Since $\max \alpha(c_k) < \min \alpha(c_{k+1})$ for every $k \in \omega$, $\max \alpha(c_k) \neq \max \alpha(c_l)$ for distinct $k, l \in \omega$. Consequently, $\phi \restriction C$ is one-to-one, so we can find $k_n > k_{n-1}$ such that $\phi(c_{k_n}) > 2^n$. That finishes the inductive construction of k_n.

Let $D = \{c_{k_n} : n \in \omega\}$. If we show that $\phi[\mathrm{FS}(D)] \in \mathcal{I}_{1/n}$, the proof of this case will be finished. Using the properties of c_{k_n}'s we can see that $\phi[c_{k_n} + \mathrm{FS}(\{c_{k_i} : i < n\})] = \{\phi(c_{k_n})\}$, for every $n \in \omega$, so

$$\sum_{y \in \phi[\mathrm{FS}(D)]} \frac{1}{y+1} = \sum_{n \in \omega} \left(\sum_{y \in \{\phi(c_{k_n})\} \cup \phi[c_{k_n} + \mathrm{FS}(\{c_{k_i} : i < n\})]} \frac{1}{y+1} \right)$$
$$= \sum_{n \in \omega} \frac{1}{\phi(c_{k_n}) + 1} \leq \sum_{n \in \omega} \frac{1}{2^n + 1} < \infty.$$

Case 4. We construct a strictly increasing sequence $\{k_n : n \in \omega\}$ such that

$$\forall n \in \omega\, \forall i < n\ (\phi(c_{k_n}) > n2^n \wedge \phi(c_{k_n} + c_{k_i}) > n2^n).$$

Suppose that k_i are constructed for $i < n$. Since $\max \alpha(c_k) < \min \alpha(c_{k+1})$ for every $k \in \omega$, we obtain that $\min \alpha(c_k + c_{k_i}) \neq \min \alpha(c_k + c_{k_j})$ and $\min \alpha(c_k) \neq$

$\min\alpha(c_k+c_{k_i})$ for every $k>k_{n-1}$ and $i<j\leq n-1$. Consequently, the function $\phi\restriction(\{c_k+c_{k_i}:k>k_{n-1},i<n\}\cup\{c_k:k>k_{n-1}\})$ is one-to-one, so using pigeonhole principle we can find $k_n>k_{n-1}$ such that $\phi(c_{k_n})>n2^n$ and $\phi(c_{k_n}+c_{k_i})>n2^n$ for every $i<n$. That finishes the inductive construction of k_n.

Let $D=\{c_{k_n}:n\in\omega\}$. If we show that $\phi[\mathrm{FS}(D)]\in\mathcal{I}_{1/n}$, the proof of this case will be finished. Using the properties of c_{k_n}'s we can see that $\phi[c_{k_m}+\mathrm{FS}(\{c_{k_i}:m<i<n\})+c_{k_n}]=\{\phi(c_{k_m}+c_{k_n})\}$ for every $m<n$, $m,n\in\omega$, so

$$\begin{aligned}\sum_{y\in\phi[\mathrm{FS}(D)]}\frac{1}{y+1}&=\sum_{n\in\omega}\frac{1}{\phi(c_{k_n})+1}\\&\quad+\sum_{n\in\omega}\sum_{m<n}\left(\sum_{y\in\{\phi(c_{k_m}+c_{k_n})\}\cup\phi[c_{k_m}+\mathrm{FS}(\{c_{k_i}:m<i<n\})+c_{k_n}]}\frac{1}{y+1}\right)\\&=\sum_{n\in\omega}\frac{1}{\phi(c_{k_n})+1}+\sum_{n\in\omega}\sum_{m<n}\frac{1}{\phi(c_{k_m}+c_{k_n})+1}\\&\leq\sum_{n\in\omega}\frac{1}{n2^n+1}+\sum_{n\in\omega}\sum_{m<n}\frac{1}{n2^n+1}<\infty.\end{aligned}$$

Case 5. We construct inductively a strictly increasing sequence $\{k_n:n\in\omega\}$ such that

$$\forall n\in\omega\,\forall x\in\mathrm{FS}(\{c_{k_i}:i<n\})\ \left(\phi(c_{k_n})>2^{2n}\wedge\phi(c_{k_n}+x)>2^{2n}\right).$$

Suppose that k_i are constructed for $i<n$. Let $m\in\omega$ be such that $m>2^{2n}$ and $m>\phi(x)$ for every $x\in\mathrm{FS}(\{c_{k_i}:i<n\})$. Since $\phi\restriction\mathrm{FS}(C)$ is one-to-one, the set $F=\phi^{-1}[\{0,1,\dots,m\}]$ is finite. Let $k_n\in\omega$ be such that $c_{k_n}>\max F$. Since $c_{k_n}>\max F$, we obtain that $c_{k_n}\notin F$ and consequently $\phi(c_{k_n})>m>2^{2n}$. Similarly, for every $x\in\mathrm{FS}(\{c_{k_i}:i<n\})$ we have $c_{k_n}+x>c_{k_n}>\max F$, so $\phi(c_{k_n}+x)>m>2^{2n}$. That finishes the inductive construction of k_n.

Let $D=\{c_{k_n}:n\in\omega\}$. If we show that $\phi[\mathrm{FS}(D)]\in\mathcal{I}_{1/n}$, the proof of this case will be finished. Using the properties of c_{k_n}'s we can see that:

$$\begin{aligned}\sum_{y\in\phi[\mathrm{FS}(D)]}\frac{1}{y+1}&=\sum_{n\in\omega}\left(\frac{1}{\phi(c_{k_n})+1}+\sum_{x\in\mathrm{FS}(\{c_{k_i}:i<n\})}\frac{1}{\phi(c_{k_n}+x)+1}\right)\\&\leq\sum_{n\in\omega}\left(\frac{1}{2^{2n}+1}+\sum_{x\in\mathrm{FS}(\{c_{k_i}:i<n\})}\frac{1}{2^{2n}+1}\right)\\&\leq\sum_{n\in\omega}\left(\frac{1}{2^{2n}+1}+(2^n-1)\cdot\frac{1}{2^{2n}+1}\right)<\infty.\end{aligned}$$ □

6. Summable ideal is not below Ramsey ideal

Theorem 6.1. $\mathcal{I}_{1/n}\not\leq_K\mathcal{R}$.

Proof. Let $\phi:[\omega]^2\to\omega$ be an arbitrary function. We will show that ϕ is not a witness for $\mathcal{I}_{1/n}\leq_K\mathcal{R}$ i.e. we will find an infinite set $H\subseteq\omega$ such that $\phi[[H]^2]\in\mathcal{I}_{1/n}$. Using Canonical Ramsey Theorem ([9, Theorem II], see also [15, Theorem 2 at p. 129]), there is an infinite set $T\subseteq\omega$ such that one of the following four cases holds:

(1) $\forall x,y\in[T]^2(\phi(x)=\phi(y))$,
(2) $\forall x,y\in[T]^2(\phi(x)=\phi(y)\iff\min x=\min y)$,
(3) $\forall x,y\in[T]^2(\phi(x)=\phi(y)\iff\max x=\max y)$,
(4) $\forall x,y\in[T]^2(\phi(x)=\phi(y)\iff x=y)$.

Case 1. We take $H = T$ and see that the set $\phi[[H]^2]$ has only one element, so it belongs to $\mathcal{I}_{1/n}$.

Case 2. In this case, for every $t \in T$ the restriction $\phi \restriction \{\{t, s\} : s \in T, s > t\}$ is constant with distinct values for distinct t. Thus, for every $t \in T$ there is k_t such that $\{k_t\} = \phi[\{\{t, s\} : s \in T, s > t\}]$.

Since k_{t_n} are pairwise distinct, we can find a one-to-one sequence $\{t_n : n \in \omega\} \subseteq T$ such that $k_{t_n} > 2^n$ for every $n \in \omega$.

Now, we take $H = \{t_n : n \in \omega\}$ and notice that

$$\sum_{k\in\phi[[H]^2]} \frac{1}{k+1} = \sum_{n=0}^{\infty} \left(\sum_{k\in\phi[\{\{t_n,t_i\}:i>n\}]} \frac{1}{k+1} \right) = \sum_{n=0}^{\infty} \frac{1}{k_{t_n}+1} \leq \sum_{n=0}^{\infty} \frac{1}{2^n} < \infty,$$

so $\phi[[H]^2] \in \mathcal{I}_{1/n}$.

Case 3. In this case, for every $t \in T$ the restriction $\phi \restriction \{\{s, t\} : s \in T, s < t\}$ is constant with distinct values for distinct t. Thus, for every $t \in T$ there is k_t such that $\{k_t\} = \phi[\{\{t, s\} : s \in T, s < t\}]$.

Since k_{t_n} are pairwise distinct, we can find a one-to-one sequence $\{t_n : n \in \omega\} \subseteq T$ such that $k_{t_n} > 2^n$ for every $n \in \omega$.

Now, we take $H = \{t_n : n \in \omega\}$ and notice that

$$\sum_{k\in\phi[[H]^2]} \frac{1}{k+1} = \sum_{n=0}^{\infty} \left(\sum_{k\in\phi[\{\{t_i,t_n\}:i<n\}]} \frac{1}{k+1} \right) = \sum_{n=0}^{\infty} \frac{1}{k_{t_n}+1} \leq \sum_{n=0}^{\infty} \frac{1}{2^n} < \infty,$$

so $\phi[[H]^2] \in \mathcal{I}_{1/n}$.

Case 4. We construct inductively a one-to-one sequence $\{t_n : n \in \omega\} \subseteq T$ such that $\phi(\{t_i, t_n\}) > n \cdot 2^n$ for every $n \in \omega$ and every $i < n$.

Suppose that t_i are constructed for $i < n$. Since there are only finitely many numbers below $n \cdot 2^n$ and the function ϕ is one-to-one on $[T]^2$ there is $t_n \in T \setminus \{t_i : i < n\}$ such that $\phi(\{t, t_n\}) > n \cdot 2^n$ for every $t \in T$. That finishes the inductive construction of t_n.

Now, we take $H = \{t_n : n \in \omega\}$ and notice that

$$\sum_{k\in\phi[[H]^2]} \frac{1}{k+1} = \sum_{n=0}^{\infty} \sum_{i<n} \frac{1}{\phi(\{t_i,t_n\})+1} \leq \sum_{n=0}^{\infty} \sum_{i<n} \frac{1}{n\cdot 2^n+1} \leq \sum_{n=0}^{\infty} \frac{1}{2^n} < \infty,$$

so $\phi[[H]^2] \in \mathcal{I}_{1/n}$. $\square$

7. Hindman ideal is not below Ramsey ideal

Lemma 7.1. *If D is very sparse, then $\{x \in \mathrm{FS}(D) : \alpha_D(x) \cap \alpha_D(y) \neq \emptyset\} \in \mathcal{H}$ for every $y \in \mathrm{FS}(D)$.*

Proof. Let $(d_n)_{n\in\omega}$ be the increasing enumeration of all elements of D and $\alpha_D(y) = \{k_0, \ldots, k_n\}$. Since

$$\{x \in \mathrm{FS}(D) : \alpha_D(x) \cap \alpha_D(y) \neq \emptyset\} = \bigcup_{i\leq n} \{x \in \mathrm{FS}(D) : k_i \in \alpha_D(x)\},$$

we only need to show that $\{x \in \mathrm{FS}(D) : k_i \in \alpha_D(x)\} \in \mathcal{H}$ for every $i \leq n$.

If $y, z \in \{x \in \mathrm{FS}(D) : k_i \in \alpha_D(x)\}$, then $k_i \in \alpha_D(y) \cap \alpha_D(z) \neq \emptyset$, so $y + z \notin \mathrm{FS}(D)$ (since D is very sparse). Thus, there is no infinite (even two-element) set C such that $\mathrm{FS}(C) \subseteq \{x \in \mathrm{FS}(D) : k_i \in \alpha_D(x)\}$. $\square$

Theorem 7.2. $\mathcal{H} \not\leq_K \mathcal{R}$.

Proof. Let $D \subseteq \omega$ be a very sparse set (which exists by Lemma 2.3). Since the ideal $\mathcal{H}$ is homogeneous (see Proposition 2.2), it suffices to show that $\mathcal{H} \restriction \mathrm{FS}(D) \not\leq_K \mathcal{R}$.

Assume to the contrary that there exists $f : [\omega]^2 \to \mathrm{FS}(D)$ which witnesses $\mathcal{H} \restriction \mathrm{FS}(D) \leq_K \mathcal{R}$.

We will recursively define infinite sets $B_n \subseteq \omega$ and pairwise distinct elements $b_n \in \omega$ such that for all $n \in \omega$ the following conditions are satisfied:

(a) $b_n \in B_n$, $b_{n+1} > b_n$,
(b) $B_{n+1} \subseteq B_n$, $B_0 = \omega$,
(c) for each $y \in f\left[[\{b_i : i < n\}]^2\right]$ we have
$$f[[B_n]^2] \cap \{x \in \mathrm{FS}(D) : \ \alpha_D(x) \cap \alpha_D(y) \neq \emptyset\} = \emptyset,$$
(d) for each $y \in f\left[[\{b_i : i < n\}]^2\right]$ and $i < n$ we have
$$f\left[\{\{b_i, b\} : b \in B_n\}\right] - y \in \mathcal{H}.$$

Let $b_0 = 0$ and $B_0 = \omega$. Then b_0 and B_0 are as required. Assume that b_i and B_i have been constructed for $i < n$ and satisfy items (b)–(d).

Since $\{x \in \mathrm{FS}(D) : \alpha_D(x) \cap \alpha_D(y) \neq \emptyset\} \in \mathcal{H}$ for every $y \in f\left[[\{b_i : i < n\}]^2\right] \subseteq \mathrm{FS}(D)$ (by Lemma 7.1), $[B_{n-1}]^2 \in \mathcal{R}^+$ and we assumed that f witnesses $\mathcal{H} \restriction \mathrm{FS}(D) \leq_K \mathcal{R}$, there exists an infinite set $B \subseteq \omega$ such that
$$[B]^2 \subseteq [B_{n-1}]^2 \setminus \bigcup_{y \in f[[\{b_i : i<n\}]^2]} f^{-1}\left[\{x \in \mathrm{FS}(D) : \ \alpha_D(x) \cap \alpha_D(y) \neq \emptyset\}\right].$$

Observe that for each infinite set $E \subseteq \omega$ and $b, y \in \omega$ there exists an infinite set $C \subseteq E$ such that $f[\{\{b, c\} : c \in C\}] - y \in \mathcal{H}$. Indeed, let $g : E \setminus \{b\} \to \omega$ be given by $g(x) = f(\{b, x\}) - y$. Since $\mathcal{H}$ is a tall ideal (Proposition 2.1), $\mathcal{H} \not\leq_K \mathrm{Fin}(E \setminus \{b\})$. Thus, there is $C \notin \mathrm{Fin}(E \setminus \{b\})$ such that $C \subseteq E \setminus \{b\}$ and $g[C] = f[\{\{b, c\} : c \in C\}] - y \in \mathcal{H}$.

Now, using recursively the above observation we can find an infinite set $C \subseteq B$ such that $f[\{\{b_i, c\} : c \in C\}] - y \in \mathcal{H}$ for every $i < n$ and $y \in f\left[[\{b_i : i < n\}]^2\right]$.

We put $B_n = C$ and pick any $b_n \in B_n$ with $b_n > b_{n-1}$.

The construction of the sequences $(B_n)_{n\in\omega}$ and $(b_n)_{n\in\omega}$ is finished.

Let $B = \{b_n : \ n \in \omega\}$. Since B is infinite, $[B]^2 \in \mathcal{R}^+$. Since we assumed that f witnesses $\mathcal{H} \restriction \mathrm{FS}(D) \leq_K \mathcal{R}$, $f[[B]^2] \in \mathcal{H}^+ \restriction \mathrm{FS}(D)$, and consequently there exists an infinite set $C \subseteq \omega$ such that $\mathrm{FS}(C) \subseteq f[[B]^2]$.

Pick any $c \in C$ and let $j, n \in \omega$ be such that $c = f(\{b_j, b_n\})$ and $j < n$.

Since $X = [\{b_i : i \leq n\}]^2$ is finite, $f[X] - c \in \mathcal{H}$.

Let $Y = \{\{b_i, b_k\} : i \leq n < k\}\}$. Since $\{b_k : k > n\} \subseteq B_{n+1}$ and B_{n+1} satisfies item (d) applied to $y = c$, we have $f\left[Y\right] - c \in \mathcal{H}$.

Let $Z = [\{b_i : i > n\}]^2$. We claim that $\mathrm{FS}(C \setminus \{c\}) \cap (f\left[Z\right] - c) = \emptyset$. Suppose to the contrary that there exists $a \in \mathrm{FS}(C \setminus \{c\}) \cap (f\left[Z\right] - c)$. Then $a + c \in \mathrm{FS}(C) \cap f[Z] \subseteq \mathrm{FS}(D) \cap f[[B_{n+1}]^2]$, so by item (c) applied to $y = c$, $\alpha_D(c) \cap \alpha_D(a + c) = \emptyset$. On the other hand, $a, c \in \mathrm{FS}(D)$, D is very sparse and $a + c \in \mathrm{FS}(D)$, so $\alpha_D(a) \cap \alpha_D(c) = \emptyset$. Consequently, $\alpha_D(a + c) = \alpha_D(a) \cup \alpha_D(c)$, so $\alpha_D(c) \cap \alpha_D(a + c) = \alpha_D(c) \neq \emptyset$, a contradiction.

Since $[B]^2 = X \cup Y \cup Z$ and $\mathrm{FS}(C \setminus \{c\}) \subseteq \mathrm{FS}(C) - c \subseteq f[[B]^2] - c$, we have
$$\begin{aligned}\mathrm{FS}(C \setminus \{c\}) &\subseteq (f[X] - c) \cup (f[Y] - c) \cup ((f[Z] - c) \cap \mathrm{FS}(C \setminus \{c\})) \\ &= (f[X] - c) \cup (f[Y] - c) \cup \emptyset \in \mathcal{H},\end{aligned}$$
a contradiction. □

8. Ramsey ideal is not below Hindman ideal

Theorem 8.1. *$\mathcal{R} \not\leq_K \mathcal{H}$.*

Proof. By Γ we will denote the set $\Gamma = \{(z_0, z_1) \in \omega^2 : z_0 > z_1\}$. In this proof we will view $\mathcal{R}$ as an ideal on Γ consisting of those $A \subseteq \Gamma$ that do not contain any $B^2 \cap \Gamma$, for infinite $B \subseteq \omega$.

By Lemma 2.3, there is a very sparse $X \in [\omega]^\omega$. The ideal $\mathcal{H}$ is homogeneous (see Proposition 2.2), so it suffices to show that $\mathcal{R} \not\leq_K \mathcal{H} \restriction \mathrm{FS}(X)$. Fix any $f : \mathrm{FS}(X) \to \Gamma$ and assume to the contrary that it witnesses $\mathcal{R} \leq_K \mathcal{H}$. There are two possible cases.

Case 1. There are $k \in \omega$ and very sparse $D \in [\omega]^\omega$, $\mathrm{FS}(D) \subseteq \mathrm{FS}(X)$, such that for all $n > k$ and $x \in \mathrm{FS}(D)$ we have: $(f^{-1}[(\omega \times \{n\}) \cap \Gamma] \cap \{y \in FS(D) : \alpha_D(x) \subseteq \alpha_D(y)\}) - x \in \mathcal{H} \restriction \mathrm{FS}(X)$.

In this case we recursively pick $\{x_n : n \in \omega\} \subseteq \mathrm{FS}(D)$ and $\{D_n : n \in \omega \cup \{-1\}\} \subseteq [\omega]^\omega$ such that $D_{-1} = D$ and for all $n \in \omega$ we have:

(a) $x_n \in \mathrm{FS}(D_{n-1}) \setminus \big(\{x_i : i < n\} \cup \bigcup_{i<n} \bigcup_{j<n} \{y \in \mathrm{FS}(D_j) : \alpha_{D_j}(y) \cap \alpha_{D_j}(x_i) \neq \emptyset\}\big)$ (here we put $\alpha_{D_j}(x_i) = \emptyset$ whenever $x_i \notin D_j$);
(b) D_n is very sparse;
(c) $\mathrm{FS}(\{x_0, \dots, x_n\}) \subseteq \mathrm{FS}(D)$;
(d) $\mathrm{FS}(D_n) \subseteq \mathrm{FS}(D_{n-1}) \subseteq \mathrm{FS}(D)$;
(e) $\big(f^{-1}[(\omega \times \{k+i\}) \cap \Gamma] - x\big) \cap \mathrm{FS}(D_n) = \emptyset$ for every $x \in \mathrm{FS}(\{x_0, \dots, x_n\})$ and $1 \leq i \leq n+1$;
(f) $f^{-1}[(\omega \times \{k+i\}) \cap \Gamma] \cap \mathrm{FS}(D_n) = \emptyset$ for all $1 \leq i \leq n+1$.

The initial step of the construction is given by the requirement $D_{-1} = D$. Suppose now that x_i and D_i for all $i < n$ are defined.

Find $x_n \in \mathrm{FS}(D_{n-1})$ such that $\mathrm{FS}(\{x_0, \dots, x_n\}) \subseteq \mathrm{FS}(D)$ and $x_n \neq x_i$ for all $i < n$. This is possible since it suffices to pick any point from the set

$$\mathrm{FS}(D_{n-1}) \setminus \bigcup_{i<n} \bigcup_{j<n} \{y \in \mathrm{FS}(D_j) : \alpha_{D_j}(y) \cap \alpha_{D_j}(x_i) \neq \emptyset\},$$

which is nonempty as $\mathrm{FS}(D_{n-1}) \notin \mathcal{H} \restriction \mathrm{FS}(X)$ and

$$\bigcup_{i<n} \bigcup_{j<n} \{y \in \mathrm{FS}(D_j) : \alpha_{D_j}(y) \cap \alpha_{D_j}(x_i) \neq \emptyset\} \in \mathcal{H} \restriction \mathrm{FS}(X)$$

by Lemma 7.1 and item (b) for all $j < n$ (here we put $\alpha_{D_j}(x_i) = \emptyset$ whenever $x_i \notin D_j$).

Enumerate $\mathrm{FS}(\{x_0, \dots, x_n\}) = \{c_0, c_1, \dots, c_{2^{n+1}-2}\}$. We will define sets $E_t \in [\omega]^\omega$ for $-1 \leq t \leq n$ such that $E_{-1} = D_{n-1}$ and for all $0 \leq t \leq n$:

- $\mathrm{FS}(E_t) \subseteq \mathrm{FS}(E_{t-1}) \subseteq \mathrm{FS}(D_{n-1})$,
- $\Big(\bigcup_{1 \leq i \leq n+1} f^{-1}[(\omega \times \{k+i\}) \cap \Gamma] - c_l\Big) \cap \mathrm{FS}(E_t) = \emptyset$ for every $0 \leq l \leq 2^{n+1} - 2$.

Such construction is possible. Indeed, since we are on Case 1 and each $c_l \in \mathrm{FS}(\{x_0, \dots, x_n\}) \subseteq \mathrm{FS}(D)$, we know that:

$$\left(\bigcup_{1 \leq i \leq n+1} f^{-1}[(\omega \times \{k+i\}) \cap \Gamma] - c_l\right) \cap (\{y \in \mathrm{FS}(D) : \alpha_D(c_l) \subseteq \alpha_D(y)\} - c_l)$$

$$= \left(\bigcup_{1 \leq i \leq n+1} f^{-1}[(\omega \times \{k+i\}) \cap \Gamma] \cap \{y \in \mathrm{FS}(D) : \alpha_D(c_l) \subseteq \alpha_D(y)\}\right) - c_l \in \mathcal{H} \restriction \mathrm{FS}(D).$$

On the other hand, we get:

$$\begin{aligned}&\mathrm{FS}(E_{t-1}) \cap (\{y \in \mathrm{FS}(D) : \alpha_D(c_l) \subseteq \alpha_D(y)\} - c_l)\\ &\supseteq \mathrm{FS}(E_{t-1}) \cap (\mathrm{FS}(D) \setminus \{y \in \mathrm{FS}(D) : \alpha_D(c_l) \cap \alpha_D(y) \neq \emptyset\})\\ &\supseteq \mathrm{FS}(E_{t-1}) \setminus \{y \in \mathrm{FS}(D) : \alpha_D(c_l) \cap \alpha_D(y) \neq \emptyset\} \notin \mathcal{H} \restriction \mathrm{FS}(D),\end{aligned}$$

as $\{y \in \mathrm{FS}(D) : \alpha_D(c_l) \cap \alpha_D(y) \neq \emptyset\} \in \mathcal{H} \restriction \mathrm{FS}(D)$ (by Lemma 7.1). Then

$$\begin{aligned}&\mathrm{FS}(E_{t-1}) \setminus \left(\bigcup_{1\leq i\leq n+1} f^{-1}[(\omega \times \{k+i\}) \cap \Gamma] - c_l\right)\\ &\supseteq (\mathrm{FS}(E_{t-1}) \cap (\{y \in \mathrm{FS}(D) : \alpha_D(c_l) \subseteq \alpha_D(y)\} - c_l))\\ &\setminus \left(\left(\bigcup_{1\leq i\leq n+1} f^{-1}[(\omega \times \{k+i\}) \cap \Gamma] - c_l\right)\right.\\ &\left.\cap\ (\{y \in \mathrm{FS}(D) : \alpha_D(c_l) \subseteq \alpha_D(y)\} - c_l)\right) \notin \mathcal{H} \restriction \mathrm{FS}(D).\end{aligned}$$

Thus, there is $E_t \in [\omega]^\omega$ as needed.

Once all E_t are defined, observe that

$$\bigcup_{1\leq i\leq n+1} (\omega \times \{k+i\}) \cap \Gamma \in \mathcal{R}.$$

Since we assumed that f witnesses $\mathcal{R} \leq_K \mathcal{H}$,

$$\mathrm{FS}(E_n) \setminus \bigcup_{1\leq i\leq n+1} f^{-1}[(\omega \times \{k+i\}) \cap \Gamma] \notin \mathcal{H}.$$

Hence, there is a very sparse $D_n \in [\omega]^\omega$ such that

$$\mathrm{FS}(D_n) \subseteq \mathrm{FS}(E_n) \setminus \bigcup_{1\leq i\leq n+1} f^{-1}[(\omega \times \{k+i\}) \cap \Gamma]$$

(by Lemma 2.3). Note that $\mathrm{FS}(D_n) \subseteq \mathrm{FS}(E_n) \subseteq \mathrm{FS}(D_{n-1}) \subseteq \mathrm{FS}(D)$ and

$$\bigcup_{1\leq i\leq n+1} \left(f^{-1}[(\omega \times \{k+i\}) \cap \Gamma] - x\right) \cap \mathrm{FS}(D_n) = \emptyset$$

for all $x \in \mathrm{FS}(\{x_0, \ldots, x_n\})$.

This finishes the construction of $\{x_n : n \in \omega\} \subseteq \mathrm{FS}(D)$ and $\{D_n : n \in \omega \cup \{-1\}\} \subseteq [\omega]^\omega$.

Define $B = \mathrm{FS}(\{x_n : n \in \omega\})$. Obviously, $B \notin \mathcal{H} \restriction \mathrm{FS}(X)$ as $\mathrm{FS}(\{x_0, \ldots, x_n\}) \subseteq \mathrm{FS}(D) \subseteq \mathrm{FS}(X)$ for all $n \in \omega$. We will show that $f[B] \cap (\omega \times \{n\}) \cap \Gamma$ is finite for all $n > k$. This will finish the proof in this case as $\bigcup_{n\leq k}(\omega \times \{n\}) \cap \Gamma \in \mathcal{R}$ and any set finite on each $(\omega \times \{n\}) \cap \Gamma$ belongs to $\mathcal{R}$.

Assume that $f(x) \in (\omega \times \{k+m+1\}) \cap \Gamma$ for some $m \in \omega$ and $x = x_{n_0} + \ldots + x_{n_t} \in B$, where $n_0 < \ldots < n_t$. If $n_0 > m$, then $x \in \mathrm{FS}(\{x_n : n > m\}) \subseteq \mathrm{FS}(D_m)$ which contradicts $f(x) \in (\omega \times \{k+m+1\}) \cap \Gamma$ (by item (f)). If $n_0 \leq m$ but $J = \{j \leq t : n_j > m\} \neq \emptyset$, then let $j = \min J$ and note that $x \in \sum_{i<j} x_{n_i} + \mathrm{FS}(D_m)$. As $\sum_{i<j} x_{n_i} \in \mathrm{FS}(\{x_0, \ldots, x_m\})$, item (e) gives us a contradiction with $f(x) \in (\omega \times \{k+m+1\}) \cap \Gamma$. Hence, the only possibility is that $n_j \leq m$ for all $j \leq t$. Thus, $f[B] \cap (\omega \times \{k+m+1\}) \cap \Gamma \subseteq f[\mathrm{FS}(\{x_0, \ldots, x_m\})]$, which is a finite set.

Case 2. For every $k \in \omega$ and very sparse $D \in [\omega]^\omega$, $\mathrm{FS}(D) \subseteq \mathrm{FS}(X)$, there are $n > k$ and $x \in \mathrm{FS}(D)$ such that:

$$(f^{-1}[(\omega \times \{n\}) \cap \Gamma] \cap \{y \in \mathrm{FS}(D) : \alpha_D(x) \subseteq \alpha_D(y)\}) - x \notin \mathcal{H} \restriction \mathrm{FS}(X).$$

In this case we will pick $\{n_i : i \in \omega\} \subseteq \omega$, $\{j_i : i \in \omega\} \subseteq \{0, 1\}$, $\{x_i : i \in \omega\} \subseteq \mathrm{FS}(D)$, $\{D_i : i \in \omega \cup \{-1\}\} \subseteq [\omega]^\omega$, $\{k_i : i \in \omega\} \subseteq \omega \cup \{-1\}$ and $\{F_i : i \in \omega\} \subseteq \mathrm{Fin}$ such that $D_{-1} = X$ and for each $i \in \omega$:

(a) (a1) $n_i > n_{i-1}$ (here we put $n_{-1} = -1$);
  (a2) $n_i > \min\{a \in \omega : f[\mathrm{FS}(\{x_j : j < i\})] \subseteq \{0, 1, \ldots, a\}^2 \cap \Gamma\}$;
(b) (b1) $\mathrm{FS}(D_i) \subseteq \mathrm{FS}(D_{i-1}) \subseteq \mathrm{FS}(X)$;
  (b2) D_i is very sparse;
(c) if $j_i = 0$, then:
  (c1) $k_i = -1$;
  (c2) $F_i = \emptyset$;
  (c3) $x_i \in \mathrm{FS}(D_{i-1}) \cap f^{-1}[(\omega \times \{n_i\}) \cap \Gamma]$;
  (c4) $x_i + \mathrm{FS}(D_i) \subseteq f^{-1}[(\omega \times \{n_i\}) \cap \Gamma]$;
(d) if $j_i = 1$, then:
  (d1) $k_i \in \{0 \leq u < i : u \notin \bigcup_{q<i} F_q, j_u = 0\}$;
  (d2) $F_i = \{k_i, k_i + 1, \ldots, i - 1\}$;
  (d3) (d3a) $x_i \in f^{-1}[\{(n_i, n_{k_i})\}]$;
    (d3b) $x_i \in x_{k_i} + (\{0\} \cup \mathrm{FS}(\{x_r : k_i < r < i, r \notin \bigcup_{q<i} F_q\})) + \mathrm{FS}(D_{i-1})$;
  (d4) $x_i + \mathrm{FS}(D_i) \subseteq f^{-1}[\{(n_i, n_{k_i})\}]$;
(e) if $x = \sum_{b \leq a} x_{t_b}$ for some $0 \leq t_0 < \ldots < t_a < i$, $t_b \notin \bigcup_{q \leq i} F_q$ (so $x \in \mathrm{FS}(\{x_t : t < i,\ t \notin \bigcup_{q \leq i} F_q\})$), then:
  (e1) $(x + x_i + \mathrm{FS}(D_i)) \cap f^{-1}[\{(n_i, n_{t_0})\}] = \emptyset$;
  (e2) $(x + \mathrm{FS}(D_i)) \cap f^{-1}[\{(n_i, n_{t_0})\}] = \emptyset$;
  (e3) $(x + x_i) \cap f^{-1}[\{(n_i, n_{t_0})\}] = \emptyset$;
(f) $\mathrm{FS}(D_i) \cap \{y \in \mathrm{FS}(D_t) : \alpha_{D_t}(y) \cap \alpha_{D_t}(x_u) \neq \emptyset\} = \emptyset$ for all $-1 \leq t < i$ and $0 \leq u \leq i$ such that $x_u \in \mathrm{FS}(D_t)$;
(g) (g1) $\mathrm{FS}(\{x_t : t \leq i, t \notin \bigcup_{q \leq i} F_q\}) \subseteq \mathrm{FS}(X)$;
  (g2) $\sum_{b \leq a} x_{t_b} \in x_{t_0} + \mathrm{FS}(D_{t_0})$ for every $a > 0$, $0 \leq t_0 < \ldots < t_a \leq i$, $t_b \notin \bigcup_{q \leq i} F_q$;

At first step, since we are in Case 2, for $k = 0$ and $D = X$ there are $n_0 > k$ (note that (a) is satisfied) and $x'_0 \in \mathrm{FS}(X)$ such that: $(f^{-1}[(\omega \times \{n_0\}) \cap \Gamma] \cap \{y \in \mathrm{FS}(X) : \alpha_X(x'_0) \subseteq \alpha_X(y)\}) - x'_0 \notin \mathcal{H} \restriction \mathrm{FS}(X)$. Hence, there is $D'_0 \in [\omega]^\omega$ such that: $x'_0 + \mathrm{FS}(D'_0) \subseteq f^{-1}[(\omega \times \{n_0\}) \cap \Gamma] \cap \{y \in \mathrm{FS}(X) : \alpha_X(x'_0) \subseteq \alpha_X(y)\} \subseteq \mathrm{FS}(X)$. Put $j_0 = 0$, $k_0 = -1$ and $F_0 = \emptyset$ (note that (c1) and (c2) are satisfied). Moreover, define $x_0 = x'_0 + \min(D'_0)$ (note that (c3) and (g1) are satisfied, because $x_0 \in x'_0 + \mathrm{FS}(D'_0) \subseteq \mathrm{FS}(X)$ and $x'_0 + \mathrm{FS}(D'_0) \subseteq f^{-1}[(\omega \times \{n_0\}) \cap \Gamma]$) and using Lemma 2.3 find a very sparse $D_0 \in [\omega]^\omega$ such that

$$\mathrm{FS}(D_0) \subseteq \mathrm{FS}(D'_0 \setminus \{\min(D'_0)\}) \setminus \{y \in \mathrm{FS}(X) : \alpha_X(y) \cap \alpha_X(x_0) \neq \emptyset\},$$

which is possible as $\{y \in \mathrm{FS}(X) : \alpha_X(y) \cap \alpha_X(x_0) \neq \emptyset\} \in \mathcal{H} \restriction \mathrm{FS}(X)$ by Lemma 7.1 (note that (c4), (f) and (b) are satisfied, because $x_0 + \mathrm{FS}(D_0) \subseteq x'_0 + \mathrm{FS}(D'_0) \subseteq f^{-1}[(\omega \times \{n_0\}) \cap \Gamma]$ and $\mathrm{FS}(D_0) \subseteq \mathrm{FS}(D'_0) \subseteq \{y \in \mathrm{FS}(X) : \alpha_X(x'_0) \subseteq \alpha_X(y)\} - x'_0 \subseteq \mathrm{FS}(X)$). In conditions (e) and (g2) there is nothing to check. Thus, all the requirements are met.

At ith step, where $i > 0$, since we are in Case 2, if $k = \max\{n_{i-1}, \min\{a \in \omega : f[\mathrm{FS}(\{x_j : j < i\})] \subseteq \{0, 1, \ldots, a\}^2 \cap \Gamma\}\}$ and $D = D_{i-1}$, then there are $n_i > k$ (so (a) is satisfied) and $x'_i \in \mathrm{FS}(D_{i-1})$ such that

$$\big(f^{-1}[(\omega \times \{n_i\}) \cap \Gamma] \cap \{y \in \mathrm{FS}(D_{i-1}) : \alpha_{D_{i-1}}(x'_i) \subseteq \alpha_{D_{i-1}}(y)\}\big) - x'_i \notin \mathcal{H} \restriction \mathrm{FS}(X).$$

Hence, there is $D'_i \in [\omega]^\omega$ such that: $x'_i + \mathrm{FS}(D'_i) \subseteq f^{-1}[(\omega \times \{n_i\}) \cap \Gamma] \cap \{y \in \mathrm{FS}(D_{i-1}) : \alpha_{D_{i-1}}(x'_i) \subseteq \alpha_{D_{i-1}}(y)\}$. In particular, $\mathrm{FS}(D'_i) \subseteq \{y \in \mathrm{FS}(D_{i-1}) : \alpha_{D_{i-1}}(x'_i) \subseteq \alpha_{D_{i-1}}(y)\} - x'_i = \{y \in \mathrm{FS}(D_{i-1}) : \alpha_{D_{i-1}}(x'_i) \cap \alpha_{D_{i-1}}(y) = \emptyset\} \subseteq \mathrm{FS}(D_{i-1})$. There are two possibilities.

Assume first that there is $x = \sum_{b\leq a} x_{t_b}$ for some $t_0 < \ldots < t_a < i$, $t_b \notin \bigcup_{q<i} F_q$ such that either $x+x'_i+\mathrm{FS}(\bar{D}_i) \subseteq f^{-1}[\{(n_i, n_{t_0})\}]$ or $x+\mathrm{FS}(\bar{D}_i) \subseteq f^{-1}[\{(n_i, n_{t_0})\}]$ for some $\bar{D}_i \in [\omega]^\omega$ such that $\mathrm{FS}(\bar{D}_i) \subseteq \mathrm{FS}(D'_i)$. Define $j_i = 1$ and let k_i be minimal such that there is (one or more) x as above with $k_i = t_0$.

Notice that $k_i \in \{0 \leq u < i : u \notin \bigcup_{q<i} F_q\}$. We will show that $j_{k_i} = 0$ (i.e., (d1) is satisfied). Suppose that $j_{k_i} = j_{t_0} = 1$. Observe that $x'_i + \mathrm{FS}(\bar{D}_i) \subseteq \mathrm{FS}(D_{i-1})$ (by $\mathrm{FS}(\bar{D}_i) \cap \{y \in \mathrm{FS}(D_{i-1}) : \alpha_{D_{i-1}}(y) \cap \alpha_{D_{i-1}}(x'_i) \neq \emptyset\} = \emptyset$) and consequently $x'_i + \mathrm{FS}(\bar{D}_i) \subseteq \mathrm{FS}(D_{t_0})$ (by item (b1)). Then items (b1), (f) and (g2) give us:

- $x + x'_i + \mathrm{FS}(\bar{D}_i) \subseteq x_{t_0} + \mathrm{FS}(D_{t_0})$,
- $x + \mathrm{FS}(\bar{D}_i) \subseteq x_{t_0} + \mathrm{FS}(D_{t_0})$.

Then from (d4) we have:

- $f[x + x'_i + \mathrm{FS}(\bar{D}_i)] \subseteq \{(n_{t_0}, n_{k_{t_0}})\}$,
- $f[x + \mathrm{FS}(\bar{D}_i)] \subseteq \{(n_{t_0}, n_{k_{t_0}})\}$.

This contradicts $x+x'_i+\mathrm{FS}(\bar{D}_i) \subseteq f^{-1}[\{(n_i, n_{t_0})\}]$ or $x+\mathrm{FS}(\bar{D}_i) \subseteq f^{-1}[\{(n_i, n_{t_0})\}]$, because $n_{t_0} < n_i$ (by $t_0 < i$ and item (a1)).

Define $F_i = \{k_i, k_i+1, \ldots, i-1\}$ (so (d2) is satisfied) and $\bar{x}_i = x + x'_i$ (or $\bar{x}_i = x$ if $x + \mathrm{FS}(\bar{D}_i) \subseteq f^{-1}[\{(n_i, n_{t_0})\}]$).

To define D_i and x_i, note that by the choice of k_i, for each $y = \sum_{b\leq a} x_{t_b}$, $t_0 < \ldots < t_a < i$, $t_b \notin \bigcup_{q\leq i} F_q$ (so in fact $t_a < k_i$) we know that $(y+\bar{x}_i+\mathrm{FS}(E)) \not\subseteq f^{-1}[\{(n_i, n_{t_0})\}]$ and $(y + \mathrm{FS}(E)) \not\subseteq f^{-1}[\{(n_i, n_{t_0})\}]$ for every $E \in [\omega]^\omega$ such that $\mathrm{FS}(E) \subseteq \mathrm{FS}(D'_i)$. In other words, $f^{-1}[\{(n_i, n_{t_0})\}] - (y + \bar{x}_i) \in \mathcal{H} \restriction \mathrm{FS}(D'_i)$ and $f^{-1}[\{(n_i, n_{t_0})\}] - y \in \mathcal{H} \restriction \mathrm{FS}(D'_i)$, for every such y. Thus, we can find $\tilde{D}_i \in [\omega]^\omega$ such that:

- $\mathrm{FS}(\tilde{D}_i) \subseteq \mathrm{FS}(\bar{D}_i)$;
- $(y+\bar{x}_i+\mathrm{FS}(\tilde{D}_i))\cap f^{-1}[\{(n_i, n_{t_0})\}] = \emptyset$ and $(y+\mathrm{FS}(\tilde{D}_i))\cap f^{-1}[\{(n_i, n_{t_0})\}] = \emptyset$ for every $y = \sum_{b\leq a} x_{t_b}$, where $t_0 < \ldots < t_a < i$, $t_b \notin \bigcup_{q\leq i} F_q$;
- $\mathrm{FS}(\tilde{D}_i) \cap \{y \in \mathrm{FS}(D_{i-1}) : \alpha_{D_{i-1}}(y) \cap \alpha_{D_{i-1}}(x'_i) \neq \emptyset\} = \emptyset$;

(the last item is trivial, as $\mathrm{FS}(\tilde{D}_i) \subseteq \mathrm{FS}(\bar{D}_i) \subseteq \mathrm{FS}(D'_i)$ and $\mathrm{FS}(D'_i) \subseteq \{y \in \mathrm{FS}(D_{i-1}) : \alpha_{D_{i-1}}(x'_i) \cap \alpha_{D_{i-1}}(y) = \emptyset\}$).

Define $x_i = \bar{x}_i + \min(\tilde{D}_i)$ and let $D_i \in [\omega]^\omega$ be very sparse such that $\mathrm{FS}(D_i) \subseteq \mathrm{FS}(\tilde{D}_i \setminus \{\min(\tilde{D}_i)\})$ and D_i satisfies item (f). It is possible using Lemma 2.3, as $\{y \in \mathrm{FS}(D_t) : \alpha_{D_t}(y) \cap \alpha_{D_t}(x_u) \neq \emptyset\} \in \mathcal{H} \restriction \mathrm{FS}(X)$ by Lemma 7.1 and item (b2) for all $-1 \leq t < i$. Then (b2) is satisfied. Observe that other conditions are met:

- (b1) $\mathrm{FS}(D_i) \subseteq \mathrm{FS}(\tilde{D}_i) \subseteq \mathrm{FS}(\bar{D}_i) \subseteq \mathrm{FS}(D'_i) \subseteq \mathrm{FS}(D_{i-1}) \subseteq \mathrm{FS}(X)$;
- (d3b) if $\bar{x}_i = x + x'_i$, then $x_i = \bar{x}_i + \min(\tilde{D}_i) = x_{k_i} + (x - x_{k_i}) + x'_i + \min(\tilde{D}_i) \in x_{k_i} + (\{0\} \cup \mathrm{FS}(\{x_r : k_i < r < i, r \notin \bigcup_{q<i} F_q\})) + \mathrm{FS}(D_{i-1})$ by the fact that $\mathrm{FS}(\tilde{D}_i) \cap \{y \in \mathrm{FS}(D_{i-1}) : \alpha_{D_{i-1}}(y) \cap \alpha_{D_{i-1}}(x'_i) \neq \emptyset\} = \emptyset$ (if $\bar{x}_i = x$ this is even easier to show);
- (d3a) if $\bar{x}_i = x + x'_i$, then $x_i \in x + x'_i + \mathrm{FS}(\bar{D}_i) \subseteq f^{-1}[\{(n_i, n_{k_i})\}]$ (if $\bar{x}_i = x$ this is also true);
- (d4) if $\bar{x}_i = x + x'_i$, then $x_i + \mathrm{FS}(D_i) \subseteq x + x'_i + \mathrm{FS}(\bar{D}_i) \subseteq f^{-1}[\{(n_i, n_{k_i})\}]$ (if $\bar{x}_i = x$ this is also true);
- (e) for (e3), if $y = \sum_{b\leq a} x_{t_b}$, $t_0 < \ldots < t_a < i$, $t_b \notin \bigcup_{q\leq i} F_q$, then note that $y + x_i = y + \bar{x}_i + \min(\tilde{D}_i) \in y + \bar{x}_i + \mathrm{FS}(\tilde{D}_i)$ and recall that $(y + \bar{x}_i + \mathrm{FS}(\tilde{D}_i))\cap f^{-1}[\{(n_i, n_{t_0})\}] = \emptyset$, thus $y+x_i \notin f^{-1}[\{(n_i, n_{t_0})\}]$ ((e1) and (e2) are similar);
- (g1) $\mathrm{FS}(\{x_t : t \leq i, t \notin \bigcup_{q\leq i} F_q\}) \subseteq \mathrm{FS}(\{x_t : t < i, t \notin \bigcup_{q<i} F_q\})\cup(x_i+\mathrm{FS}(\{x_t : t < i, t \notin \bigcup_{q<i} F_q\}) \subseteq \mathrm{FS}(X)$ by items (f) and (g1) applied to $i-1$ and item (d3b) applied to i;

(g2) if $a > 0$, $t_0 < \ldots < t_a \leq i$, $t_b \notin \bigcup_{q\leq i} F_q$ then either $t_a < i$ and $\sum_{b\leq a} x_{t_b} \in x_{t_0} + \mathrm{FS}(D_{t_0})$ (by (g2) applied to $i-1$) or $t_a = i$ and $\sum_{b\leq a} x_{t_b} = \sum_{b<a} x_{t_b} + x_i \in x_{t_0} + (\{0\} \cup \mathrm{FS}(\{x_r : t_0 < r < k_i, r \notin \bigcup_{q<i} F_q\})) + x_{k_i} + (\{0\} \cup \mathrm{FS}(\{x_r : k_i < r < i, r \notin \bigcup_{q<i} F_q\})) + \mathrm{FS}(D_{i-1}) \subseteq x_{t_0} + \mathrm{FS}(D_{t_0})$ by items (b1), (d3b), (f) and (g2) for $i-1$.

Hence, all the requirements are met. This finishes the case of $j_i = 1$.

Assume now that for all $x = \sum_{b\leq a} x_{t_b}$, $t_0 < \ldots < t_a < i$, $t_b \notin \bigcup_{q<i} F_q$ we have $x + x'_i + \mathrm{FS}(E) \not\subseteq f^{-1}[\{(n_i, n_{t_0})\}]$ and $x + \mathrm{FS}(E) \not\subseteq f^{-1}[\{(n_i, n_{t_0})\}]$ for all $E \in [\omega]^\omega$ such that $\mathrm{FS}(E) \subseteq \mathrm{FS}(D'_i)$. Put $j_i = 0$, $k_i = -1$ and $F_i = \emptyset$ (note that (c1) and (c2) are satisfied).

Similarly as above (in the construction of $\tilde{D}_i$), we can find $\tilde{D}_i \in [\omega]^\omega$ such that:

- $\mathrm{FS}(\tilde{D}_i) \subseteq \mathrm{FS}(D'_i)$;
- $(x + x'_i + \mathrm{FS}(\tilde{D}_i)) \cap f^{-1}[\{(n_i, n_{t_0})\}] = \emptyset$ and $(x + \mathrm{FS}(\tilde{D}_i)) \cap f^{-1}[\{(n_i, n_{t_0})\}] = \emptyset$ for every $x = \sum_{b\leq a} x_{t_b}$, $t_0 < \ldots < t_a < i$, $t_b \notin \bigcup_{q\leq i} F_q$;
- $\mathrm{FS}(\tilde{D}_i) \cap \{y \in \mathrm{FS}(D_{i-1}) : \alpha_{D_{i-1}}(y) \cap \alpha_{D_{i-1}}(x'_i) \neq \emptyset\} = \emptyset$.

Define $x_i = x'_i + \min(\tilde{D}_i)$ and let $D_i \in [\omega]^\omega$ be very sparse such that $\mathrm{FS}(D_i) \subseteq \mathrm{FS}(\tilde{D}_i \setminus \{\min(\tilde{D}_i)\})$ and D_i satisfies item (f) (which is possible by Lemmas 2.3 and 7.1 and (b2) applied to all $-1 \leq t < i$). Note that (b2) is satisfied. Observe that other conditions are met:

(b1) $\mathrm{FS}(D_i) \subseteq \mathrm{FS}(\tilde{D}_i) \subseteq \mathrm{FS}(D'_i) \subseteq \mathrm{FS}(D_{i-1}) \subseteq \mathrm{FS}(X)$;

(c3) $x_i \in \mathrm{FS}(D_{i-1})$ as $\mathrm{FS}(\tilde{D}_i) \cap \{y \in \mathrm{FS}(D_{i-1}) : \alpha_{D_{i-1}}(y) \cap \alpha_{D_{i-1}}(x'_i) \neq \emptyset\} = \emptyset$, $x_i \in x'_i + \mathrm{FS}(D'_i) \subseteq f^{-1}[(\omega \times \{n_i\}) \cap \Gamma]$;

(c4) $x_i + \mathrm{FS}(D_i) \subseteq x'_i + \mathrm{FS}(D'_i) \subseteq f^{-1}[(\omega \times \{n_i\}) \cap \Gamma]$;

(e) for (e3), if $x = \sum_{b\leq a} x_{t_b}$, $t_0 < \ldots < t_a < i$, $t_b \notin \bigcup_{q\leq i} F_q$, then note that $x + x_i = x + x'_i + \min(\tilde{D}_i) \in x + x'_i + \mathrm{FS}(\tilde{D}_i)$ and recall that $(x + x'_i + \mathrm{FS}(\tilde{D}_i)) \cap f^{-1}[\{(n_i, n_{t_0})\}] = \emptyset$, thus $x + x_i \notin f^{-1}[\{(n_i, n_{t_0})\}]$ ((e1) and (e2) are similar);

(g1) by items (f) and (g1) applied to $i-1$ and item (c3) applied to i, $\mathrm{FS}(\{x_t : t \leq i, t \notin \bigcup_{q\leq i} F_q\}) = \mathrm{FS}(\{x_t : t < i, t \notin \bigcup_{q<i} F_q\}) \cup (x_i + \mathrm{FS}(\{x_t : t < i, t \notin \bigcup_{q<i} F_q\})) \subseteq \mathrm{FS}(X)$;

(g2) if $a > 0$, $t_0 < \ldots < t_a \leq i$ and $t_b \notin \bigcup_{q\leq i} F_q$, then either $t_a < i$ and $\sum_{b\leq a} x_{t_b} \in x_{t_0} + \mathrm{FS}(D_{t_0})$ (by item (g2) applied to $i-1$) or $t_a = i$ and $\sum_{b\leq a} x_{t_b} \in x_{t_0} + \mathrm{FS}(D_{t_0})$ as $\sum_{b<a} x_{t_b} \in x_{t_0} + \mathrm{FS}(D_{t_0})$ and $x_i \in \mathrm{FS}(D_{i-1}) \subseteq \mathrm{FS}(D_{t_0}) \setminus \{y \in \mathrm{FS}(D_{t_0}) : \exists_{j<i}\ \alpha_{D_{t_0}}(y) \cap \alpha_{D_{t_0}}(x_j) \neq \emptyset\}$ by item (c3) for i and items (b1), (f) and (g2) for $i-1$.

Hence, all the requirements are met. This finishes the case of $j_i = 0$.

Note that $x_i \neq x_j$ for $i \neq j$ (it follows from items (a2), (c3) and (d3a)). Once the whole recursive construction is completed, define $A = \{x_i : i \notin \bigcup_{q\in\omega} F_q\}$. We need to show two facts:

(i) A is infinite;

(ii) $f[\mathrm{FS}(A)] \in \mathcal{R}$.

Note that this will finish the proof as item (i) together with $\mathrm{FS}(A) \subseteq \mathrm{FS}(X)$ (by item (g1)) guarantee that $\mathrm{FS}(A) \notin \mathcal{H} \restriction \mathrm{FS}(X)$.

(i): Since $x_i \neq x_j$ for $i \neq j$, we only need to show that there are infinitely many $t \in \omega$ such that $x_t \in A$. Assume to the contrary that there is $p \in \omega$ such that $x_t \notin A$ for all $t \geq p$. Without loss of generality we may assume that p is minimal with that property. Since $x_p \notin A$, we have that $p \in \bigcup_{q\in\omega} F_q$, hence there is q such that $p \in F_q$. By items (c2) and (d2), we know that $j_q = 1$, $q > p$ and, by minimality of p, $F_q = \{p, p+1, \ldots q-1\}$. Again, as $x_q \notin A$ (because $q > p$), there should be r such that $q \in F_r = \{k_r, k_r+1, \ldots, r-1\}$ (so $k_r \leq q < r$) and $k_r \geq p$ (by minimality of p). However, this is impossible as item (d1) gives us:

$$k_r \in \{u < r : u \notin \bigcup_{w<r} F_w, j_u = 0\} \cap \{0,1,\ldots,q\} \subseteq$$

$$\subseteq \{u \leq q : u \notin F_q\} \cap \{u \leq q : j_u = 0\} =$$

$$= (\{0,1,\ldots,p-1\} \cup \{q\}) \cap \{u \leq q : j_u = 0\} \subseteq \{0,1,\ldots,p-1\}$$

(here, if $p=0$ then $\{0,1,\ldots,p-1\} = \emptyset$).

(ii): We have:

$$\mathrm{FS}(A) = \bigcup_{i\in B} (\{x_i\} \cup (x_i + \mathrm{FS}(A \setminus \{0,1,\ldots,x_i\})))$$
$$\cup \bigcup_{i\in C} (\{x_i\} \cup (x_i + \mathrm{FS}(A \setminus \{0,1,\ldots,x_i\}))),$$

where $B = \{i \in \omega : i \notin \bigcup_{q\in\omega} F_q, j_i = 0\}$ and $C = \{i \in \omega : i \notin \bigcup_{q\in\omega} F_q, j_i = 1\}$.

At first we will show that $f[\bigcup_{i\in C}(\{x_i\} \cup (x_i + \mathrm{FS}(A \setminus \{0,1,\ldots,x_i\})))] \in \mathcal{R}$. Note that $f(x_i) = (n_i, n_{k_i})$ (by item (d3a)) and $f[x_i + \mathrm{FS}(A \setminus \{0,1,\ldots,x_i\})] \subseteq f[x_i + \mathrm{FS}(D_i)] = \{(n_i, n_{k_i})\}$ for each $i \in C$ (by items (d4) and (g2)). Moreover, the sequence $(n_i)_{i\in\omega}$ is injective (by item (a1)). Hence, $f[\bigcup_{i\in C}(\{x_i\} \cup (x_i + \mathrm{FS}(A \setminus \{0,1,\ldots,x_i\})))] \in \mathcal{R}$, as any set intersecting each $(\{n\} \times \omega) \cap \Gamma$ on at most one point belongs to $\mathcal{R}$.

Now we will show that $f[\bigcup_{i\in B}(\{x_i\} \cup (x_i + \mathrm{FS}(A \setminus \{0,1,\ldots,x_i\})))] \in \mathcal{R}$. By items (c3), (c4) and (g2), $f[\{x_i\} \cup (x_i + \mathrm{FS}(A \setminus \{0,1,\ldots,x_i\}))] \subseteq f[\{x_i\} \cup (x_i + \mathrm{FS}(D_i))] \subseteq (\omega \times \{n_i\}) \cap \Gamma$ for all $i \in B$. Note that $\bigcup_{i\in B} f[\{x_i\}] \in \mathcal{R}$ (from (a1), as each set intersecting each $(\omega \times \{n\}) \cap \Gamma$ on at most one point belongs to $\mathcal{R}$). Suppose that $Z^2 \cap \Gamma \subseteq \bigcup_{i\in B} f[x_i + \mathrm{FS}(A \setminus \{0,1,\ldots,x_i\})]$ for some $Z \in [\omega]^\omega$.

Firstly, we will show that $|Z \setminus \{n_i : i \in B\}| \leq 1$. Suppose that there are $z, w \in Z \setminus \{n_i : i \in B\}$ such that $z > w$. Then there is $i \in B$ such that $(z,w) \in f[x_i + \mathrm{FS}(A \setminus \{0,1,\ldots,x_i\})]$, hence $x_i + \mathrm{FS}(A \setminus \{0,1,\ldots,x_i\}) \subseteq f^{-1}[\{(z,w)\}]$. But by (c4) and (g2) we have $x_i + \mathrm{FS}(A \setminus \{0,1,\ldots,x_i\}) \subseteq x_i + \mathrm{FS}(D_i) \subseteq f^{-1}[(\omega \times \{n_i\}) \cap \Gamma]$. So $(z,w) \in (\omega \times \{n_i\}) \cap \Gamma$, i.e., $w = n_i$. A contradiction.

By the previous paragraph, since Z is infinite, there are $i, j \in B$ such that $j < i$ and $n_i, n_j \in Z$. We will show that $(n_i, n_j) \notin \bigcup_{k\in B} f[x_k + \mathrm{FS}(A \setminus \{0,1,\ldots,x_k\})]$. This will contradict $Z^2 \cap \Gamma \subseteq \bigcup_{k\in B} f[x_k + \mathrm{FS}(A \setminus \{0,1,\ldots,x_k\})]$ and finish the proof.

Suppose that $(n_i, n_j) \in \bigcup_{k\in B} f[x_k + \mathrm{FS}(A \setminus \{0,1,\ldots,x_k\})]$. From (c4) and (g2), for every $k \neq j$, $k \in B$ we have $f[x_k + \mathrm{FS}(A \setminus \{0,1,\ldots,x_k\})] \subseteq f[x_k + \mathrm{FS}(D_k)] \subseteq (\omega \times \{n_k\}) \cap \Gamma$, so $(n_i, n_j) \notin f[x_k + \mathrm{FS}(A \setminus \{0,1,\ldots,x_k\})]$. Hence, $(n_i, n_j) \in f[x_j + \mathrm{FS}(A \setminus \{0,1,\ldots,x_j\})]$. Let $y \in x_j + \mathrm{FS}(A \setminus \{0,1,\ldots,x_j\})$ be such that $f(y) = (n_i, n_j)$. Then $y = x_j + x_{s_0} + \ldots + x_{s_p}$ for some $j < s_0 < \ldots < s_p$. We have five cases:

- If $s_p < i$, then from item (a2) we have $f(y) \in [\{0,\ldots,n_i-1\}]^2$. A contradiction.
- If $s_p = i$, then $y = (x_j + \ldots + x_{s_{p-1}}) + x_i$ and from item (e3) (applied to $x = (x_j + \ldots + x_{s_{p-1}})$) we get $f(y) \neq (n_i, n_j)$, a contradiction.
- If there exists $k < p$ such that $s_k = i$, then $y = (x_j + \ldots + x_{s_{k-1}}) + x_i + (x_{s_{k+1}} + \ldots + x_{s_p}) \in (x_j + \ldots + x_{s_{k-1}}) + x_i + \mathrm{FS}(D_i)$ by item (g2) and from item (e1) (applied to $x = (x_j + \ldots + x_{s_{k-1}})$) we get a contradiction.
- If there exists $k \leq p$ such that $s_{k-1} < i < s_k$, then $y = (x_j + \ldots + x_{s_{k-1}}) + (x_{s_k} + \ldots + x_{s_p}) \in (x_j + \ldots + x_{s_{k-1}}) + \mathrm{FS}(D_i)$ by item (g2) and from item (e2) we get a contradiction.

- If $i < s_0$, then $y = x_j + (x_{s_0} + \ldots + x_{s_p}) \in x_j + \mathrm{FS}(D_i)$ by item (g2) and from item (e2) we get a contradiction.

Thus, $f[\mathrm{FS}(A)] \in \mathcal{R}$ and the proof is finished. $\square$

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