Differentiation in Banach Spaces.

Problems to solve.

PWP Interdysciplinary Doctoral Studies in Mathematical Modeling
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Problem 1. Check if the functional $f : \mathbb{R}^2 \to \mathbb{R}$ is differentiable at $(0,0)$ (check both Frechét and Gâteaux differentiability)

a) $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases};$

b) $f(x,y) = \begin{cases} \frac{(xy)^2}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}.$

Problem 2. Let E be the Hilbert space. Prove that the map $f : E \to \mathbb{R}$ given by $f(x) = \|\cdot\|^2$ is Frechét differentiable. Find its derivative.

Problem 3. Find the Fréchet derivative of $A : C[0,1] \to C[0,1]$ given by

$$A(u)(t) = \int_0^t u(s)ds.$$

Problem 4. Prove that the norm $\|\cdot\|$ in Hilbert space E is Fréchet differentiable at any point $x_0 \neq 0$. Find the derivative.

Problem 5. Prove that the norm $\|\cdot\|$ in l^1 is not Gâteaux differentiable at any point.

Problem 6. Find the derivative of the functional $I : C[a,b] \to \mathbb{R}$ given by

$$f(u) = \int_a^b \varphi(t)u^2(t)dt,$$

where $\varphi : [a,b] \to \mathbb{R}$ is a continuous function.

Problem 7. Let $U \subset \mathbb{R}^k$ be the open and bounded subset of the Euclidean space and the functional $f : C(\overline{U}) \to \mathbb{R}$ is given by

$$f(u) = \int_U |u(x)|dx.$$

Show that f is Gâteaux differentiable at u_0 if and only if $u_0(x) = 0$ for almost every $x \in U$.

Problem 8. Let $\phi : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a class C^2 function. Find the Fréchet derivative of the map $\Phi : C[a,b] \to C[a,b]$ given by

$$\Phi(u)(t) = \phi(t, u(t)).$$

Problem 9. Let $\phi : \mathbb{R} \to \mathbb{R}$ is the function of the class C^1. Prove that $\Phi : C[a,b] \to C[a,b]$ given by

$$\Phi(u)(t) = \phi(u(t))$$

Gâteaux differentiable. Find the Gâteaux derivative.