Differentiation in Banach Spaces.

Sample problems.

PWP Interdysciplinary Doctoral Studies in Mathematical Modeling
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Problem 1. Check if the functional $f : \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = \begin{cases} \frac{x_1^3 x_2}{x_1^6 + x_2^2} & \text{for } (x_1, x_2) \neq (0,0) \\ 0 & \text{for } (x_1, x_2) = (0,0), \end{cases}$$

is differentiable at $(0, 0)$.

Solution: Let us take any vector $v = (v_1, v_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$ and consider the function $\varphi_v : \mathbb{R} \to \mathbb{R}$ given by

$$\varphi_v(t) = f(tv_1, tv_2) = \begin{cases} \frac{t^4 v_1^3 v_2}{t^6 v_1^6 + t^2 v_2^2} & \text{for } t \neq 0 \\ 0 & \text{for } t = 0. \end{cases}$$

If $v_2 \neq 0$ there is $\varphi_v(t) = t^2 \frac{v_1^3 v_2}{t^4 v_1^6 + v_2^2}$. If $v_2 = 0$, then φ_v is just constantly equal 0. In both situations we have $\varphi_v'(0) = 0$ implying that the Gâteaux derivative exists and equals 0.

On the other hand the function is not continuous: let us observe that $f(\frac{1}{n}, \frac{1}{n^3}) = \frac{1}{2} \not\to 0 = f(0,0)$. So the function is not Fréchet differentiable at $(0, 0)$.

Problem 2. Let E be Hilbert space and $B : E \to E$ be the bounded linear map. Let us find the Fréchet derivative of the functional $f : E \to \mathbb{R}$ given by $f(x) = \langle x, Bx \rangle$.

Solution: Let us consider

$$f(x+h) - f(x) = \langle h, Bx \rangle + \langle x, Bh \rangle + \langle h, Bh \rangle.$$

Then assuming $Df(x)(h) = \langle h, Bx \rangle + \langle x, Bh \rangle$, we can see that

$$\left(f(x+h) - f(x) - DF(x)h\right)/\|h\| = \langle h, Bh \rangle/\|h\| = \langle h, B(h/\|h\|) \rangle \to 0,$$

as $h \to 0$.

Problem 3. Let $I : C[0,1] \to \mathbb{R}$ be given by

$$I(u) = \int_0^1 (s + u(s))^2 ds.$$

Find the derivative of the functional I.

Solution: Let us start with the Gâteaux derivative: let us fix the function $u \in C[0,1]$ and direction $v \in C[0,1]$. We are now going to check if there exists the limit

$$\lim_{t \to 0} \frac{I(u + tv) - I(u)}{t} = \lim_{t \to 0} \frac{\int_0^1 (s + u(s) + th(s))^2 - (s + u(s))^2 ds}{t}.$$

We can see that

$$\lim_{t \to 0} \frac{\int_0^1 (s + u(s) + th(s))^2 - (s + u(s))^2 ds}{t} = \lim_{t \to 0} \int_0^1 \frac{2t(s + u(s))h(s) + t^2 h^2(s)}{t} ds$$

$$= \lim_{t \to 0} \int_0^1 2(s + u(s))h(s) + th^2(s) ds =$$

$$= \int_0^1 \lim_{t \to 0} 2(s + u(s))h(s) + th^2(s) ds = \int_0^1 2(s + u(s))h(s) ds$$

by the Dominated Convergence Theorem. As we can see this was relatively easy calculation.

Let us now check how it works with the Frechét derivate. We know that if the Frechét derivative exists it equals to

$$DI(u)(h) = \int_0^1 2(s + u(s))h(s) ds.$$

Hence we have to check is

$$\lim_{\|h\| \to 0} \frac{\| \int_0^1 (s + u(s) + th(s))^2 ds - \int_0^1 (s + u(s))^2 ds - \int_0^1 2(s + u(s))h(s) ds \|}{\|h\|} = 0,$$

where $\|f\| = \sup_{t \in [0,1]} = |f(t)|$ denotes the supremum norm in $C[0,1]$.

We can see that

$$\int_0^1 (s + u(s) + h(s))^2 ds - \int_0^1 (s + u(s))^2 ds - \int_0^1 2(s + u(s))h(s) ds = \int_0^1 h^2(s) ds$$

hence

$$\frac{\| \int_0^1 (s + u(s) + th(s))^2 ds - \int_0^1 (s + u(s))^2 ds - \int_0^1 2(s + u(s))h(s) ds \|}{\|h\|} = \frac{\| \int_0^1 h^2(s) ds \|}{\|h\|} \leq$$

$$\leq \frac{\|h\|^2\|}{\|h\|} = \|h\| \to 0.$$

This completes the proof.

Problem 4. Find the derivative of the map $F : C[0,1] \to C[0,1]$ given by

$$F(u)(t) = u^3(t).$$

Solution: Let us first review the difference

$$(u(t) + h(t))^3 - u^3(t) = 3u^2(t)h(t) + 3u(t)h^2(t) + h^3(t).$$

We can see that the linear term equals to $3u^2(t)h(t)$, so we will try to prove that $DF(u)(h) = 3u^2(t)h(t)$. The map $DF(u)$ is of course the linear and continuous map between $C[0,1]$ and $C[0,1]$.

Let $\|f\|$ denotes the supremum norm of the function $f \in C[0,1]$ and let us estimate

$$\frac{1}{\|h\|} \left\|(u(t) + h(t))^3 - u^3(t) - 3u^2(t)h(t)\right\| = \frac{1}{\|h\|} \left\|3u(t)h^2(t) + h^3(t)\right\| \leq$$

$$\leq 3\|u\|\|h\| + \|h\|^2$$

The last term converges to 0 when $h \to 0$. This shows that F is Frechét differentiable.

Problem 5. Let $\phi : \mathbb{R} \to \mathbb{R}$ be the real function of class C^2. Show that $\Phi : C[a,b] \to C[a,b]$ given by

$$\Phi(u)(t) = \phi(u(t))$$

is Frechét differentiable.

Solution: Let us fix $u, h \in C[a,b]$ and let us consider the difference

$$\Phi(u+h)(t) - \Phi(u)(t) = \varphi(u(t) + h(t)) - \varphi(u(t)).$$

We may assume that for certain constant $M > 0$ both $\|u\| \leq M/2$ and $\|h\| \leq M/2$ implying that $\|u + h\| \leq M$. We can also assume that there exists such constant $K > 0$, that for $x \in [-M, M]$

$$|\varphi(x)| \leq K \qquad |\varphi'(x)| \leq K \qquad |\varphi''(x)| \leq K.$$

By Taylor's theorem for any $t \in [a,b]$ there exists such $c(t) \in [-M, M]$ that

$$\varphi(u(t) + h(t)) = \varphi(u(t)) + \varphi'(u(t))h(t) + \frac{1}{2}\varphi''(c(t))h^2(t).$$

So taking $D\Phi(u)(h) = \varphi'(u)h$ we can see that

$$\Phi(u+h)(t) - \Phi(u)(t) - \varphi'(u(t))h(t) = \frac{1}{2}\varphi''(c(t))h^2(t)$$

and

$$\|\Phi(u+h) - \Phi(u) - \varphi'(u)h\| = \frac{1}{2}K\|h\|^2,$$

what implies that

$$\lim_{h \to 0} \frac{\|\Phi(u+h) - \Phi(u) - \varphi'(u)h\|}{\|h\|} = 0.$$