Dimensions of measures and sets - the list of exercises

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Exercise 1. Let $f : A \to [0,\infty]$ be given. Set

$$\mathcal{F}_f = \{g : A \to [0,\infty] : g \text{ is Borel–measurable and } g(x) \geq f(x),\ x \in A\}$$

and define the upper Lebesgue integral by the formula

$$\overline{\int}_A f(x)dx = \inf_{g\in\mathcal{F}_f} \int_A g(x)dx.$$

Show that:

- $\overline{\int}_A f(x)dx = \int_A f(x)dx$ if f is Borel measurable;
- $\overline{\int}_A f(x)dx \leq \overline{\int}_A g(x)dx$ if $0 \leq f(x) \leq g(x)$ for $x \in A$;
- $\overline{\int}_A f(x)dx = \int_A g(x)dx$ for some $g \in \mathcal{F}_f$;
- $\overline{\int}_A f(x)dx > 0$ if $f(x) > 0$ for every $x \in A$;
- $\overline{\int}_A \liminf_{n\to\infty} f_n(x)dx \leq \liminf_{n\to\infty} \overline{\int}_A f_n(x)dx$ for a sequence of nonnegative functions $(f_n)_{n\geq 1}$.

Exercise 2. Let l^∞ denote the space of all bounded sequences. A functional $\mathbb{L} : l^\infty \to \mathbb{R}$ is called a *Banach limit* if it is nonnegative, bounded with the norm 1 and such that

$$\mathbb{L}((a_1, a_2, \ldots)) = \mathbb{L}((a_2, a_3, \ldots)) \quad \text{for all } (a_1, a_2, \ldots) \in l^\infty.$$

- Using the Hahn–Banach theorem prove that a Banach limit exists.
- Show that if $\mathbb{L}$ is a Banach limit, then

  $$\liminf_{n\to\infty} a_n \leq \mathbb{L}((a_1, a_2, \ldots)) \leq \limsup_{n\to\infty} a_n$$

  for $(a_1, a_2, \ldots) \in l^\infty$.

Exercise 3. Let $\mathbb{Q} = \{q_1, q_2, \ldots\}$ be an enumaration of the rational numbers. For $A \subset \mathbb{R}$ define

$$\mu(A) = \sum_{q_i \in A} 2^{-i}.$$

Verify that μ is a measure with all subsets of $\mathbb{R}$ measurable. Show that $\operatorname{supp}\mu + \mathbb{R}$, even though $\mu(\mathbb{R} \setminus \mathbb{Q}) = 0$.

Exercise 4. Let μ be a measure on $\mathbb{R}^d$ such that for all $x \in \mathbb{R}^d$ there is a ball $B(x,r)$ with $\mu(B(x,r) < \infty$. Show that μ is locally finite.

Exercise 5. Find the Hausdorff dimension of the set $\{(1/p, 1/q) : p, q \in \mathbb{N}\} \subset \mathbb{R}^2$.

Exercise 6. Fix $0 < \lambda < 1/2$ and let $F_1, F_2 : \mathbb{R} \to \mathbb{R}$ be given by $F_1(x) = \lambda x$, $F_2(x) = \frac{1}{2}x + \frac{1}{2}$. Describe the attractor of $\{F_1, F_2\}$ and find an expression for its Hausdorff and packing dimension.

Exercise 7. Let $E \subset \mathbb{R}^d$ be a Borel set, let μ be a finite Borel measure on $\mathbb{R}^d$ and $0 < c < \infty$. Prove

- If $\limsup_{r \to 0} \mu(B(x,r))/r^s \leq c$ for all $x \in E$ then $\mathcal{H}^s(E) \geq \mu(E)/c$.
- If $\limsup_{r \to 0} \mu(B(x,r))/r^s \geq c$ for all $x \in E$ then $\mathcal{H}^s(E) \leq 2^s \mu(E)/c$.

Exercise 8. Let $E \subset \mathbb{R}^d$ be a Borel set and let μ be a finite measure. Define

$$\underline{\dim}_{loc}\mu(x) := \liminf_{r \to 0} \frac{\log \mu(B(x,r))}{\log r} \qquad \text{for } x \in E.$$

Show that

- If $\underline{\dim}_{loc}\mu(x) \geq s$ for all $x \in E$ and $\mu(E) > 0$ then $\dim_H E \geq s$.
- If $\underline{\dim}_{loc}\mu(x) \leq s$ for all $x \in E$ then $\dim_H E \leq s$.