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Dimension of measures and sets

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Definitions

Let (X, ρ) be a Polish space. We introduce the notion of topological dimensions. Recall that if X is a separable metric space, the three principal topological dimensions (small inductive dimension, large inductive dimension and covering dimension) are equal. This common quantity we denote by $\dim_T X$ and we call *topological dimension* of X.The value $\dim_T X$ is an integer greater than or equal to -1 or equal to ∞. It can be defined by the following reccurent scheme:
(i) $\dim_T X = -1$ if and only if $X = \emptyset$;
(ii) $\dim_T X \leq n$, $n = 0, 1, \dots,$ if for every point $x \in X$ and every neighbourhood U of x there is a neighbourhood V of x such that $V \subset U$ and $\dim_T \partial V \leq n - 1$;
(iii) $\dim_T X = n$ if and only if $\dim_T X \leq n$ and it is not true that $\dim_T X \leq n - 1$;
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Definitions

Now we are in a position to recall the definition of Hausdorff dimension of a set. To do it we must introduce Hausdorff measure.

For $A \subset X$ and $s, \delta > 0$ define

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_{i=1}^\infty (\operatorname{diam} U_i)^s : A \subset \bigcup_{i=1}^\infty U_i \text{ and } \operatorname{diam} U_i \le \delta\right\}$$

and

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The restriction of $\mathcal{H}^s$ to the σ–algebra of $\mathcal{H}^s$–measurable sets is called the *Hausdorff s–dimensional measure*. Note that all Borel sets are $\mathcal{H}^s$–measurable. The value

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$$\underline{\dim}_L \mu = \liminf_{r \to 0} \frac{\log Q_\mu(r)}{\log r},$$

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The *concentration dimension* of X is defined by the formula

$$\dim_L X = \sup_{\mu \in \mathcal{M}_1(X)} \underline{\dim}_L \mu.$$

Results

We are in a position to present some reletaions between different dimensions of measures and sets.

Proposition 1.

For every $\mu \in \mathcal{M}_1(X)$ we have

$$\dim_H \mu \geq \underline{\dim}_L \mu$$

and consequently

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Results

The following lemma is crucial for proving the main result of our lecture.

Lemma 1.

Suppose that $\dim_{\mathcal{T}} X \geq d$, where $d \in \mathbb{N} \cup \{0\}$. Then there exists a Borel probability measure μ such that

$$\mu(B(x, r)) \leq Cr^d \quad \text{for every} \quad x \in X \quad \text{and} \quad r > 0,$$

where $C > 0$ is some positive constant independent of x and r.

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From Lemma 1 it follows.

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Let X be a Polish space space. Then there exists a measure $\mu_* \in \mathcal{M}_1(X)$ such that

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From Propositions 1 and 2 it follows a generalization of the well known Marczewski theorem.

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Let X be a Polish space space. Then we have

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Moreover, we obtain the following.

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Suppose that $\dim_T X \geq d$, where $d \in \mathbb{N} \cup \{0\}$. Then there exists a compact subspace $Y \subset X$ such that $\dim_H Y \geq d$.

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