# Dimensions of measures and sets

Tomasz Szarek

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# 1 Introduction

Let $A \subset \mathbb{R}$ be a Borel set. Let $f : A \to [0, \infty]$ be given. Set

$$\mathcal{F}_f = \{g : A \to [0, \infty] : g \text{ is Borel–measurable and } g(x) \geq f(x),\ x \in A\}$$

and define the upper Lebesgue integral by the formula

$$\overline{\int_A} f(x)dx = \inf_{g\in\mathcal{F}_f} \int_A g(x)dx.$$

The following facts can be easily derived from the above definition:

- $\overline{\int}_A f(x)dx = \int_A f(x)dx$ if $f$ is Borel measurable;
- $\overline{\int}_A f(x)dx \leq \overline{\int}_A g(x)dx$ if $0 \leq f(x) \leq g(x)$ for $x \in A$;
- $\overline{\int}_A f(x)dx = \int_A g(x)dx$ for some $g \in \mathcal{F}_f$;
- $\overline{\int}_A f(x)dx > 0$ if $f(x) > 0$ for every $x \in A$;
- $\overline{\int}_A \liminf_{n\to\infty} f_n(x)dx \leq \liminf_{n\to\infty} \overline{\int}_A f_n(x)dx$ for a sequence of nonnegative functions $(f_n)_{n\geq 1}$.

By $B(x, r)$ (resp. $S(x, r)$) we denote the closed ball (resp. the sphere) with center in $x$ and radius $r$. We introduce the notion of topological dimensions. Recall that if $X$ is a separable metric space, the three principal topological dimensions (small inductive dimension, large inductive dimension and covering dimension) are equal. This common quantity we denote by $\dim_T X$ and we call *topological dimension* of $X$. The value $\dim_T X$ is an integer greater than or equal to $-1$ or equal to $\infty$. It can be defined by the following reccurent scheme:

(i) $\dim_T X = -1$ if and only if $X = \emptyset$;

(ii) $\dim_T X \leq n$, $n = 0, 1, \ldots,$ if for every point $x \in X$ and every neighbourhood $U$ of $x$ there is a neighbourhood $V$ of $x$ such that $V \subset U$ and $\dim_T \partial V \leq n - 1$;

(iii) $\dim_T X = n$ if and only if $\dim_T X \leq n$ and it is not true that $\dim_T X \leq n - 1$;

(iv) $\dim_T X = \infty$ if $\dim_T X \geq n$ for every $n \in \mathbb{N}$.

Given a measure $\mu \in \mathcal{M}_1(X)$ we define the *lower* and *upper concentration dimension* of $\mu$ by the formulas

$$\underline{\dim}_L \mu = \liminf_{r\to 0} \frac{\log Q_\mu(r)}{\log r},$$

$$\overline{\dim}_L \mu = \limsup_{r\to 0} \frac{\log Q_\mu(r)}{\log r},$$

where

$$Q_\mu(r) = \sup\{\mu(A) : \mathrm{diam} A \leq r,\ A \in \mathcal{B}(X)\} \quad \text{for} \quad r > 0.$$

Recall that $Q_\mu$ is a well known Lévy concentration function frequently used in the theory of random variables.

The *concentration dimension* of $X$ is defined by the formula

$$\dim_L X = \sup_{\mu \in \mathcal{M}_1(X)} \underline{\dim}_L \mu.$$

Now we are in a position to recall the definition of Hausdorff dimension of a set. To do it we must introduce Hausdorff measure. For $A \subset X$ and $s, \delta > 0$ define

$$\mathcal{H}^s_\delta(A) = \inf\left\{\sum_{i=1}^{\infty}(\operatorname{diam} U_i)^s : A \subset \bigcup_{i=1}^{\infty} U_i \text{ and } \operatorname{diam} U_i \leq \delta\right\}$$

and

$$\mathcal{H}^s(A) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(A).$$

The restriction of $\mathcal{H}^s$ to the $\sigma$–algebra of $\mathcal{H}^s$–measurable sets is called the *Hausdorff $s$–dimensional measure*. Note that all Borel sets are $\mathcal{H}^s$–measurable. The value

$$\dim_H A = \inf\{s > 0 : \mathcal{H}^s(A) = 0\}$$

is called the *Hausdorff dimension* of the set $A$. (As usual, we admit $\inf \emptyset = +\infty$).

The *Hausdorff dimension* of a measure $\mu \in \mathcal{M}_1(X)$ is defined by the formula

$$\dim_H \mu = \inf\left\{\dim_H A \ : \ A \in \mathcal{B}(X),\ \mu(A) = 1\right\}.$$

## 2 Auxilliary results

**Lemma 1.** *If* $\dim_T X \geq d + 1$, *where $d$ is an integer greater than or equal to* $-1$, *then there exists* $x_0 \in X$ *and* $\lambda_0 > 0$ *such that* $\dim_T S(x_0, \lambda) \geq d$ *for every* $\lambda \in (0, \lambda_0]$.

*Proof:* Suppose, for the contradiction, that for every $x_0 \in X$ and $\lambda_0 > 0$ there exists $\lambda \in (0, \lambda_0]$ such that $\dim_T S(x_0, \lambda) \leq d-1$. Then by the definition of topological dimension we have $\dim_T X \leq d$, which is impossible. □

**Lemma 2.** *Suppose that* $\dim_T X \geq d$, *where* $d \in \mathbb{N} \cup \{0\}$. *Then there exists a Borel probability measure $\mu$ such that*

(1) $$\mu(B(x, r)) \leq Cr^d \quad \text{for every} \quad x \in X \quad \text{and} \quad r > 0,$$

*where $C > 0$ is a positive constant independent of $x$ and $r$.*

*Proof:* We use the induction argument with respect to $d$. For $d = 0$ condition (1) obviously holds for every Borel probability measure $\mu$. Assume that the statement holds for $d = m$. We will prove that it holds for $d = m + 1$. By Lemma 1 there exists $x_0 \in X$ and $\lambda_0 > 0$ such that $\dim_T S(x_0, \lambda) \geq m$ for every $\lambda \in (0, \lambda_0]$. Fix $\lambda \in (0, \lambda_0]$ and set

$X_\lambda = S(x_0, \lambda)$. By the induction hypothesis there exists a Borel probability measure $\tilde{\mu}_\lambda$ on $X_\lambda$ such that

$$\tilde{\mu}_\lambda(B_\lambda(x,r)) \le C_\lambda r^m \quad \text{for every} \quad x \in X_\lambda \quad \text{and} \quad r > 0,$$

where $B_\lambda(x,r)$ stands for the closed ball in the space $X_\lambda$ with centre at $x \in X_\lambda$ and radius $r$, and $C_\lambda$ is independent of $x$ and $r$. Without loss of generality we may assume that $C_\lambda \ge 1$. Define the Borel measure $\mu_\lambda : \mathcal{B}(X) \to [0,1]$ by the formula

$$\mu_\lambda(A) = \tilde{\mu}_\lambda(A \cap X_\lambda)/(2^m C_\lambda) \qquad \text{for} \quad A \in \mathcal{B}(X).$$

Clearly $\operatorname{supp} \mu_\lambda \subset X_\lambda$ and

$$\mu_\lambda(B(x,r)) \le r^m \quad \text{for every} \quad x \in X \quad \text{and} \quad r > 0. \tag{2}$$

Set

$$\beta = \overline{\int}_{(0,\lambda_0]} \mu_\lambda(X) d\lambda$$

and observe that $\beta > 0$, by the property of the upper Lebesgue integral. From properties of the upper Lebesgue integral it follows that we can define a decreasing sequence of closed sets $(X_n)_{n\ge1}$, $X_n \subset X$, such that

$$\overline{\int}_{(0,\lambda_0]} \mu_\lambda(X_n) d\lambda > \beta/2$$

and $X_n$ has $2^{-n}$–net for $n \in \mathbb{N}$. Indeed, let $n = 1$ and let $\{x_k\}_{k\ge1}$ be a dense subset of $X$. Applying the Fatou lemma for the upper Lebesgue integral to $f_n(\lambda) = \mu_\lambda(\bigcup_{i=1}^{n} B(x_i, 1/2))$ we obtain

$$\overline{\int}_{(0,\lambda_0]} \mu_\lambda \left( \bigcup_{i=1}^{i_1} B(x_i, 1/2) \right) d\lambda > \beta/2$$

for some $i_1 \in \mathbb{N}$. Set $X_1 = \bigcup_{i=1}^{i_1} B(x_i, 1/2)$. Further, assume that we have defined $X_1, \dots, X_k$. Analogously as before we find $i_{k+1}$ such that

$$\overline{\int}_{(0,\lambda_0]} \mu_\lambda \left( X_k \cap \bigcup_{i=1}^{i_{k+1}} B(x_i, 1/2^{k+1}) \right) d\lambda > \beta/2.$$

Setting $X_{k+1} = X_k \cap \bigcup_{i=1}^{i_{k+1}} B(x_i, 1/2^{k+1})$ finishes the induction.

For $k \in \mathbb{N}$ and $i \in \{1, \dots, k\}$ we define

$$\alpha_{k,i} = \sup \left\{ \mu_\lambda(X_k) : \lambda \in \left( \frac{(i-1)\lambda_0}{k}, \frac{i\lambda_0}{k} \right] \right\}.$$

Let

$$\nu_k = \frac{\lambda_0}{k} \sum_{i=1}^{k} \mu_{k,i},$$

where $\mu_{k,i} = \mu_{\lambda_{k,i}}$ with $\lambda_{k,i} \in ((i-1)\lambda_0/k, i\lambda_0/k]$ and such that

$$\mu_{\lambda_{k,i}}(X_k) \geq \alpha_{k,i}/2.$$

By the definition of the upper Lebesgue integral we have

$$2\nu_k(X_k) \geq \frac{\lambda_0}{k}\sum_{i=1}^{k}\alpha_{k,i} \geq \int_0^{\lambda_0} \mu_\lambda(X_k)d\lambda > \beta/2. \tag{3}$$

Define now the positive linear functional $\Lambda : C(X) \to \mathbb{R}$ by the formula

$$\Lambda(f) = \mathbb{L}\left(\left(\int_{X_k} f d\nu_k\right)_{k\in\mathbb{N}}\right) \quad \text{for } f \in C(X),$$

where $\mathbb{L}$ is a Banach limit and $C(X)$ stands for the space of continuous functions $f : X \to \mathbb{R}$. From (3) it follows that $\Lambda$ is nontrivial. Let $K = \bigcap_{k=1}^{\infty} X_k$. Observe that $K$ is a compact set and $\Lambda(f) = 0$ for $0 \leq f \leq \mathbf{1}_{X\setminus K}$. Hence $\Lambda$ is a Riesz functional. Let $\mu_*$ be the Borel measure such that

$$\Lambda(f) = \int_X f(x)\mu_*(dx) \qquad \text{for } f \in C(X).$$

Obviously $\operatorname{supp}\mu_* \subset K$. We end the proof with showing that

$$\mu_*(B(x,r)) \leq 2r^{m+1}$$

for arbitrary $x \in X$ and $r > 0$. Fix $x \in X$ and $r > 0$ and consider the ball $B(x,r)$. For $k \in \mathbb{N}$ define

$$i(k) = \min J_k \qquad \text{and} \qquad I(k) = \max J_k,$$

where

$$J_k = \{1 \leq i \leq k : B(x,r) \cap S(x_0, \lambda_{k,i}) \neq \emptyset\}.$$

If $J_k = \emptyset$ we set $i(k) = I(k) = 0$. Further we have

$$\frac{\lambda_0}{k}(I(k) - i(k)) \leq 2r + \frac{\lambda_0}{k}.$$

On the other hand, by the construction of the measures $\nu_k$ we obtain

$$\nu_k(B(x,r)) \leq \frac{\lambda_0}{k}r^m(I(k) - i(k) + 1) \leq 2r^{m+1} + \frac{\lambda_0}{k}2r^m. \tag{4}$$

Fix now $\eta > 0$ and let $f \in C(X)$ be such that $f(y) = 1$ for $y \in B(x,r)$, $f(y) = 0$ for $y \notin B(x, r+\eta)$ and $0 \leq f \leq 1$. Then

$$\mu_*(B(x,r)) \leq \Lambda(f) \leq \liminf_{k\to\infty} \nu_k(B(x, r+\eta)).$$

By (4) and the fact that $\eta > 0$ may be arbitrary small, we obtain

$$\mu_*(B(x,r)) \leq 2r^{m+1}$$

and the proof is complete. $\square$

## 3 Main Results

**Proposition 1.** *For every* $\mu \in \mathcal{M}_1(X)$ *we have*

$$\dim_H \mu \geq \underline{\dim}_L \mu$$

*and consequently*

$$\dim_H X \geq \dim_L X.$$

*Proof:* Let $A \in \mathcal{B}(X)$ be such that $\mu(A) = 1$. Set $d = \underline{\dim}_L \mu$. If $d = 0$, the statement is obvious.

Suppose now that $0 < d \leq +\infty$ and choose a positive number $s < d$. Define

$$\omega(r) = \frac{\log Q_\mu(r)}{\log r} \quad \text{for} \quad r > 0.$$

Then obviously

$$Q_\mu(r) = r^{\omega(r)} \quad \text{and} \quad \liminf_{r \to 0} \omega(r) > s.$$

Let $r_0 \in (0, 1)$ be such that

$$\omega(r) > s \quad \text{for every} \quad r \in (0, r_0).$$

Fix $r \in (0, r_0)$ and let $\{U_i\}$ be an arbitrary cover of $A$ satisfying $\operatorname{diam} U_i \leq r$, $i \in \mathbb{N}$. We have

$$1 = \mu(A) \leq \sum \mu(U_i) \leq \sum (\operatorname{diam} U_i)^s.$$

Therefore,

$$\mathcal{H}^s_r(A) \geq 1 \quad \text{for} \quad r \in (0, r_0).$$

Consequently $\mathcal{H}^s(A) \geq 1$ and so $\dim_H A \geq s$. Since $s < d$ was arbitrary, we Have

$$\dim_H \mu \geq \underline{\dim}_L \mu.$$

Taking supremum over all $\mu \in \mathcal{M}_1(X)$ and using the obvious inequality $\dim_H X \geq \dim_H \mu$ for any $\mu \in \mathcal{M}_1(X)$ we obtain the second statement of our Proposition. □

We are in a position to formulate the main results of our lecture that generalizes the Marczewski theorem..

**Theorem 1.** *Let* $(X, \rho)$ *be a Polish space. We have*

$$\dim_H X \geq \dim_L X \geq \dim_T X.$$

*Proof:* Set $d := \dim_T X$. From Lemma it follows that there exist a Borel measure $\mu_* \in \mathcal{M}_1(X)$ and a positive constant $C$ such that

$$\mu_*(B(x, r)) \leq C r^d \qquad \text{for } x \in X \text{ and } r > 0.$$

Hence $\underline{\dim}_L \mu_* \geq d$ and consequently $\dim_L X \geq d$. From Proposition it follows the second inequality:

$$\dim_H X \geq \dim_L X.$$

This completes the proof. $\Box$

**Theorem 2.** *Suppose that* $\dim_T X \geq d$, *where* $d \in \mathbb{N} \cup \{0\}$. *Then there exists a compact subspace* $Y \subset X$ *such that* $\dim_H Y \geq d$.

*Proof:* From Lemma 3 it follows that there exists a Borel measure, say, $\mu_*$ and a positive constant $C > 0$ such that

$$\mu_*(B(x,r)) \leq Cr^d \qquad \text{for } x \in X \text{ and } r > 0.$$

Further there exists a compact set $Y \subset X$ such that $\mu_*(Y) > 0$, by Ulam's lemma. Then $\dim_H Y \geq d$, by Frostman's lemma (see [3, 4]). $\Box$

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