Numerical Continuation Methods.

Sample problems.

PWP Interdysciplinary Doctoral Studies in Mathematical Modeling
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Problem 1. Find the structure of the set of solutions of problems

$$f(x) = 0$$

and

$$f(x) = c$$

for any $c \neq 0$, where $f : \mathbb{R}^3 \to \mathbb{R}^2$ is given by

$$f(x, y, z) = (x^2 + y^2 - z^2, x^2 + y^2).$$

and $c = (c_1, c_2) \in \mathbb{R}^2$.

Solution: Let us first consider the system

$$\begin{cases} x^2 + y^2 - z^2 = 0 \\ x^2 + y^2 = 0. \end{cases}$$

The only zero of this system is $(0, 0, 0)$.

Now let us have a look at

$$\begin{cases} x^2 + y^2 - z^2 = c_1 \\ x^2 + y^2 = c_2. \end{cases}$$

If $c_2 < 0$, then there is no solution of the system. If $c_2 = 0$, then the first equation turns into

$$-z^2 = c_1.$$

This means that for $c_1 > 0$ there is no solution and for $c_1 < 0$ there are two solutions $(0, 0, \sqrt{|c_1|})$ and $(0, 0, -\sqrt{|c_1|})$.

Let us now assume that $c_2 > 0$. Then the first equation turns into

$$c_2 - z^2 = c_1$$

and

$$z^2 = c_2 - c_1.$$

When $c_2 < c_1$, then there is no solution of the system.

If $c_2 = c_1$, then the solution set is the closed curve – the circle $x^2 + y^2 = c_2$ in $z = 0$ plane.

If $c_2 > c_1$, then the solution set consists of two closed curves – two circles $x^2 + y^2 = c_2$ in the plane $z = \sqrt{c_2 - c_1}$ and in the plane $z = -\sqrt{c_2 - c_1}$.

Problem 2. Find the regular and critical points in the set of zeroes of the map $f : \mathbb{R}^2 \to \mathbb{R}^1$ given by

$$f(x, y) = y^3 - x^2.$$

Solution: The zeroes of the map form the curve being the graph of the function $g : \mathbb{R}^1 \to \mathbb{R}^1$ given by

$$g(x) = \sqrt[3]{x^2}.$$

The solution set is a curve, without any intersections – so locally homoeomorphic to the interval. So it is a 1-dimensional manifold, but it is not a submanifold of $\mathbb{R}^2$ (due to singularity in $(0, 0)$).

Let us check the value of the derivative of the map f:

$$Df(x, y) = \left[-2x, 3y^2\right].$$

Point $(0, 0)$ is not a regular point, but every other zero is a regular point of the map f.

Problem 3. Find the regular and critical points in the set of zeroes of the map $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$f(x, y, z) = (x^2 + y^2 + z^2 - 4, (x-1)^2 + y^2 - 1),$$

(i.e. in the Viviani's curve).

Solution: The derivative of the map f equals

$$Df(x, y, z) = \begin{bmatrix} 2x & 2y & 2z \\ 2(x-1) & 2y & 0 \end{bmatrix}.$$

Now we should check if there is any zero of this map, where the derivative's rank is less than 2. Let us look at the determinant

$$\det \begin{bmatrix} 2y & 2z \\ 2y & 0 \end{bmatrix} = -4yz.$$

So whenever $y \neq 0$ and $z \neq 0$ the derivative's rank equals 2.

Let us check what happens if $z = 0$. Then the zero $(x, y, 0)$ of f satisfies the system of equations

$$\begin{cases} x^2 + y^2 = 4 \\ (x-1)^2 + y^2 = 1, \end{cases}$$

what gives

$$x^2 - (x-1)^2 = 3$$

and $x = 2$. This implies that $y = 0$ and we have one point of this type: $(2, 0, 0)$. Then we have

$$Df(2, 0, 0) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In this case the derivative's rank equals 1, so this zero is not the regular zero.

Let us check what happens if $y = 0$. Then the zero $(x, 0, z)$ satisfies the system of equations:

$$\begin{cases} x^2 + z^2 = 2 \\ (x-1)^2 = 1, \end{cases}$$

This means that there is either $x = 0$ or $x = 2$. This leads to the following solutions: $(0, 0, 2)$, $(0, 0, -2)$, $(2, 0, 0)$. Let us check how the derivative of the map f looks like for zero $(0, 0, \pm 2)$.

$$Df(0, 0, \pm 2) = \begin{bmatrix} 0 & 0 & \pm 4 \\ -2 & 0 & 0 \end{bmatrix}.$$

This matrix has rank 2 meaning that the two zeroes $(0, 0, \pm 2)$ are regular zeroes.

Hence all zeroes of f are regular, except for the zero $(2, 0, 0)$.

Problem 4. Prove that the definition of affine independent points

Definition. We say that the points $v_0, v_1, ..., v_l \in \mathbb{R}^{k+1}$ are *affine independent* when vectors $v_1 - v_0, v_2 - v_0, ..., v_l - v_0$ are linearly independent.

does not depend on the selection of the point v_0.
Solution: Let us fix $j \in \{1, 2, ..., l\}$ and denote

$$w_i = v_{i-1} - v_l,$$

for $i = 1, 2, ..., j$ and

$$w_i = v_i - v_l,$$

for $i = j+1, ..., l$.
Assume that

$$\alpha_1 w_1 + \alpha_2 w_2 + ... + \alpha_l w_l = 0.$$

This means that

$$\alpha_1(v_0 - v_j) + \alpha_2(v_1 - v_j) + ... + \alpha_j(v_{j-1} - v_j) + \alpha_{j+1}(v_{j+1} - v_j) + ... + \alpha_l(v_l - v_j) = 0.$$

We may rearrange the terms a little bit to have

$$\alpha_1 v_0 + \alpha_2 v_1 + ... + \alpha_j v_{j-1} + (-\alpha_1 - \alpha_2 - ... - \alpha_l) v_j + \alpha_{j+1} v_{j+1} + ... \alpha_l v_l = 0$$

$$\alpha_2(v_1 - v_0) + ... + \alpha_j(v_{j-1} - v_0) + (-\alpha_1 - \alpha_2 - ... - \alpha_l)(v_j - v_0) + \alpha_{j+1}(v_{j+1} - v_0) + ... \alpha_l(v_l - v_0) = 0$$

But we know that vectors $v_1 - v_0, v_2 - v_0, ..., v_l - v_0$ are independent, so

$$\alpha_2 = ... = \alpha_j = (-\alpha_1 - \alpha_2 - ... - \alpha_l) = \alpha_{j+1} = ... = \alpha_l = 0.$$

This means that all $\alpha_i = 0$ for $i \in \{1, 2, ..., l\}$ implying that the vectors w_i are independent.

Problem 5. Assume the triangulation $\mathbb{T}$ in $\mathbb{R}^{k+1}$ is given. Let us define the map $f_{\mathbb{T}}$ for the vertex v belonging to the triangulation $\mathbb{T}$ – by the formula:

$$f_{\mathbb{T}}(v) = f(v).$$

Then the map $f_{\mathbb{T}}$ is extended the the affine map for each simplex $\sigma \in \mathbb{T}$ separately by:

$$f_{\mathbb{T}}(t_0 v_0 + t_1 v_1 + ... + t_{k+1} v_{k+1}) = t_0 f_{\mathbb{T}}(v_0) + t_1 f_{\mathbb{T}}(v_1) + ... + t_{k+1} f_{\mathbb{T}}(v_{k+1}).$$

Prove that the map $f_{\mathbb{T}}$ is well defined – i.e. on each common face of two simplices belonging to the triangulation the values induced by the two simplices coincide.

Solution: Let us take $x \in \sigma_1 \cap \sigma_2$, where $\sigma_1, \sigma_2 \in \mathbb{T}$. Let $\sigma_1 = \text{conv}\{v_0, v_1, ..., v_{k+1}\}$ and $\sigma_2 = \text{conv}\{w_0, w_1, ..., w_{k+1}\}$. We know that $\sigma_1 \cap \sigma_2$ is a common face of the two simplices, so we may assume that $v_i = w_i$ for $i = 0, 1, ..., l$ and $\sigma_1 \cap \sigma_2 = \text{conv}\{v_0, ..., v_l\}$.

Hence each $x \in \text{conv}\{v_0, ..., v_l\}$ is represented by the barycentric coordinates $x = \tau_0 v_0 + ... \tau_l v_l$, $\tau_i \geq 0$, $\tau_0 + \tau_1 + ... + \tau_l = 1$.

That is why the barycentric coordinates of x in σ_1 are the same as in σ_2 and equal to $(\tau_0, ... \tau_l, 0, 0, ..., 0)$. It means that the formulas for $f_{\mathbb{T}}$ induced by σ_1 and σ_2 are the same and equal

$$f_{\sigma_1}(x) = f_{\sigma_2}(x) = \tau_0 f(v_0) + \tau_1 f(v_1) + ... + \tau_l f(v_l).$$

Problem 6. Find the PL approximation of the map $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$f(x, y, z) = (x^2 + y^2 + z^2 - 4, (x - 1)^2 + y^2 - 1)$$

in the simplex σ spanned by the vertices $(0.5, 0.5, 1)$, $(1.5, 1.5, 1)$, $(0.5, 1.5, 1)$, $(1, 1, 1.5)$.

Solution: The values in the vertices of the simplex are

$$f(0.5, 0.5, 1) = (-2.5, -0.5),$$

$$f(1.5, 1.5, 1) = (1.5, 1.5),$$

$$f(0.5, 1.5, 1) = (-0.5, 1.5),$$

$$f(1, 1, 1.5) = (0.25, 0).$$

The PL approximation of f is given by

$$f_\sigma(x, y, z) = f_\sigma(s(0.5, 0.5, 1) + t(1.5, 1.5, 1) + u(0.5, 1.5, 1) + v(1, 1, 1.5)) =$$

$$= s(-2.5, -0.5)+t(1.5, 1.5)+u(-0.5, 1.5)+v(0.25, 0) = (-2.5s+1.5t-0.5u+0.25v, -0.5s+1.5t+1.5u),$$

where $s, t, u, v \geq 0$ and $s + t + u + v = 1$.

Problem 7. Find the set of zeroes in the simplex spanned by the vertices $(0.5, 0.5, 1)$, $(1.5, 1.5, 1)$, $(0.5, 1.5, 1)$, $(1, 1, 1.5)$ for the PL approximation of the map $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$f(x, y, z) = (x^2 + y^2 + z^2 - 4, (x-1)^2 + y^2 - 1)$$

Solution: The PL approximation found in the previous problem leads us to the following system of linear equations

$$\begin{cases} -2.5s + 1.5t - 0.5u + 0.25v = 0 \\ -0.5s + 1.5t + 1.5u = 0 \\ s + t + u + v = 1 \end{cases}$$

The solution of this system may be represented as

$$(s, t, u, v) = (3/4 - 3/4v, 1 - 9/8v, 7/8v - 6/8, v).$$

But we want all variables to be nonnegative: $s, t, u, v \geq 0$. This leads to the series of conditions:

$$v \leq 1, v \leq 8/9, v \geq 6/7, v \geq 0.$$

This gives $v \in [6/7, 8/9]$. So the zero set of the PL approximation in the simplex is the line segment given by

$$(3/4 - 3/4v, 1 - 9/8v, 7/8v - 6/8, v), v \in [6/7, 8/9].$$