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Elements of Spectral Theory

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Linear operator

Let H be a complex Hilbert space.

Definition

A **linear operator** A is a mapping

$$A : \mathcal{D}(A) \to \mathcal{R}(A),$$

where $\mathcal{D}(A) \subset H$ and $\mathcal{R}(A) \subset H$ are linear subspaces, such that

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay, \qquad x, y \in \mathcal{D}(A), \quad \alpha, \beta \in \mathbb{C}.$$

We say that $\mathcal{D}(A)$ is the **domain** of A and $\mathcal{R}(A)$ is the **range** of A.

Definition

- We say that a linear operator B is an **extension** of A if $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $Ax = Bx$ for every $x \in \mathcal{D}(A)$. Then we write $A \subset B$.
- Two operators A and B are **equal**, i.e. $A = B$, if $A \subset B$ and $B \subset A$.

Examples of linear operators

1. **Zero operator Θ**

   H – any Hilbert space $\qquad \mathcal{D}(\Theta) = H \qquad \theta x = 0, \quad x \in H$

2. **Unit operator $\mathbb{1}$**

   H – any Hilbert space $\qquad \mathcal{D}(\mathbb{1}) = H \qquad \mathbb{1}x = x, \quad x \in H$

3. **Position operator Q**

   $$H = L_2(\mathbb{R}) \qquad \mathcal{D}(Q) = \left\{ x \in L_2(\mathbb{R}) : \int_{-\infty}^{\infty} t^2 |x(t)|^2 \mathrm{d}t < \infty \right\}$$

   $$(Qx)(t) = tx(t), \quad x \in \mathcal{D}(Q).$$

4. **Momentum operator P**

   $$H = L_2(\mathbb{R}) \qquad \mathcal{D}(P) = \left\{ x \in L_2(\mathbb{R}) : x \in C^{(1)}(\mathbb{R}) \ \wedge \int_{-\infty}^{\infty} |x'(t)|^2 \mathrm{d}t \right\}$$

   $$(Px)(t) = -ix'(t), \quad x \in \mathcal{D}(P).$$

Operations on linear operators

Let A and B be linear operators, $\alpha \in \mathbb{C}$.

- **Scalar multiplication** αA

$$\mathcal{D}(\alpha A) = \mathcal{D}(A) \qquad\qquad (\alpha A)x = \alpha(Ax), \quad x \in \mathcal{D}(A)$$

- **Sum** $A + B$

$$\mathcal{D}(A+B) = \mathcal{D}(A) \cap \mathcal{D}(B) \qquad\qquad (A+B)x = Ax + Bx, \quad x \in \mathcal{D}(A+B)$$

- **Composition (multiplication)** AB

$$\mathcal{D}(AB) = \{x \in H : x \in \mathcal{D}(B) \ \wedge \ Bx \in \mathcal{D}(A)\}$$

$$(AB)x = A(Bx), \qquad x \in \mathcal{D}(AB).$$

Operations on linear operators

Remark

In some cases the addition and multiplication of general linear operators may be a very subtle issue.
For example, given an operator A, it may happen that $\mathcal{D}(A^2) \neq \mathcal{D}(A)$.

Example

Let us consider the momentum operator Q:

$$\mathcal{D}(Q^2) =$$

$$= \left\{ x \in L_2(\mathbb{R}) : x \in C^{(1)} \wedge \int |x'(t)|^2 \mathrm{d}t < \infty \wedge x' \in C^{(1)} \wedge \int |x''(t)|^2 \mathrm{d}t < \infty \right\}$$

$$= \left\{ x \in L_2(\mathbb{R}) : x \in C^{(2)} \wedge \int \left(|x'(t)|^2 + |x''(t)|^2 \right) \mathrm{d}t < \infty \right\}$$

Obviously, it is a proper subspace of $\mathcal{D}(Q)$.

In the worst case it may be $\mathcal{D}(A^2) = \{0\}$ even if $\mathcal{D}(A)$ is a dense subspace of H.

Operations on linear operators

Exercise

Given linear operators A, B, C prove that

$$AB + AC \subset A(B + C).$$

Show that the converse inclusion need not be satisfied.

Definition

We say that a linear operator A is **invertible** if there is an operator A^{-1} such that $\mathcal{D}(A^{-1}) = \mathcal{R}(A)$, $\mathcal{R}(A^{-1}) = \mathcal{D}(A)$ and $A^{-1}(Ax) = x$ for every $x \in \mathcal{D}(A)$.

Remark

It follows from the above definition that A is invertible if and only if $x = 0$ is the only element $x \in \mathcal{D}(A)$ satisfying the equation $Ax = 0$.
Note, that invertibility of a linear operator does not mean that A is a bijection from H onto itself! It is just a linear isomorphism of subspaces $\mathcal{D}(A)$ and $\mathcal{R}(A)$.

Graph of linear operator

Definition

Let A be a linear operator. We define a **graph** of A as

$$\Gamma(A) = \{(x, Ax) : x \in \mathcal{D}(A)\} \subset H \oplus H.$$

Obviously, $\Gamma(A)$ is a linear subspace of the direct sum of Hilbert spaces $H \oplus H$.

Exercise

Show that a linear subspace $S \subset H \oplus H$ is a graph of some linear operator if and only if for every $x \in H$ there is at most one element $y \in H$ such that $(x, y) \in S$.

Remark

One can easily show that B is extension of A ($A \subset B$) if and only if $\Gamma(A) \subset \Gamma(B)$.

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Closure of linear operator

In general, graph $\Gamma(A)$ of a linear operator A need not be a closed subspace of $H \oplus H$. Is the closure $\overline{\Gamma(A)}$ a graph of some operator?

Theorem

The subspace $\overline{\Gamma(A)}$ is a graph of a linear operator if and only if

$$\forall\,(x_n) \subset \mathcal{D}(A) : \left\{ \begin{array}{l} \lim\limits_{n\to\infty} x_n = 0 \\ (Ax_n) \text{ is convergent} \end{array} \right\} \Rightarrow \lim_{n\to\infty} Ax_n = 0$$

Proof.

Exercise. □

Definition

1. We say that A is **closable** if $\overline{\Gamma(A)}$ is a graph of some operator.
2. The **closure** of a closable operator A is an operator $\overline{A}$ such that $\Gamma(\overline{A}) = \overline{\Gamma(A)}$.

Closed operators

Definition

We say that an operator A is **closed** if A is closable and $\overline{A} = A$.

Theorem

Let A be a linear operator. The following conditions are equivalent:

1. A *is closed,*
2. $\Gamma(A)$ *is a closed subspace of* $H \oplus H$*,*
3. $\forall\,(x_n) \subset \mathcal{D}(A)\ \forall\, x, y \in H : \left\{ \begin{array}{c} \lim\limits_{n\to\infty} x_n = x \\ \lim\limits_{n\to\infty} Ax_n = y \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} x \in \mathcal{D}(A) \\ Ax = y \end{array} \right\}$

Theorem

If A is closed and invertible, then A^{-1} is closed.

Bounded operators

Definition

An operator A is **bounded** if

$$\exists\, m \geq 0 \;\forall\, x \in \mathcal{D}(A): \; \|Ax\| \leq m\|x\|.$$

The **operator norm** of a bounded operator A

$$\|A\| = \inf\{m \geq 0 : \; \|Ax\| \leq m\|x\| \text{ for every } x \in \mathcal{D}(A)\}$$

Proposition

Let A be an operator. The following conditions are equivalent:

1. *A is bounded,*
2. *A is continuous at some point of $\mathcal{D}(A)$,*
3. *A is continuous at each point of $\mathcal{D}(A)$.*

Proposition

$$\|A\| = \sup\{\|Ax\| : \; x \in \mathcal{D}(A), \, \|x\| \leq 1\} = \sup\{\|Ax\| : \; x \in \mathcal{D}(A), \, \|x\| \leq 1\}$$

Bounded operators

Proposition

- *Each bounded operator A is closable and $\overline{A}$ is bounded. Moreover, $\|\overline{A}\| = \|A\|$.*
- *If A is bounded then A is closed if and only if $\mathcal{D}(A)$ is closed.*

By $\mathfrak{L}(H)$ we denote the set of all bounded operators A such that $\mathcal{D}(A) = H$. It is known, that $\mathfrak{L}(H)$ equipped with the operator norm has the structure of a Banach algebra with unit. In particular:

$$\|A + B\| \leq \|A\| + \|B\|, \qquad \|AB\| \leq \|A\|\|B\|$$

Proposition

- *If $A \in \mathfrak{L}(H)$ then A is closed.*
- *If $A \in \mathfrak{L}(H)$ and B is closed, then both $A + B$ and BA are closed operators.*
- *If $A \in \mathfrak{L}(H)$ and B is closable, then $A + B$ is closable and $\overline{A + B} = A + \overline{B}$.*

Topologies on $\mathfrak{L}(H)$

- **Uniform topology**

$$\lim_{n\to\infty} A_n = A \quad \Leftrightarrow \quad \lim_{n\to\infty} \|A_n - A\| = 0$$

- **Strong topology**

$$A_n \to A \quad \Leftrightarrow \quad \forall\, x \in H : \lim_{n\to\infty} A_n x = Ax$$

- **Weak topology**

$$A_n \rightharpoonup A \quad \Leftrightarrow \quad \forall\, x, y \in H : \lim_{n\to\infty} \langle y, A_n x\rangle = \langle y, Ax\rangle$$

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Point spectrum of an operator

Given an operator A and $\lambda \in \mathbb{C}$, let $A_\lambda = \lambda \mathbb{1} - A$.

Definition

We say that λ is an **eigevalue** of A if A_λ is not invertible.
The set of all eigenvalues is called a **point spectrum** of A and is denoted by $\sigma_{\mathrm{p}}(A)$

Remark

A number λ is an eigenvalue if and only if $Ax = \lambda x$ for some $x \in \mathcal{D}(A)$.

Proposition

If $H = \mathbb{C}^n$ and $A = (a_{ij})$, then λ is an eigenvalue if and only if $\det(\lambda \mathbb{1} - A) = 0$.

Residual spectrum of an operator

Definition

Residual spectrum is the set of all numbers $\lambda \in \mathbb{C}$ such that A_λ is invertible and $\mathcal{R}(A_\lambda)$ is not dense in H.

Example

Let $H = \ell_2$ and let A be defined as

$$A(a_1, a_2, a_3, \ldots) = (0, a_1, a_2, a_3, \ldots),$$

and $\mathcal{D}(A) = \ell_2$. Assume that $Ax = \lambda x$ for some $x = (a_1, a_2, \ldots) \in \ell_2$. Then

$$(0, a_1, a_2, a_3, \ldots) = (\lambda a_1, \lambda a_2, \lambda a_3, \ldots)$$

Observe that this equation is satisfied only if $a_1 = a_2 = a_3 = \ldots = 0$. Hence, $\sigma_{\mathrm{p}}(A) = \emptyset$. On the other hand, the operator $A_0 = -A$ is invertible and the range

$$\mathcal{R}(A) = \{(a_1, a_2, a_3, \ldots) \in \ell_2 : \ a_1 = 0\}$$

is not dense in ℓ_2. So, $0 \in \sigma_{\mathrm{r}}(A)$.

Continuous spectrum of an operator

Definition

Continuous spectrum is the set of all numbers $\lambda \in \mathbb{C}$ such that A_λ is invertible, $\mathcal{R}(A_\lambda)$ is dense in H and A_λ^{-1} is unbounded.

Example

Let $H = L_2(\mathbb{R})$ and Q be the position operator, i.e.

$$\mathcal{D}(Q) = \{x \in L_2(\mathbb{R}) : \int_{\mathbb{R}} t^2 |x(t)|^2 \mathrm{d}t < \infty\}, \quad Qx(t) = tx(t).$$

We will show that $\mathbb{R} \subset \sigma_c(Q)$. Let $\lambda \in \mathbb{R}$. Then $Q_\lambda x(t) = (\lambda - t)x(t)$ for $x \in \mathcal{D}(Q)$.

- Q_λ is invertible.

$$Q_\lambda x = 0 \Leftrightarrow (\lambda - t)x(t) = 0 \text{ a.e.} \Leftrightarrow x(t) = 0 \text{ a.e.} \Leftrightarrow x = 0.$$

Continuous spectrum of an operator

Definition

Continuous spectrum is the set of all numbers $\lambda \in \mathbb{C}$ such that A_λ is invertible, $\mathcal{R}(A_\lambda)$ is dense in H and A_λ^{-1} is unbounded.

Example

We will show that $\mathbb{R} \subset \sigma_c(Q)$. Let $\lambda \in \mathbb{R}$. Then $Q_\lambda x(t) = (\lambda - t)x(t)$ for $x \in \mathcal{D}(Q)$.

- $\mathcal{R}(Q_\lambda)$ is dense in $L_2(\mathbb{R})$. Let $y \in L_2(\mathbb{R})$. For $\varepsilon > 0$ let $E_\varepsilon = \{t \in \mathbb{R} : |t| \geq \varepsilon\}$ and

$$x_\varepsilon(t) = \frac{y(t)}{\lambda - t}\chi_{E_\varepsilon}(\lambda - t).$$

Then $x_\varepsilon \in L_2(\mathbb{R})$ because

$$\int_{\mathbb{R}} |x_\varepsilon(t)|^2 \mathrm{d}t = \int_{\{|\lambda - t| \geq \varepsilon\}} \frac{|y(t)|^2}{(\lambda - t)^2}\mathrm{d}t \leq \frac{1}{\varepsilon^2}\int_{\mathbb{R}} |y(t)|^2 \mathrm{d}t < \infty.$$

$$\|y - Q_\lambda x_\varepsilon\|^2 = \int_{\mathbb{R}} |y(t) - y(t)\chi_{E_\varepsilon}(\lambda - t)|^2 \mathrm{d}t = \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} |y(t)|^2 \mathrm{d}t \xrightarrow[\varepsilon \to 0]{} 0.$$

Continuous spectrum of an operator

Definition

Continuous spectrum is the set of all numbers $\lambda \in \mathbb{C}$ such that A_λ is invertible, $\mathcal{R}(A_\lambda)$ is dense in H and A_λ^{-1} is unbounded.

Example

We will show that $\mathbb{R} \subset \sigma_c(Q)$. Let $\lambda \in \mathbb{R}$. Then $Q_\lambda x(t) = (\lambda - t)x(t)$ for $x \in \mathcal{D}(Q)$.

- Q_λ^{-1} is unbounded. Let $y_n(t) = \chi_{[\frac{1}{n},1]}(\lambda - t)$ and $x_n(t) = \dfrac{y_n(t)}{\lambda - t}$ for $n \in \mathbb{N}$. Then $x_n \in \mathcal{D}(Q)$, hence $y_n \in \mathcal{R}(Q_\lambda)$. Moreover

$$\|y_n\|^2 = \int_{\lambda-1}^{\lambda-\frac{1}{n}} \mathrm{d}t = 1 - \frac{1}{n} \le 1$$

$$\|Q_\lambda^{-1} y_n\|^2 = \int_{\mathbb{R}} \left|\frac{y_n(t)}{\lambda - t}\right|^2 \mathrm{d}t = \int_{\lambda-1}^{\lambda-\frac{1}{n}} \frac{1}{(\lambda-t)^2}\mathrm{d}t = n - 1 \underset{n\to\infty}{\longrightarrow} \infty$$

Hence $\lambda \in \sigma_c(Q)$.

Spectrum and resolvent set

Definition

Spectrum of a linear operator A is the set

$$\sigma(A) = \sigma_{\mathrm{p}}(A) \cup \sigma_{\mathrm{r}}(A) \cup \sigma_{\mathrm{c}}(A).$$

Definition

Resolvent set of A is the set $\varrho(A) = \mathbb{C} \setminus \sigma(A)$.

Remark

It follows from the definition that

$$\varrho(A) = \{\lambda \in \mathbb{C} : A_\lambda \text{ is invertible, } \mathcal{R}(A_\lambda) \text{ is dense in } H,\ A_\lambda^{-1} \text{ is bounded}\}$$

Definition

We define the **resolvent** of A as the family of bounded operators

$$R(\lambda, A) = A_\lambda^{-1}, \quad \lambda \in \varrho(A).$$

Properties of the resolvent

Theorem (Hilbert equation)

If A is a closed operator, then

$$R(\lambda, A) - R(\mu, A) = -(\lambda - \mu) R(\lambda, A) R(\mu, A), \qquad \lambda, \mu \in \varrho(A).$$

Proof.

From the definition of the resolvent the following hold for any $x \in H$

$$R(\lambda, A)(\lambda \mathbb{1} - A) R(\mu, A) x = R(\mu, A) x, \quad R(\lambda, A)(\mu \mathbb{1} - A) R(\mu, A) x = R(\lambda, A) x$$

$$\begin{aligned} R(\lambda, A) x &= R(\lambda, A)(\mu \mathbb{1} - A) R(\mu, A) x \\ &= R(\lambda, A)(-(\lambda - \mu)\mathbb{1} + \lambda \mathbb{1} - A) R(\mu, A) x \\ &= -(\lambda - \mu) R(\lambda, A) R(\mu, A) x + R(\lambda, A)(\lambda \mathbb{1} - A) R(\mu, A) x \\ &= -(\lambda - \mu) R(\lambda, A) R(\mu, A) x + R(\mu, A) x \end{aligned}$$

□

Properties of the resolvent

Theorem

If A is a closed operator and $\dim H > 0$ then for every $\lambda_0 \in \varrho(A)$ we have

$$O_{\lambda_0} := \left\{ \lambda \in \mathbb{C} : \, |\lambda - \lambda_0| < \frac{1}{\|R(\lambda_0, A)\|} \right\} \subset \varrho(A).$$

Moreover, for every $\lambda \in O_{\lambda_0}$

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1},$$

where the series is uniformly convergent.

Properties of the resolvent

Proof

Let $\lambda_0 \in \varrho(A)$. Assume that $\lambda \in O_{\lambda_0}$ i.e. $|\lambda - \lambda_0| \|R(\lambda_0, A)\| \le q$ for some $q < 1$.

$$\delta_n := \left\|(\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1}\right\| \le |\lambda - \lambda_0|^n \|R(\lambda_0, A)\|^{n+1} \le q^n \|R(\lambda_0, A)\|$$

$$\sum_{n=0}^{\infty} \delta_n < \infty \;\Rightarrow\; \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1} \text{ is convergent uniformly to } S(\lambda) \in \mathfrak{L}(H)$$

We will show that $S(\lambda) = (\lambda \mathbb{1} - A)^{-1}$.

$$S(\lambda)(\lambda \mathbb{1} - A)x = \lim_{N\to\infty} \sum_{n=0}^{N} (\lambda_0 - \lambda) R^{n+1} (\lambda \mathbb{1} - A)x, \quad x \in \mathcal{D}(A)$$

$$R^{n+1}(\lambda \mathbb{1} - A)x = R^{n+1}((\lambda_0 \mathbb{1} - A)x + (\lambda - \lambda_0)x) = R^n x + (\lambda - \lambda_0) R^{n+1} x$$

$$\begin{aligned}\sum_{n=0}^{N} (\lambda_0 - \lambda)^n R^{n+1} (\lambda \mathbb{1} - A)x &= \sum_{n=0}^{N} \left[(\lambda_0 - \lambda)^n R^n x - (\lambda_0 - \lambda)^{n+1} R^{n+1}\right] \\ &= x - (\lambda_0 - \lambda)^{N+1} R^{N+1} x\end{aligned}$$

Properties of the resolvent

Proof

$$\left\|(\lambda_0-\lambda)^{N+1}R^{N+1}x\right\| \leq q^{N+1}\|x\| \underset{N\to\infty}{\longrightarrow} 0 \;\Rightarrow$$

$$\Rightarrow \quad S(\lambda)(\lambda\mathbb{1}-A)x = \lim_{N\to\infty}\left(x-(\lambda_0-\lambda)^{N+1}R^{N+1}x\right) = x$$

Thus we proved that $S(\lambda)(\lambda\mathbb{1}-A)x = x$ for $x\in\mathcal{D}(A)$.

Now, let $x\in H$ and define

$$x_N = \sum_{n=0}^{N}(\lambda_0-\lambda)^n R^{n+1}x = R\sum_{n=0}^{N}(\lambda_0-\lambda)^n R^n x \qquad x_N\in\mathcal{R}(R)=\mathcal{D}(A).$$

$$\begin{aligned}(\lambda\mathbb{1}-A)x_N &= [\lambda_0\mathbb{1}-A+(\lambda-\lambda_0)\mathbb{1}]\,R\sum_{n=0}^{N}(\lambda_0-\lambda)^n R^n x\\ &= \sum_{n=0}^{N}(\lambda_0-\lambda)^n R^n x-\sum_{n=0}^{N}(\lambda_0-\lambda)^{n+1}R^{n+1}x\\ &= x-(\lambda_0-\lambda)^{N+1}R^{N+1}x \underset{N\to\infty}{\longrightarrow} x\end{aligned}$$

Properties of the resolvent

Proof.

$\lambda\mathbb{1} - A$ is closed, so

$$x_N \to S(\lambda)x \;\wedge\; (\lambda\mathbb{1} - A)x_N \to x \quad\Rightarrow\quad S(\lambda)x \in \mathcal{D}(A) \;\wedge\; (\lambda\mathbb{1} - A)S(\lambda)x = x.$$

We showed that

$$S(\lambda)(\lambda\mathbb{1} - A)x = x,\ x \in \mathcal{D}(A) \quad \text{and} \quad (\lambda\mathbb{1} - A)S(\lambda)x = x,\ x \in H.$$

Therefore,

$$S(\lambda) = (\lambda\mathbb{1} - A)^{-1} \qquad \text{and} \qquad \lambda \in \varrho(A).$$

□

Corollary

If A is a closed operator, then $\varrho(A)$ is an open subset of $\mathbb{C}$. Consequently, $\sigma(A)$ is a closed subset of the complex plane.

Spectrum of bounded operator

Remark

There are examples of closed operators such that one of sets $\varrho(A)$ and $\sigma(A)$ is empty.

Theorem

Assume that $A \in \mathfrak{L}(H)$. If $|\lambda| > \|A\|$, then $\lambda \in \varrho(A)$.

Proof.

Define

$$B_\lambda = \frac{1}{\lambda} A_\lambda = \mathbb{1} - \frac{1}{\lambda} A.$$

$$|\lambda| > \|A\| \;\Rightarrow\; \|\mathbb{1} - B_\lambda\| = \left\| \frac{1}{\lambda} A \right\| < 1 \;\Rightarrow\; \sum_{n=0}^{\infty} (\mathbb{1} - B_\lambda)^n \text{ is convergent uniformly}$$

$$\sum_{n=0}^{\infty} (\mathbb{1} - B_\lambda)^n \in \mathfrak{L}(H) \quad \text{and} \quad \sum_{n=0}^{\infty} (\mathbb{1} - B_\lambda)^n = B_\lambda^{-1} = \lambda A_\lambda^{-1}$$

□

Spectrum of bounded operator

Lemma

If $A \in \mathfrak{L}(H)$ *then* $\lim\limits_{\lambda\to\infty} \|R(\lambda, A)\| = 0.$

Proof.

$$B_\lambda^{-1} = \sum_{n=0}^{\infty} (\mathbb{1} - B_\lambda)^n = \mathbb{1} + \sum_{n=1}^{\infty} (\mathbb{1} - B_\lambda)^n$$

$$\|B_\lambda^{-1}\| \le 1 + \sum_{n=1}^{\infty} \|\mathbb{1} - B_\lambda\|^n = 1 + \frac{\|\mathbb{1} - B_\lambda\|}{1 - \|\mathbb{1} - B_\lambda\|}$$

$$\|R(\lambda, A)\| = \|A_\lambda^{-1}\| = \frac{1}{|\lambda|}\|B_\lambda^{-1}\| \le \frac{1}{|\lambda|}\left(1 + \frac{\frac{1}{|\lambda|}\|A\|}{1 - \frac{1}{|\lambda|}\|A\|}\right) \xrightarrow[|\lambda|\to\infty]{} 0$$

□

Spectrum of bounded operator

Theorem

If $\dim H > 0$ *and* $A \in \mathfrak{L}(H)$ *then* $\sigma(A)$ *is nonempty.*

Proof.

For fixed $\lambda_0 \in \varrho(A)$ and for $\lambda \in O_{\lambda_0}$ we have

$$R(\lambda, A) = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n R(\lambda_0, A)^{n+1}.$$

$$f_{x,y}(\lambda) := \langle x, R(\lambda, A)y\rangle = \sum_{n=0}^{\infty} (\lambda_0 - \lambda)^n \langle x, R(\lambda_0, A)^{n+1} y\rangle, \qquad x, y \in H$$

$f_{x,y}$ is an analytical function. Suppose that $\sigma(A) = \emptyset$. Then $f_{x,y}$ is an entire function.

$$|f_{x,y}(\lambda)| = |\langle x, R(\lambda, A)y\rangle| \leq \|R(\lambda, A)\| \|x\| \|y\| \underset{|\lambda|\to\infty}{\longrightarrow} 0.$$

$f_{x,y}$ is bounded. By Liouville theorem, $f_{x,y}$ is constatnt, and $f_{x,y} \equiv 0$. Thus $R(\lambda, A)x = 0$ for every $x \in H$ and $\lambda \in \mathbb{C}$. It contradicts invertibility of $R(\lambda, A)$. □

Spectrum of bounded operator

Corollary

If $A \in \mathfrak{L}(H)$, then $\sigma(A)$ is a nonempty compact set contained in the circle $\{\lambda \in \mathbb{C} : |\lambda| \leq \|A\|\}$.

Contents

Bilinear forms

Let $\mathcal{D} \subset H$ be a linear subspace.

Definition

A map $f : H \times \mathcal{D} \to \mathbb{C}$ is called a **bilinear form** if

$$f(x, \alpha y + \beta z) = \alpha f(x,y) + \beta f(x,z), \quad x \in H,\ y,z \in \mathcal{D},\ \alpha,\beta \in \mathbb{C}$$

$$f(\alpha x + \beta y, z) = \overline{\alpha} f(x,z) + \overline{\beta} f(y,z), \quad x,y \in H,\ z \in \mathcal{D},\ \alpha,\beta \in \mathbb{C}$$

Remark

Let A be a linear operator. The map $f_A : H \times \mathcal{D}(A) \to \mathbb{C}$ defined as $f_A(x,y) = \langle x, Ay \rangle$ for $x \in H$ and $y \in \mathcal{D}(A)$, is a bilinear form.
Moreover, f_A uniquely determines A, i.e. if $f_A = f_B$ for some operators A and B, then $A = B$.

Operators from bilinear forms

Theorem

Let $\mathcal{D} \subset H$ *be a subspace and let* $f : H \times \mathcal{D} \to \mathbb{C}$ *be a bilinear form. Assume*

$$\forall y \in \mathcal{D} \ \exists m(y) \geq 0 \ \forall x \in H : \ |f(x,y)| \leq m(y)\|x\|.$$

Then there is the unique operator A *such that* $\mathcal{D}(A) = \mathcal{D}$ *and* $\langle x, Ay \rangle = f(x,y)$ *for all* $x \in H$ *and* $y \in \mathcal{D}$.

Operators from bilinear forms

Proof.

For $y \in \mathcal{D}$ define $\xi_y : H \to \mathbb{C}$ by $\xi_y(x) = \overline{f(x,y)}$. ξ_y is a linear functional on H. Moreover, $|\xi_y(x)| \le m(y)\|x\|$, hence ξ_y is continuous. Due to Riesz representation theorem there is uniquely determined $\bar{y} \in H$ such that $\xi_y(x) = \langle \bar{y}, x\rangle$ for $x \in H$. Let $Ay = \bar{y}$ for $y \in \mathcal{D}$. Then

$$f(x,y) = \overline{\xi_y(x)} = \langle x, \bar{y}\rangle = \langle x, Ay\rangle, \quad x \in H,\ y \in \mathcal{D}.$$

For any $x \in H$, $y, z \in \mathcal{D}$ and $\alpha, \beta \in \mathbb{C}$ we have

$$\begin{aligned}\langle x, A(\alpha y + \beta z)\rangle &= f(x, \alpha y + \beta z) = \alpha f(x,y) + \beta f(x,z) \\ &= \alpha\langle x, Ay\rangle + \beta\langle x, Az\rangle = \langle x, \alpha Ay + \beta Az\rangle\end{aligned}$$

Since it holds for any x, we conclude that

$$A(\alpha y + \beta z) = \alpha Ay + \beta Az.$$

Therefore, A is linear. □

Bounded bilinear forms

Definition

A bilinear form $f : H \times \mathcal{D} \to \mathbb{C}$ is called **bounded** if there is a constant $m \geq 0$ such that

$$|f(x, y)| \leq m\|x\|\|y\| \qquad \text{for all } x \in H,\ y \in \mathcal{D}.$$

The norm of f is defined as

$$f = \inf\{m \geq 0 :\ |f(x, y)| \leq m\|x\|\|y\| \text{ for } x \in H,\ y \in \mathcal{D}\}$$

Remark

Observe that a bounded bilinear form satisfies condition from the previous theorem.
Hence there is the unique operator A such that $\mathcal{D}(A) = \mathcal{D}$ and $\langle x, Ay\rangle = f(x, y)$ for $x \in H$ and $y \in \mathcal{D}$.
Due to continuity of bounded operators, A can be extended to the unique operator from $\mathcal{L}(H)$.

Theorem

There is a linear isomorphism $A \mapsto f_A$ from $\mathcal{L}(H)$ to bounded bilinear forms on $H \times H$, such that $\langle x, Ay\rangle = f_A(x, y)$ for $x, y \in H$. Moreover, it is isometry, i.e. $\|f_A\| = \|A\|$.

Contents

Adjoint linear operator

Let A be a linear operator and assume $\overline{\mathcal{D}(A)} = H$. For $x \in H$ define

$$\xi_x : \mathcal{D}(A) \to \mathbb{C}, \qquad \xi_x(y) = \langle x, Ay \rangle, \quad y \in \mathcal{D}(A)$$

$$\mathcal{D} := \{x \in H : \xi_x \text{ is a continuous functional}\}$$

Proposition

There is the unique linear operator B such that $\mathcal{D}(B) = \mathcal{D}$ and $\langle Bx, y \rangle = \langle x, Ay \rangle$ for $x \in \mathcal{D}$ and $y \in \mathcal{D}(A)$.

Proof.

Since $\mathcal{D}(A)$ is dense in H, the functional ξ_x has the unique continuous extension $\tilde{\xi}_x$ to the whole H for any $x \in \mathcal{D}$. Hence, there are constants $m(x)$ such that $|\tilde{\xi}_x(y)| \leq m(x)\|y\|$ for $y \in H$.

$$f(y,x) := \overline{\tilde{\xi}_x(y)} \ \Rightarrow \ f \text{ is bilinear}, \quad |f(y,x)| \leq m(x)\|y\|, \ \ x \in \mathcal{D},\ y \in H.$$

$$\exists\, B : \mathcal{D} \to H \ \forall\, x \in \mathcal{D},\ y \in \mathcal{D}(A) :\ \langle y, Bx \rangle = f(y,x) = \overline{\tilde{\xi}_x(y)} = \overline{\langle x, Ay \rangle} = \langle Ay, x \rangle.$$

□

Adjoint linear operator

Definition

The operator B is called the **adjoint operator** and is denoted by A^*.

$$\langle A^*x, y\rangle = \langle x, Ay\rangle, \qquad x \in \mathcal{D}(A^*),\ y \in \mathcal{D}(A)$$
$$\mathcal{D}(A^*) = \{x \in H : \text{there is } z \in H \text{ such that } \langle x, Ay\rangle = \langle z, y\rangle \text{ for } y \in \mathcal{D}(A)\}$$

Exercise

Show that
$$(\alpha A)^* = \overline{\alpha} A^*, \quad A^* + B^* \subset (A+B)^*, \quad B^*A^* \subset (AB)^*$$

Proposition

If A^* exists, then it is closed operator.

Proof.

Exercise. *Hint:* Show that $\Gamma(A^*)$ is the orthogonal complement of $U\overline{\Gamma(A)}$, where $U : H \oplus H \to H \oplus H : (x, y) \mapsto (y, -x)$. □

Symmetric and selfadjoint operators

Definition

We say that a densely defined operator A is **symmetric** if $A \subset A^*$, i.e.

$$\langle Ax, y\rangle = \langle x, Ay\rangle, \qquad x, y \in \mathcal{D}(A).$$

A is called **selfadjoint** if $A = A^*$.

Exercise

Let $H = L_2(a,b)$. Define an operator A by conditions

$$\mathcal{D}(A) = \left\{ x \in L_2(a,b) : \quad \begin{array}{l} x \text{ is absolutely continuous in } [a,b] \\ x' \in L_2(a,b) \\ x(a) = x(b) = 0 \end{array} \right\}$$

$$(Ax)(t) = ix'(t), \qquad x \in \mathcal{D}(A)$$

Show that A is symmetric but it isn't selfadjoint.

Spectra of symmetric operators

Lemma

If A is a symmetric operator and $\lambda \in \mathbb{C} \setminus \mathbb{R}$ then $\lambda \mathbb{1} - A$ is invertible, $(\lambda \mathbb{1} - A)^{-1}$ is bounded and

$$\|(\lambda \mathbb{1} - A)^{-1}\| \leq \frac{1}{|\mathrm{Im}\lambda|}.$$

Proof

$$\begin{aligned}
\|(\lambda \mathbb{1} - A)x\|^2 &= \langle (\lambda \mathbb{1} - A)x, (\lambda \mathbb{1} - A)x \rangle \\
&= |\lambda|^2 \|x\|^2 - \langle \lambda x, Ax \rangle - \langle Ax, \lambda x \rangle + \langle Ax, Ax \rangle \\
&= |\lambda|^2 \|x\|^2 - 2(\mathrm{Re}\lambda)\langle x, Ax \rangle + \langle Ax, Ax \rangle \\
&= (\mathrm{Im}\lambda)^2 \|x\|^2 + (\mathrm{Re}\lambda)^2 \|x\|^2 - 2(\mathrm{Re}\lambda)\langle x, Ax \rangle + \langle Ax, Ax \rangle \\
&= (\mathrm{Im}\lambda)^2 \|x\|^2 + \|((\mathrm{Re}\lambda)\mathbb{1} - A)x\|^2 \geq (\mathrm{Im}\lambda)^2 \|x\|^2.
\end{aligned}$$

Spectra of symmetric operators

Theorem

If A is symmetric, then $\sigma_{\mathrm{p}}(A) \cup \sigma_{\mathrm{c}}(A) \subset \mathbb{R}$.

Spectra of selfadjoint operators

Remark

It follows from the previous theorem that the spectrum of a symmetric operator can contain nonreal numbers in its residual part.
In next theorem we will show that for selfadjoint operators it is not the case.

Theorem

If A is selfadjoint then $\sigma_{\mathrm{r}}(A) = \emptyset$.

Proof

If B is a linear operator, then we define its null subspace $\mathcal{N}(B)$ as

$$\mathcal{N}(B) = \{x \in \mathcal{D}(B) :\ Bx = 0\}.$$

If B is densely defined then (exercise)

$$\overline{\mathcal{R}(B)} \oplus \mathcal{N}(B^*) = H.$$

Spectra of selfadjoint operators

Proof.

$$B = \lambda \mathbb{1} - A \quad \Rightarrow \quad \overline{\mathcal{R}(\lambda \mathbb{1} - A)} \oplus \mathcal{N}(\bar{\lambda}\mathbb{1} - A^*) = H$$

For any densely defined A we have $\sigma_{\mathrm{r}}(A)^* \subset \sigma_{\mathrm{p}}(A^*)$:

$$\lambda \in \sigma_{\mathrm{r}}(A) \quad \Rightarrow \quad \overline{\mathcal{R}(\lambda \mathbb{1} - A)} \neq H \ \Rightarrow \ \mathcal{N}(\bar{\lambda}\mathbb{1} - A^*) \neq \{0\} \ \Rightarrow$$
$$\Rightarrow \quad \bar{\lambda}\mathbb{1} - A^*) \text{ is not invertible } \Rightarrow \ \bar{\lambda} \in \sigma_{\mathrm{p}}(A^*)$$

$$A = A^* \ \Rightarrow \ \sigma_{\mathrm{r}}(A)^* \subset \sigma_{\mathrm{p}}(A)$$

Remind that $\sigma_{\mathrm{p}}(A) \subset \mathbb{R}$. Hence $\sigma_{\mathrm{r}}(A) \subset \sigma_{\mathrm{p}}(A) \ \Rightarrow \ \sigma_{\mathrm{r}}(A) = \emptyset$. □

Spectra of selfadjoint operators

Theorem

Let A be a symmetric operator. If x_1 and x_2 are eigenvector of A corresponding to different eigenvalues, then x_1 and x_2 are orthogonal.

Proof

Exercise.

Contents

Spectral measure

By $\mathcal{B}(\mathbb{C})$ we denote the σ-field of Borel subsets of $\mathbb{C}$.

Definition

Spectral resolution of identity in a Hilbert space H is a map which to every $M \in \mathcal{B}(\mathbb{C})$ assigns an orthogonal projection $E(M)$ on H such that the following conditions are satisfied:

1. $E(\emptyset) = 0$ and $E(\mathbb{C}) = \mathbb{1}$,
2. $E(M \cap N) = E(M)E(N)$ for $M, N \in \mathcal{B}(\mathbb{C})$,
3. $E\left(\bigcup_{k=1}^{\infty} M_k\right) = \sum_{k=1}^{\infty} E(M_k)$ for mutually disjoint subsets $M_1, M_2, \ldots \in \mathcal{B}(\mathbb{C})$, where the series is strongly convergent.

Projections $E(M)$ are called **spectral projections**.

Definition

Support of a spectral measure E is a set

$$\mathrm{supp} E = \bigcap \{\bar{M} : \; M \in \mathcal{B}(\mathbb{C}), \; E(M) = \mathbb{1}\}.$$

Family of spectral measures

If E is a spectral resolution of identity, then for any $x, y \in H$ define

$$\mu_{x,y}(M) = \langle x, E(M)y \rangle, \qquad M \in \mathcal{B}(\mathbb{C})$$

Proposition

If $x, y \in H$ then $\mu_{x,y}$ is a complex measure on $\mathcal{B}(\mathbb{C})$. If $x = y$ then $\mu_x = \mu_{x,x}$ is positive probability measure.

Proof.

Easy exercise. □

If μ is a (complex) measure, then $\|\mu\|$ will denote its total variation.

Family of spectral measures

Proposition

$$\|\mu_{x,y}\| \leq \|x\|\|y\|, \qquad \|\mu_x\| = \|x\|^2$$

.

Proof

$$\|\mu_{x,y}\| = \sup\left\{\sum_{k=1}^{N} |\mu_{x,y}(M_k)| : \ M_1, \ldots, M_N \in \mathcal{B}(\mathbb{C}), \text{ mutually disjoint}, \bigcup_{k=1}^{N} M_k = \mathbb{C}\right\}$$

$$\begin{aligned}
\sum_k |\mu_{x,y}(M_k)| &= \sum_k |\langle x, E(M_k)y\rangle| = \sum_k |\langle E(M_k)x, E(M_k)y\rangle| \\
&\leq \sum_k \|E(M_k)x\|\|E(M_k)y\| \\
&\leq \left(\sum_k \|E(M_k)x\|^2\right)^{\frac{1}{2}} \left(\sum_k \|E(M_k)y\|^2\right)^{\frac{1}{2}}
\end{aligned}$$

Family of spectral measures

Proof.

$$\begin{aligned}\sum_k \|E(M_k)x\|^2 &= \sum_k \langle E(M_k)x, E(M_k)x\rangle = \sum_k \langle x, E(M_k)x\rangle \\ &= \left\langle x, E\left(\bigcup_k M_k\right) x \right\rangle = \langle x, x\rangle = \|x\|^2\end{aligned}$$

$$\sum_k |\mu_{x,y}(M_k)| \le \|x\|\|y\|$$

μ_x is positive $\quad\Rightarrow\quad \|\mu_x\| = \mu_x(\mathbb{C}) = \langle x, E(\mathbb{C})x\rangle = \langle x, x\rangle = \|x\|^2.$

□

Family of spectral measures

By $B^\infty(\mathbb{C})$ we denote the set of all complex bounded Borel functions on $\mathbb{C}$.

Proposition

The family of spectral measures $\mu_{x,y}$ satisfies the following conditions:

SM1 $\mu_{x,\alpha y+\beta z} = \alpha\mu_{x,y} + \beta\mu_{x,z}$ *for any* $x, y, z \in H$ *and* $\alpha, \beta \in \mathbb{C}$,

SM2 *If* $\varphi \in B^\infty(\mathbb{C})$, *then*

$$\overline{\int \varphi(\lambda)\mathrm{d}\mu_{x,y}(\lambda)} = \int \overline{\varphi(\lambda)}\mathrm{d}\mu_{y,x}(\lambda)$$

SM3 $|\mu_{x,y}(M)| \leq \|x\|(\mu_x(M))^{\frac{1}{2}}$ *for* $x, y \in H$ *and* $M \in \mathcal{B}(\mathbb{C})$,

$$\mathrm{supp} E = \overline{\bigcup_{x,y\in H} \mathrm{supp}\,\mu_{x,y}} = \overline{\bigcup_{x\in H} \mathrm{supp}\,\mu_x}$$

Proof.

Easy exercise. □

Family of spectral measures

Definition

The set $\{\mu_{x,y} : x, y \in H\}$ of measures is called a **family of semi-spectral measures** if it satisfies conditions SM1, SM2 and SM3 from the previous proposition.

Contents

Spectral integrals

Proposition

If $\{\mu_{x,y}\}$ is a family of semi-spectral measures, then for every $\varphi \in B^\infty(\mathbb{C})$ there is $A_\varphi \in \mathcal{L}(H)$ such that

$$\langle x, A_\varphi y\rangle = \int \varphi(\lambda)\mathrm{d}\mu_{x,y}\lambda, \quad x, y \in H.$$

Proof

Let us define $f : H \times H \to \mathbb{C}$ by $f(x,y) = \int \varphi(\lambda)\mathrm{d}\mu_{x,y}(\lambda)$.
f is a bounded bilinear form:

$$f(x, \alpha y + \beta z) = \int \varphi \mathrm{d}\mu_{x,\alpha y+\beta z} = \alpha \int \varphi \mathrm{d}\mu_{x,y} + \beta \int \varphi \mathrm{d}\mu_{x,z} = \alpha f(x,y) + \beta f(x,z)$$

$$\begin{aligned} f(\alpha x + \beta y, z) &= \int \varphi \mathrm{d}\mu_{\alpha x+\beta y,z} = \overline{\int \overline{\varphi}\, \mathrm{d}\mu_{z,\alpha x+\beta y}} = \overline{\alpha}\overline{\int \overline{\varphi}\, \mathrm{d}\mu_{z,x}} + \overline{\beta}\overline{\int \overline{\varphi}\, \mathrm{d}\mu_{z,y}} \\ &= \overline{\alpha}\int \varphi\, \mathrm{d}\mu_{x,z} + \overline{\beta}\int \varphi\, \mathrm{d}\mu_{y,z} = \overline{\alpha} f(x,z) + \overline{\beta} f(y,z) \end{aligned}$$

Spectral integrals

Proof.

$$|f(x,y)| = \left|\int \varphi(\lambda)\mathrm{d}\mu_{x,y}(\lambda)\right| \leq \|\varphi\|_\infty \|\mu_{x,y}\| \leq \|\varphi\|_\infty \|x\|\|y\|$$

Thus there is the unique $A_\varphi \in \mathcal{L}(H)$ such that

$$\langle x, A_\varphi y\rangle = f(x,y) = \int \varphi(\lambda)\mathrm{d}\mu_{x,y}(\lambda)$$

□

Definition

Let $\mu_{x,y}$ be a family of semi-spectral measures. For $M \in \mathcal{B}(\mathbb{C})$ let $E(M) = A_{\chi_M}$. We say that $\mu_{x,y}$ is a family of spectral measures if

SM4 $\mu_{x,E(M)y}(N) = \mu_{x,y}(M \cap N)$ for every $x, y \in H$ and $M, N \in \mathcal{B}(\mathbb{C})$.

Spectral integrals

Definition

We say the sequence $\varphi_n \in B^\infty(\mathbb{C})$ **converges in a bounded way** to $\varphi \in B^\infty(\mathbb{C})$ if $\varphi_n(\lambda) \to \varphi(\lambda)$ for every $\lambda \in \mathbb{C}$ and there $\psi \in B^\infty(\mathbb{C})$ such that $|\varphi_n(\lambda)| \leq \psi(\lambda)$ for every $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$.

Proposition

If $\varphi_n \to \varphi$ in a bounded way, then $A_{\varphi_n} \rightharpoonup A_\varphi$ (weakly).

Proof.

Exercise □

Spectral integrals

Proposition

Let $\mu_{x,y}$ be a family of spectral measures and $\varphi, \psi \in B^\infty(\mathbb{C})$. Then

$$A_{\varphi\psi} = A_\varphi A_\psi, \qquad A_\varphi^* = A_{\overline{\varphi}}.$$

Proof.

The second equality follows directly from SM2. We will prove the first.
Assume that $\varphi = \chi_M$ for $M \in \mathcal{B}(\mathbb{C})$. Then $A_\varphi = E(M)$. It follows from second equality that $E(M)$ is selfadjoint.

$$\begin{aligned} \langle x, A_\varphi A_\psi y\rangle &= \langle x, E(M)A_\psi y\rangle = \langle E(M)x, A_\psi y\rangle = \int \psi \mathrm{d}\mu_{E(M)x,y} \\ &= \overline{\int \overline{\psi} \mathrm{d}\mu_{y,E(M)x}} = \overline{\int \chi_M \overline{\psi} \mathrm{d}\mu_{y,x}} = \int \chi_M \psi \mathrm{d}\mu_{x,y} = \langle x, A_{\varphi\psi} y\rangle \end{aligned}$$

Hence the first equality is satisfied for all simple functionsand by the limit procedure, for all functions from $B^\infty(\mathbb{C})$. □

Spectral integrals

Corollary

If $\varphi_n \to \varphi$ in a bounded way then $A_{\varphi_n} \to A_\varphi$ strongly, i.e. $\lim_{n\to\infty} A_{\varphi_n} x = A_\varphi x$ for every $x \in H$.

Proof.

We can assume that $\varphi = 0$.

$$\|A_{\varphi_n} x\|^2 = \langle x, A^*_{\varphi_n} A_{\varphi_n} x\rangle = \langle x, A_{|\varphi_n|^2} x\rangle \xrightarrow[n\to\infty]{} 0$$

because of weak convergence. □

Spectral integrals

Theorem

There is a one to one correspondence between spectral resolutions of identity and families of spectral measures. Moreover, we have the following equality of supports

$$\operatorname{supp} E = \overline{\bigcup_{x,y\in H} \operatorname{supp} \mu_{x,y}} = \overline{\bigcup_{x\in H} \operatorname{supp} \mu_x}$$

Proof

We have shown that if $\mu_{x,y}$ are maesures determined by a resolution of identity E, then it is a family of spectral measures.
Assume now that $\mu_{x,y}$ is a family of spectral measures and define $E(M) = A_{\chi_M}$ for $M \in \mathcal{B}(\mathbb{C})$. We will show that E is a resolution of identity.

$$E(M)^2 = A_{\chi_M}^2 = A_{\chi_M^2} = A_{\chi_M} = E(M) \Rightarrow E(M) \text{ is a projection}$$

$$E(M \cap N) = A_{\chi_{M\cap N}} = A_{\chi_M \chi_N} = A_{\chi_M} A_{\chi_N} = E(M)E(N).$$

Spectral integrals

Proof.

For $M_1, M_2, \ldots \in \mathcal{B}(\mathbb{C})$ are mutually disjoint define $N_k = M_1 \cup \ldots \cup M_k$ and $N = \bigcup_{n=1}^{\infty} M_n$. Then $\chi_{N_k} \to \chi_N$ in a bounded way. Hence

$$\sum_{n=1}^{\infty} E(M_n) = \lim_{k\to\infty} E(N_k) = E(N) = E\left(\bigcup_{n=1}^{\infty} M_n\right) \text{ strongly.}$$

□

Remark

If E is a spectral resolution of identity and $\varphi \in B^{\infty}(\mathbb{C})$ then we use the following notation

$$\int \varphi(\lambda)\mathrm{d}E(\lambda) := A_{\varphi}.$$

Contents

Mapping of spectra

Let $A \in \mathcal{L}(H)$. Given a polynomial $w(s) = \sum_{k=0}^{n} a_k s^k$ we define an operator $w(A) = \sum_{k=0}^{n} a_k A^k$.

Theorem

$$w(\sigma(A)) = \sigma(w(A))$$

Proof.

Assume $\deg w = n \geq 1$ and fix $\lambda_0 \in \mathbb{C}$. Let $\lambda_1, \dots, \lambda_n$ be roots of $u(s) = \lambda_0 - w(s)$, i.e.

$$u(s) = \lambda_0 - w(s) = a(\lambda_1 - s)\dots(\lambda_n - s), \qquad a \neq 0.$$

$$u(A) = \lambda_0 \mathbb{1} - w(A) = a(\lambda_1 \mathbb{1} - A)\dots(\lambda_n \mathbb{1} - A).$$

$$\lambda_0 \in \sigma(w(A)) \;\Leftrightarrow\; \lambda_i \in \sigma(A) \text{ for some } i \;\Leftrightarrow\; w(\lambda_i) \in w(\sigma(A)) \;\Leftrightarrow\; \lambda_0 \in w(\sigma(A))$$

□

Spectral radius for normal operators

Definition

An operator $A \in \mathfrak{L}(H)$ is called **normal** if $A^*A = AA^*$.

Definition

For $A \in \mathfrak{L}(H)$ we define its **spectral radius**

$$r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$$

Remark

Section 1: $\sigma(A) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|A\|\}$ for any $A \in \mathfrak{L}(H)$. Thus $r(A) \leq \|A\|$.

Theorem

If A is normal then $r(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \|A\|$.

Proof.

The proof is based on the theory of comlex analytical functions. We skip it here. □

Spectral radius for normal operators

Theorem

If $A \in \mathfrak{L}(H)$ is normal and w is a polynomial, then

$$\|w(A)\| = \sup\{|w(\lambda)| : \lambda \in \sigma(A)\}.$$

Proof.

If A is normal then $w(A)$ is also normal.

$$\begin{aligned} \|w(A)\| &= r(w(A)) = \sup\{|\mu| : \mu \in \sigma(w(A))\} = \sup\{|\mu| : \mu \in w(\sigma(A))\} \\ &= \sup\{|w(\lambda)| : \lambda \in \sigma(A)\} \end{aligned}$$

□

Spectral theorem for normal operator

Theorem

Let $A \in \mathfrak{L}(H)$ be normal. Then there is uniquely defined spectral resolution of identity E such that $\operatorname{supp} E = \sigma(A)$ and

$$A = \int \lambda \mathrm{d}E(\lambda).$$

Proof

Let $C(\sigma(A))$ denote the algebra of continuous functions on $\sigma(A)$ equipped with the supremum norm

$$\|f\|_\infty = \sup\{|f(\lambda)| : \lambda \in \sigma(A)\}.$$

Stone-Weierstrass theorem asserts that polynomials are dense in $C(\sigma(A))$.
Consider the map $w \mapsto w(A) \in \mathfrak{L}(H)$, where w are polynomials. We showed in previous theorem that $\|w(A)\| = \|w\|_\infty$. Thus the map is an isometric *-homomorphism

$$(w+v)(A) = w(A) + v(A), \qquad (wv)(A) = w(A)v(A), \qquad \overline{w}(A) = w(A)^*$$

and it can be extended to whole $C(\sigma(A))$.

Spectral theorem for normal operator

Proof

Hence we have defined an isometric correspondence

$$C(\sigma(A)) \ni f \mapsto f(A) \in \mathfrak{L}(H)$$

into some closed abelian subalgebra of $\mathfrak{L}(H)$. For $x, y \in H$ let $\xi_{x,y}$ be a functional on $C(\sigma(A))$ defined as

$$\xi_{x,y}(f) = \langle x, f(A)y \rangle.$$

$$|\xi_{x,y}(f)| \leq \|f(A)\|\|x\|\|y\| = \|f\|_\infty \|x\|\|y\|$$

$\xi_{x,y}$ is bounded, hence Riesz-Kakutani theorem asserts that there is a complex measure $\mu_{x,y}$ such that

$$\langle x, f(A)y \rangle = \int f(\lambda) \mathrm{d}\mu_{x,y}(\lambda), \qquad \|\mu_{x,y}\| \leq \|x\|\|y\|.$$

Since $\operatorname{supp} \mu_{x,y} \subset \sigma(A)$ one can show that

$$\sigma(A) = \overline{\bigcup_{x \in H} \operatorname{supp} \mu_x}.$$

Spectral theorem for normal operator

Proof.

It is left as exercise to show that $\{\mu_{x,y}\}$ is a family of spectral measures. The family determines uniquely a spectral resolution of identity E.

$$\langle x, Ay\rangle = \int \lambda \mathrm{d}\mu_{x,y}(\lambda) \quad \Rightarrow \quad A = \int \lambda \mathrm{d}E(\lambda).$$

Let us show uniqueness of E. Assume that F is a spectral resolution of identity such that $A = \int \lambda \mathrm{d}E(\lambda)$. By induction argument we show that

$$\langle x, w(A)y\rangle = \int w(\lambda)\mathrm{d}F(\lambda)$$

for any polynomial. Consequently, we get the above equality for any continuous function. Finally

$$\langle x, E(M)y\rangle = \langle x, F(M)y\rangle, \qquad M \in \mathcal{B}(\mathbb{C}).$$

□

Spectral theorem for normal operator

Corollary

If A is normal, then for every continuous function f we have

$$\sigma(f(A)) = f(\sigma(A)).$$

Definition

We say that $A \in \mathfrak{L}(H)$ is a **positive** operator if $\langle x, Ax \rangle \geq 0$ for every $x \in H$.

Remark

One can show that positive aperator is selfadjoint and $\sigma(A) \subset [0, \infty)$.

Corollary

If A is positive operator, then

$$B = \int \sqrt{\lambda} \mathrm{d}E(\lambda)$$

is the unique positive operator such that $B^2 = A$.

Spectral theorem for normal operator

Remark

B is called square root of A and is denoted by $A^{1/2}$.