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Tensor Products of Banach Spaces and Operator Spaces

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Contents

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- If x is an element in X and φ is a linear functional on X, we denote the value of φ at x either by $\varphi(x)$ or by $\langle x, \varphi \rangle$.
- By $L(X, Y)$ we will denote the space of all linear mappings from X into Y. Obviously, $L(X, \mathbb{K})$ is just the space X'.

Bilinear maps

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$$A(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 A(x_1, y) + \alpha_2 A(x_2, y),$$

$$A(x, \beta_1 y_1 + \beta_2 y_2) = \beta_1 A(x, y_1) + \beta_2 A(x, y_2)$$

for all $x_i, x \in X$, $y_i, y \in Y$ and scalars α_i, β_i, $i = 1, 2$.

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If Z is the scalar field then the corresponding space of bilinear functionals will be denoted by $B(X \times Y)$.

Tensor product of vector spaces

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Thus, the typical element of $X \otimes Y$ is of the form

$$u = \sum_{i=1}^{n} \lambda_i x_i \otimes y_i,$$

where $n \in \mathbb{N}$, $\lambda_i \in \mathbb{K}$, $x_i \in X$, $y_i \in Y$, $i = 1, 2, \ldots, n$.

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where $n \in \mathbb{N}$, $\lambda_i \in \mathbb{K}$, $x_i \in X$, $y_i \in Y$, $i = 1, 2, \ldots, n$.

Note, that the representation of u **is not unique!**

Properties

The mapping $X \times Y \ni (x, y) \mapsto x \otimes y \in X \otimes Y$ can be considered as kind of "multiplication". Namely, we have the following properties:

1. $(x_1 + x_2) \otimes y = x_1 \otimes y + x_2 \otimes y$,
2. $x \otimes (y_1 + y_2) = x \otimes y_1 + x \otimes y_2$,
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Let X and Y be linear spaces.

1. *If $E \subset X$ and $F \subset Y$ are linearly independent subsets, then $\{x \otimes y : x \in E, \ y \in F\}$ is linearly independent in $X \otimes Y$.*

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1. *If $E \subset X$ and $F \subset Y$ are linearly independent subsets, then $\{x \otimes y : x \in E,\ y \in F\}$ is linearly independent in $X \otimes Y$.*
2. *If $\{e_i : i \in I\}$ and $\{f_j : j \in J\}$ are bases of X and Y respectively, then $\{e_i \otimes f_j : (i, j) \in I \times J\}$ is a basis in $X \otimes Y$.*

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Linearization

Proposition

For every bilinear mapping $A : X \times Y \to Z$ *there exists a unique linear mapping* $\tilde{A} : X \otimes Y \to Z$ *such that* $A(x, y) = \tilde{A}(x \otimes y)$ *for all* $x \in X$ *and* $y \in Y$. *The correspondence* $A \mapsto \tilde{A}$ *is an isomorphism between the vector spaces* $B(X \times Y, Z)$ *and* $L(X \otimes Y, Z)$.

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Corollary

$$B(X \times Y) = (X \otimes Y)'.$$

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Proposition (Uniqueness of tensor product)

Suppose there exists a vector space W *and a bilinear mapping* $B : X \times Y \to W$ *such that for every vector space* Z *and every bilinear mapping* $A : X \times Y \to Z$ *there is a unique linear mapping* $L : W \to Z$ *such that* $A = L \circ B$*. Then there is an isomorphism* $J : X \otimes Y \to W$ *such that* $J(x \otimes y) = B(x, y)$ *for every* $x \in X$ *and* $y \in Y$*.*

Tensor products of linear mappings

Assume that X, Y, W, Z are vector spaces and

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Define a bilinear mapping $X \times Y \ni (x, y) \mapsto (Sx) \otimes (Ty) \in W \otimes Z$. Then from linearization property we get a linear mapping $S \otimes T : X \otimes Y \to W \otimes Z$ such that

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We will call the mapping $S \otimes T$ the tensor product of mappings S and T.

Contents

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A Banach space is a pair $(X, \|\cdot\|)$, where X is a vector space and $\|\cdot\|$ is a norm such that the above metric d turns X into a complete metric space.

Tensor norms

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We will consider the following condition:

Definition

A tensor norm on $X \otimes Y$ is a norm $\|\cdot\|$ on $X \otimes Y$ such that

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Projective tensor norm

Now, let us notice the simple consequence of the condition.

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Let $u \in X \otimes Y$. If $u = \sum_{i=1}^{n} x_i \otimes y_i$ is a representation of u, the from the triangle inequality we get

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$$\|u\| \leq \inf \left\{ \sum_{i=1}^{n} \|x_i\| \|y_i\| : u = \sum_{i=1}^{n} x_i \otimes y_i \right\}$$

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A projective norm is function $\|\cdot\|_\pi : X \otimes Y \to [0, \infty)$ defined as follows:

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Proposition

Let X and Y be Banach spaces. Then $\|\cdot\|_\pi$ is a norm on $X \otimes Y$ and

$$\|x \otimes y\|_\pi = \|x\|\|y\|$$

for any $x \in X$ and $y \in Y$.

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Example. Let ℓ_1 be the Banach space of all summable sequences, i.e.

$$\ell_1 = \left\{ (a_n)_{n=1}^\infty : \sum_{n=1}^\infty |a_n| < \infty \right\}$$

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with the norm $\|(x_n)\|_1 = \sum_{n=1}^\infty \|x_n\|$. Then

$$\ell_1 \hat{\otimes}_\pi X = \ell_1(X)$$

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Recall that for vector spaces X, Y, Z we have the following identification

$$B(X \times Y, Z) = L(X \otimes Y, Z).$$

Now, we add norms to this picture. So, assume X, Y, Z are now Banach spaces. We say that a bilinear map $B : X \times Y \to Z$ is bounded, if there is a constant $C > 0$ such that $\|B(x, y)\| \leq C\|x\|\|y\|$ for any $x \in X$ and $y \in Y$. We denote by $\mathcal{B}(X \times Y, Z)$ the Banach space of all bounded bilinear mappings from $X \times Y$ into Z, where the norm is given by

$$\|B\| = \sup\{\|B(x, y)\| : x \in X,\, y \in Y,\, \|x\| \leq 1,\, \|y\| \leq 1\}.$$

Theorem

Let $B : X \times Y \to Z$ be a bounded linear mapping. Then there exists a unique operator $\tilde{B} : X \hat{\otimes}_\pi Y \to Z$ such that $\hat{B}(x \otimes y) = B(x, y)$ for every $x \in X$ and $y \in Y$.
The correspondence $B \mapsto \hat{B}$ is an isometric isomorphism between the Banach spaces $\mathcal{B}(X \times Y, Z)$ and $\mathcal{L}(X \hat{\otimes}_\pi Y, Z)$.

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Theorem

If X and Y are Banach spaces, then

$$(X\hat{\otimes}_\pi Y)^* = \mathcal{B}(X \times Y).$$

Tensor as bilinear forms on duals

We shall now present a different approach to norming the tensor product of two Banach spaces.

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We can view elements of $X \otimes Y$ as bilinear forms on the product $X' \times Y'$ of the algebraic duals.

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$$X \otimes Y \hookrightarrow \mathcal{B}(X^* \times Y^*).$$

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The injective norm $\|\cdot\|_\varepsilon$ on $X \otimes Y$ is defined by

$$\|u\|_\varepsilon = \sup\left\{\left|\sum_{i=1}^n \varphi(x_i)\psi(y_i)\right| : \varphi \in X^*, \psi \in Y^*, \|\varphi\| \le 1, \|\psi\| \le 1\right\}$$

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Again, we denote by $X \otimes_\varepsilon Y$ the tensor product with the injective norm. And the completion, called the injective tensor product, is denoted by $X \hat{\otimes}_\varepsilon Y$.

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Proposition

Let $S : X \to W$ and $T : Y \to Z$ be bounded operators. Then there exists the unique operator $S \otimes_\varepsilon T : X \hat{\otimes}_\varepsilon Y \to W \hat{\otimes}_\varepsilon Z$ such that

$$(S \otimes_\varepsilon T)(x \otimes y) = (Sx) \otimes (Ty), \qquad x \in X, y \in Y.$$

Moreover,

$$\|S \otimes_\varepsilon T\| = \|S\|\|T\|.$$

Example

Recall that the c_0 is the Banach space of all sequences converging to zero:

$$c_0 = \left\{ (a_n)_{n=1}^{\infty} : \lim_{n \to \infty} a_n = 0 \right\}$$

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For any Banach space X we define the space $c_0(X)$ as the space of sequences with entries from X:

$$c_0(X) = \left\{ (x_n)_{n=1}^{\infty} : x_n \in X, \lim_{n\to\infty} \|x_n\| = 0 \right\}$$

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$$c_0 \hat{\otimes}_{\varepsilon} X = c_0(X).$$

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The isomorphism is defined by

$$[v_{ij}] \mapsto \sum_{i=1}^{k} \sum_{j=1}^{l} e_{ij} \otimes v_{ij},$$

where $e_{ij} \in M_{k,l}$ is the matrix with all coefficients equal to 0 except the one in i-th row and j-th column which is equal to 1.

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Then these mappings are seminorms which satisfy M1 and M2.
If, in addition, $\|\cdot\|_1$ is a norm, then $\|\cdot\|_n$ are norms for any $n \in \mathbb{N}$, and they determine an o.s. structure on V.

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A *concrete operator space* (*c.o.s.*) is a closed linear subspace $V \subset B(H)$.

V inherits the Banach space structure from $B(H)$. Moreover, since $\mathbb{M}_n(V) \subset \mathbb{M}_n(B(H))$ for $n \in \mathbb{N}$, there is a norm $\|\cdot\|_n$ on $\mathbb{M}_n(V)$ inherited from $\mathbb{M}_n(B(H))$. It is not difficult to show that this sequence of norms satisfies conditions M1 and M2.

Theorem (Choi, Effros)

If V is an abstract operator space, then there is a Hilbert space H and $V_1 \subset B(H)$ such that V is completely isometric to V_1.

Tensor products of matrices

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$$v \otimes w \in \mathbb{M}_{pq}(V \otimes W)$$

$$(v \otimes w)_{i,k;j,l} := v_{i,j} \otimes w_{k,l}, \quad i, j = 1, \ldots, p, \ k, l = 1, \ldots, q.$$

Projective operator space matrix norm

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For any operator spaces V and W we say that an operator space matrix norm $\|\cdot\|_\mu$ is a subcross matrix norm if

$$\|v \otimes w\|_\mu \leq \|v\|\|w\|$$

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Definition

Given an element $u \in \mathbb{M}_n(V \otimes W)$ we define

$$\|u\|_\wedge = \inf\{\|\alpha\|\|v\|\|w\|\|\beta\| : \ u = \alpha(v \otimes w)\beta\}$$

where the infimum is taken over arbitrary decompositions $u = \alpha(v \otimes w)\beta$ with $v \in \mathbb{M}_p(V)$, $w \in \mathbb{M}_q(W)$, $\alpha \in \mathbb{M}_{n,pq}$ and $\beta \in \mathbb{M}_{pq,n}$, $p, q \in \mathbb{N}$ arbitrary.

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Theorem

$\|\cdot\|_\wedge$ *is the largest operator space subcross matrix norm on* $V \otimes W$.

Projective tensor product of operator spaces

The completion of the space $V \otimes W$ with respect to the matrix norm $\|\cdot\|_\wedge$ is denoted by $V\hat{\otimes}W$.

Definition

The operator space $(V\hat{\otimes}W, \|\cdot\|_\wedge)$ is called the projective tensor product of spaces V and W.

Completely bounded bilinear maps

Let $\varphi : V \times W \to U$ be a bilinear mapping.

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We say that the bilinear map ϕ is completely bounded if

$$\|\varphi\|_{\mathrm{cb}} := \sup\{\|\varphi_{p;q}\| : p, q \in \mathbb{N}\} < \infty$$

Linearization for completely bounded bilinear maps

The space of all completely bounded bilinear maps from $V \times W$ into U will be denoted by $CB(V \times W, U)$.

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"$=$" in the above statement means "completely isometric".

Injective matrix norm

Recall that for Banach spaces X and Y

$$\|u\|_\varepsilon = \sup\{|(f \otimes g)(u)| : f \in X^*, g \in Y^*, \|f\| \leq 1, \|g\| \leq 1\}$$

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Definition

If V and W are operator spaces then the injective matrix norm is defined by

$$\|u\|_\vee = \sup\{\|(f \otimes g)_n(u)\| : f \in \mathbb{M}_p(V^*), g \in \mathbb{M}_q(W^*), \|f\| \leq 1, \|g\| \leq 1\}$$

for $u \in \mathbb{M}_n(V \otimes W)$.

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As in the case of Banach spaces we have

Proposition

The injective matrix norm is determined by the natural embedding

$$\theta : V \otimes W \hookrightarrow CB(V^*, W)$$

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Again, we consider the completion of $V \otimes W$ with respect to norm $\|\cdot\|_\vee$.

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Proposition

The natural embedding

$$V\check{\otimes}W \subset B(H) \otimes B(K) = B(H \otimes K)$$

is a complete isometry.

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