Topological degree and its basic applications.

Finitely dimensional case.

Problems to solve.

PWP Interdysciplinary Doctoral Studies in Mathematical Modeling
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Problem 1. Find the value of the topological degree of the map $f : [0,4] \times [-1,1] \to \mathbb{R}^2$ given by
$$f(x,y) = \left(-(1+e^y)\sin x,\ e^y \cos x - (1+y)e^y\right).$$

Problem 2. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Prove that the degree $\deg(f,(a,b))$ depends only on $f(a)$ and $f(b)$.

Problem 3. Find $\deg(f, B(0,1))$ and indices of all isolated zeroes of f where $f : \mathbb{R}^2 \to \mathbb{R}^2$ is given by
$$f(x,y) = (y - |x|, |y^2 - y|).$$

Problem 4. Let $f : [a_1,b_1] \times [a_2,b_2] \times ... \times [a_k,b_k] \to \mathbb{R}^k$ be a continuous map given by $f(x_1,x_2,...,x_k) = (f_1(x_1), f_2(x_2), ..., f_k(x_k))$, where $f_i : [a_i,b_i] \to \mathbb{R}^1$ are continuous functions and $f_i(a_i)f_i(b_i) \neq 0$ $(i = 1,...,k)$. Find the value of $\deg(f, (a_1,b_1) \times (a_2,b_2) \times ... \times (a_k,b_k))$.

Problem 5. Let us now identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ with $\mathbb{C} \ni z = x + iy \in \mathbb{R}^2$. Let us fix $z_0 = x + iy_0$, $|z_0| < r$ and define $f_n, g_n : \mathbb{R}^2 \to \mathbb{R}^2$ by $f_n(x,y) = z^n - z_0$ and $g_n(x,y) = \bar{z}^n - z_0$. Find the value of $\deg(f_n, B(0,r))$ and $\deg(g_n, B(0,r))$.

Problem 6. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by
$$f(x,y) = (a_0x^2 + 2a_1xy + a_2y^2, b_0x^2 + 2b_1xy + b_2y^2).$$
Let us define
$$D_1 = \det \begin{bmatrix} a_0 & a_1 \\ b_0 & b_1 \end{bmatrix}, \qquad D_2 = \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}, \qquad D_3 = \det \begin{bmatrix} a_0 & a_2 \\ b_0 & b_2 \end{bmatrix}$$
and $D = D_1D_2 - 4D_3^2$. Prove that for $r > 0$

a) If $D < 0$, then $\deg(f, B(0,r)) = 0$;
b) If $D > 0$ and $D_1 > 0$, then $\deg(f, B(0,r)) = 2$;
c) If $D > 0$ and $D_1 < 0$, then $\deg(f, B(0,r)) = -2$.

Problem 7. Prove that if $A \in GL(\mathbb{R}^k)$ and there does not exist any negative eigenvalue of A then A may be joined by homotopy with an identity map on any ball $\overline{B(0,R)}$.

Problem 8. Let $A : \mathbb{R}^k \to \mathbb{R}^k$ be the linear map and $\lambda \in \mathbb{R}$ is not an eigenvalue of A. Find the value of
$$\deg(f_\lambda, B(0,r)).$$
where $f_\lambda : \mathbb{R}^k \to \mathbb{R}^k$ is given by $f_\lambda(x) = (\lambda I - A)(x)$.

Problem 9. Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be bounded continuous map. Prove that there exists a fixed point of f.

Problem 10. Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be a continuous map and
$$\lim_{|x| \to +\infty} \frac{\langle f(x), x \rangle}{|x|} = -\infty.$$
Show that for each y there exists the solution of the equation $f(x) = y$.

Problem 11. Prove the following extension of the Brouwer fixed point theorem: Let $A \subset \mathbb{R}^k$ be the set homeomorphic to $\overline{B(0,1)}$ and $f : A \to A$ any continuous map. Then there exists the fixed point of f.

Problem 12. Let $B \subset A \subset \mathbb{R}^k$. We say that B is the retract of A if there exists a continuous map $r : A \to B$ (called a retraction) such that $r(A) \subset B$ and $r(x) = x$ for all $x \in B$. Prove the following extension of the Brouwer fixed point theorem: Let $A \subset \mathbb{R}^k$ be the set homeomorphic to $\overline{B(0,1)}$, and B be the retract of A. Then each contiunuous map $f : B \to B$ has a fixed point.