Topological degree and its basic applications.

Finitely dimensional case.

Sample problems.

PWP Interdysciplinary Doctoral Studies in Mathematical Modeling
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Problem 1. Let $U \subset \mathbb{R}^k$ be the open and bounded set and $y \in U$. Show that

$$\deg(I - y, U) = 1.$$

Solution: Let us take such $R > 0$ that $U \subset B(0, R)$. Then, by the excision property of the topological degree,

$$\deg(I - y, B(0, R)) = \deg(I - y, U).$$

Now we can see that the homotopy $h : [0, 1] \times \overline{B(0, R)} \to \mathbb{R}^k$ given by $h(t, x) = x - ty$ is well-defined and joins $I - y$ with an identity map on the ball $B(0, R)$ thus proving that

$$\deg(I - y, B(0, R)) = \deg(I, B(0, R)) = 1.$$

Problem 2. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (-(1 + e^y)\sin x,\ e^y \cos x - (1 + y)e^y)$. Find the value of the degree

$$\deg(f, (-(2k + \frac{1}{2})\pi,\ (2k + \frac{1}{2})\pi) \times (-1, 1)),$$

and

$$\deg(f, (-(2k + \frac{1}{2})\pi,\ (2k + \frac{1}{2})\pi) \times (-3, 3)).$$

Solution: Let us solve the system of equations

$$\begin{cases} -(1 + e^y)\sin x = 0 \\ e^y \cos x - (1 + y)e^y = 0. \end{cases}$$

As we can see there must be $\sin x = 0$ so $x = k\pi$ where $k \in \mathbb{Z}$. This means that either $\cos x = 1$ or $\cos x = -1$.

Let us first check what happens if $\cos x = -1$. Then the second equation becomes

$$-e^y - (1 + y)e^y = 0$$

what gives $y = -2$.

On the other hand if $\cos x = 1$ the second equation turns into

$$ye^y = 0,$$

what implies $y = 0$.

That is why there are two families of solutions $(2k\pi, 0)$ and $((2k + 1)\pi, -2)$ where $k \in \mathbb{Z}$.

Now we check if the zeroes of the map f are regular. The derivative of the map f is

$$Df(x, y) = \begin{bmatrix} -\cos x(1 + e^y) & -\sin xe^y \\ -\sin xe^y & e^y(\cos x - 2 - y) \end{bmatrix}.$$

For zeroes in the first family (i.e. with $y = 0$) we have

$$\det Df(2k\pi, 0) = \det \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} = 2 > 0;$$

on the other hand for zeroes of the second family $((2k + 1)\pi, -2)$ there is

$$\det Df((2k + 1)\pi, -2) = \begin{bmatrix} (1 + e^{-2}) & 0 \\ 0 & -e^{-2} \end{bmatrix} = -e^{-2}(1 + e^{-2}) < 0.$$

As we can see each zero of the map f is regular so the degree of the map f relative to the set U equals to the sum of signs of determinants of the derivative in all zeroes belonging to U.

In case $U = (-(2k + \frac{1}{2})\pi,\ (2k + \frac{1}{2})\pi) \times (-1, 1)$ all zeroes belonging to U are given by $(2n\pi, 0)$ with $n \in \{-k, -k + 1, ..., -1, 0, 1, ..., k - 1, k\}$ and for all of them the determinant of the derivative is positive. Hence $\deg(f, (-(2k + \frac{1}{2})\pi,\ (2k + \frac{1}{2})\pi) \times (-1, 1), 0) = 2k + 1$.

On the other hand if we take $V = (-(2k+\frac{1}{2})\pi,\ (2k+\frac{1}{2})\pi)\times(-3, 3)$, then the set V contains zeroes of f given by $(2n\pi, 0)$ for $n \in \{-k, -k + 1, ..., -1, 0, 1, ..., k - 1, k\}$ but also these given by $((2m + 1)\pi, -2)$ for $m \in \{-k, -k + 1, ..., -1, 0, 1, ..., k - 2, k - 1\}$. All zeroes belonging to the second family have the derivative with negative determinant. Hence $\deg(f, (-(2k + \frac{1}{2})\pi,\ (2k + \frac{1}{2})\pi) \times (-3, 3)) = 2k + 1 - 2k = 1$.

Problem 3. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$f(x,y,z) = \begin{cases} \left((x^2+y^2+z^2)\,\sin(\frac{1}{x^2+y^2+z^2}),\, y^2,\, z^2\right) & \text{for} \quad x^2+y^2+z^2 > 0 \\ (0,0,0) & \text{for} \quad x^2+y^2+z^2 = 0\,. \end{cases}$$

Find the degree $\deg(f, B(0,1))$ and indices of regular zeroes of f.

Solution: Let us first observe that the zeroes of the map f are given by $(\frac{1}{\sqrt{k\pi}}, 0, 0)$ where $k \in \mathbb{N}$. Also the point $(0,0,0)$ is the zero of the map f.

Because the set of zeroes contains infinitely many points in $B(0,1)$ we are sure that $(0,0,0)$ is not a regular value of f. Surely, the accumulation point of the set of zeroes (i.e. $(0,0,0)$) is not the regular zero. But also all isolated zeroes of f are not regular. Let us have a look at the derivative of the map f in a point $(x,0,0)$, where $x \neq 0$:

$$\begin{bmatrix} 2x\sin(\frac{1}{x^2}) + x^2\cos(\frac{1}{x^2})(-2x)x^{-4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Its determinant equals 0 for all $x \neq 0$.

In case $(0,0,0)$ is not a regular value of f, we cannot calculate the value of the degree by adding the local indices. If we follow the definition directly we may disturb f a little to become a C^1 map having 0 as a regular value. This may probably be a correct way to look at things, but not necessarily easy to follow.

We will approach the problem some other way: let us observe that the map $f$may be joined by homotopy to the map $g : \overline{B(0,1)} \to \mathbb{R}^3$ given by $g(x,y,z) = (\sin 1, y^2, z^2)$. The homotopy $h : [0,1] \times \overline{B(0,1)} \to \mathbb{R}^3$ may be given by

$$h(t,(x,y,z)) = (1-t)f(x,y,z) + tg(x,y,z).$$

As we can see

$$h(t,(x,y,z)) = (\sin 1, y^2, z^2) \neq (0,0,0),$$

for all $x^2+y^2+z^2 = 1$. That is why

$$\deg(f, B(0,1)) = \deg(g, B(0,1)) = 0,$$

because g does not achieve 0 in the closed ball $\overline{B(0,1)}$.

Problem 4. Let $f_p : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f_p(x, y) = (x^2 - y^2,\, 2xy) - (p_x, p_y),$$

where $p = (p_x, p_y)$, $p_x^2 + p_y^2 < r^2$. Find the value of $\deg(f_p, B(0, r))$.
Solution: Let us observe that for $p_x^2 + p_y^2 > 0$ the system of equations

$$\begin{cases} x^2 - y^2 = p_x \\ 2xy = p_y \end{cases}$$

has two solutions $(x_i, y_i) \neq (0, 0)$ $(i = 1, 2)$. Moreover, for all $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, the determinant of the derivative

$$\det Df_p(x_i, y_i) = \det \begin{bmatrix} 2x_i & -2y_i \\ 2y_i & 2x_i \end{bmatrix} = 2(x_i^2 + y_i^2) > 0.$$

That is why $\deg(f_p, B(0, r)) = 2$.

In case $p = (0, 0)$ there are no regular zeros of the map f_p. But we may take the homotopy $h : [0, 1] \times \overline{B(0, r)} \to \mathbb{R}^2$, given by

$$h(t, (x, y)) = (x^2 - y^2 - t\alpha, 2xy),$$

where $\alpha \in (0, 1)$ is a small, positive constant. For each t the map $h(t, \cdot)$ has two regular zeroes $(\sqrt{\alpha t}, 0)$ and $(-\sqrt{\alpha t}, 0)$.

As we have shown before

$$\deg(h(1, \cdot), B(0, r)) = 2,$$

hence, by the homotopy axiom, $\deg(f_{(0,0)}, B(0, r)) = 2$.

Problem 5. Prove that the degree is translation-invariant i.e. for a continuous map $f : \overline{U} \to \mathbb{R}^k$ satisfying $f^{-1}(0) \cap \partial U = \emptyset$, for any $x_0 \in \mathbb{R}^n$ there is $\deg(f, U) = \deg(f_{x_0}, U + x_0)$, where $f_{x_0}(x) = f(x - x_0)$ and $U + x_0 = \{x + x_0 : x \in U\}$.

Solution: Let us consider the real function $\varphi : [0, 1] \to \mathbb{R}$ given by $\varphi_{x_0}(t) = \deg(f_{tx_0}, U + tx_0)$.

First we should note that the function φ_{x_0} is well defined, i.e. $f_{tx_0}(x) \neq 0$ for each $t \in [0, 1]$ and $x \in \partial(U + tx_0)$. As we can see

$$f_{tx_0}(x) = 0 \Leftrightarrow f(x - tx_0) = 0,$$

and

$$x \in \partial(U + tx_0) \Leftrightarrow x - tx_0 \in \partial U.$$

This implies that

$$f_{tx_0}(x) = 0 \wedge x \in \partial(U + tx_0) \Leftrightarrow f(x - tx_0) = 0 \wedge x - tx_0 \in \partial U,$$

so none of it is possible.

Moreover we may show that φ_{x_0} is continuous. Let us start with the set $V = U + t_0 x_0$ for a fixed $t_0 \in [0, 1]$ and let us define the set

$$V_\varepsilon = \{x \in V : \text{dist}(x, \partial V) > \varepsilon\},$$

where $\varepsilon > 0$ and $\text{dist}(x, A) = \inf\{|x - y| : y \in A\}$ is the distance between the point x and the set $A \subset \mathbb{R}^k$. The set is obviously open, and for small ε also nonempty. Let

$$D_\varepsilon = \overline{V} \setminus V_\varepsilon = \{x \in \overline{V} : \text{dist}(x, \partial V) \leq \varepsilon\}.$$

Let us observe that we may find such small $\varepsilon > 0$, that $f(x) \neq 0$ for $x \in D_\varepsilon$. Then for $|t| < \varepsilon$

$$\overline{V_\varepsilon} + tx_0 \subset V$$

and

$$\partial(\overline{V_\varepsilon} + tx_0) \subset D_\varepsilon.$$

This means that for all $|t| < \varepsilon$

$$\deg(f, V_\varepsilon) = \deg(f, V) = \deg(f, V_\varepsilon + tx_0).$$

Problem 6. Let $x_0 \in \mathbb{R}^k$ be the isolated zero of the C^1 map $f : \overline{B(x_0, r)} \to \mathbb{R}^k$. Assume that the derivative $Df(x_0)$ is a linear isomorphism. Prove that $\deg(f, B(x_0, r)) = \deg(g, B(x_0, r))$ where $g : \mathbb{R}^k \to \mathbb{R}^k$ is given by $g(x) = Df(x_0)(x - x_0)$.

Solution: Let us first observe that for any $r_0 \in (0, r)$, by the excision property of the degree

$$\deg(f, B(x_0, r)) = \deg(f, B(x_0, r_0)).$$

Let the homotopy $h : [0,1] \times \overline{B(x_0, r_0)} \to \mathbb{R}^k$, for the appropriate $r_0 > 0$, be given by

$$h(t, x) = (1-t)Df(x_0)(x - x_0) + tf(x).$$

Then we have

$$|h(t,x)| \geq \|Df(x_0)\| \cdot |x - x_0| - t|Df(x_0)(x - x_0) - f(x)| =$$

$$\|Df(x_0)\||x - x_0| - t|f(x) - f(x_0) - Df(x_0)(x - x_0)|.$$

Because of the differentiability of f at x_0 there exists such $r_0 > 0$ that

$$|f(x) - f(x_0) - Df(x_0)(x - x_0)| \leq \frac{\|Df(x_0)\|}{2}|x - x_0|,$$

for $|x - x_0| < r_0$.

Then

$$|h(t,x)| \geq \frac{\|Df(x_0)\|}{2}|x - x_0| > 0$$

for $\|x - x_0\| = r_0$, what proves that the homotopy is well defined.

Problem 7. Prove the following Poincare-Bohl theorem:

If $f, g \in C(\overline{U})$ and for any $x \in \partial U$ the segment $[f(x), g(x)]$ does not contain 0, then $\deg(f, U) = \deg(g, U)$.

Solution: Let the homotopy $h : [0,1] \times \overline{U} \to \mathbb{R}^k$ be given by

$$h(t, x) = (1-t)f(x) + tg(x).$$

As we can see each value $h(t,x)$ for $t \in [0,1]$ and $x \in \partial U$ belongs to the segment $[f(x), g(x)]$, so it does not achieve 0. That is why the maps f and g may be joined by homotopy and by the homotopy axiom

$$\deg(f, U) = \deg(g, U),$$

what completes the proof.

Problem 8. Let $g : \mathbb{R}^k \to \mathbb{R}^k$ be a continuous map satisfying $\lim_{|x|\to+\infty} \frac{|g(x)|}{|x|} = 0$, $f : \mathbb{R}^k \to \mathbb{R}^k$ be given by $f(x) = -x + g(x)$. Prove that there exists a fixed point of f.

Solution: For $R > 0$ big enough the map $h : [0,1] \times \overline{B(0,R)} \to \mathbb{R}^k$ given by $h(t,x) = t(-f(x)) + (1-t)x$ is the homotopy joining $-f(x)$ to the identity map on $\overline{B(0,R)}$.

Let us assume that $R > 1$ and $|g(x)| \leq \frac{1}{2}|x|$ for $|x| \geq R$. Then we have

$$\langle h(t,x), x\rangle = -t\langle f(x), x\rangle + (1-t)|x|^2 = t|x|^2 - t\langle g(x), x\rangle + (1-t)|x|^2 = |x|^2 - t\langle g(x), x\rangle \geq$$

$$\geq |x|^2 - |g(x)||x| \geq \frac{1}{2}|x|^2 > 0,$$

for $|x| = R$. Hence the homotopy is well-defined, so the map $f(x)$ is homotopic to the identity map, what completes the proof.

Problem 9. Prove the following Poincare - Brouwer theorem: Let $U \subset \mathbb{R}^k$ be an open neighbourhood of zero and $k \in \mathbb{N}$ the odd number. If $f \in C(\overline{U})$, then one of conditions holds

(1) f achieves zero in the boundary ∂U,

(2) $\exists_{y \in \partial\Omega} \exists_{\lambda \neq 0} \; f(y) = \lambda y$.

Solution: Assume, contrary to our claim, that $0 \notin f(\partial U)$ and $\forall_{y \in \partial\Omega} \forall_{\lambda \neq 0} \; f(y) \neq \lambda y$. Then the homotopies $h, \tilde{h} : [0,1] \times \overline{U} \to \mathbb{R}^k$ given by

$$\begin{aligned} h(t,x) &= (1-t)f(x) - tx \\ \tilde{h}(t,x) &= (1-t)f(x) + tx \end{aligned}$$

do not achieve 0 on the boundary ∂U.

Applying the homotopy axiom twice we get

$$\begin{aligned} \deg(-I,U) &= \deg(h(1,\cdot),U) = \deg(h(0,\cdot),U) = \deg(f,U) = \\ &= \deg(\tilde{h}(0,\cdot),U) = \deg(\tilde{h}(1,\cdot),U) = \deg(I,U) = 1\,. \end{aligned}$$

But we know that $\deg(-I,U) = (-1)^k = -1$, for odd $k \in \mathbb{N}$, what gives the contradiction and completes the proof.

Problem 10. Prove the following fact: any continuous map $f : S^{2n} \to \mathbb{R}^{2n+1}$ such that $f(x) \in \{y;\ \langle y, x\rangle = 0\}$ (i.e. belongs to the tangent space to S^{2n} at x) has at least one zero. Here $S^{2n} = \{x \in \mathbb{R}^{2n+1} : |x| = 1\}$.

Solution: Assume that $f(x) \neq 0$ for $x \in S^{2n}$ and for all these points there is $\langle x, f(x)\rangle = 0$. By Poincare-Brouwer theorem (see problem 9) there exists $\lambda \neq 0$ and such $x \in S^{2n}$ that $f(x) = \lambda x$. This gives $\lambda\langle x, x\rangle = 0$, a contradiction.

Problem 11. Prove that the following theorems are equivalent

(1) Brouwer fixed point theorem;

(2) Bohl theorem : if continuous map $f : \overline{B(0,1)} \to \mathbb{R}^k$ satisfies $f\left(S^{k-1}\right) \subset \overline{B(0,1)}$, then there exists a fixed point of f. Here $S^{k-1} = \partial B(0,1)$.

Solution: Let us first prove that (2) implies (1). Let $f : \overline{B(0,1)} \to \mathbb{R}^k$ be a continuous map satisfying assumptions of the Brouwer fixed point theorem, i.e. $f(\overline{B(0,1)}) \subset \overline{B(0,1)}$. Then, of course, $f(S^{k-1}) \subset \overline{B(0,1)}$, so the assumptions of Bohl theorem are satisfied as well. By Bohl theorem there exists the fixed point of f, so the Brouwer fixed point theorem is proved.

We will now prove that (1) implies (2). Assume that continuous $f : \overline{B(0,1)} \to \mathbb{R}^k$ maps S^{k-1} to $\overline{B(0,1)}$. Assume, contrary to our claim, that f does not have a fixed point.

Let us define the map $\tilde{f} : \overline{B(0,1)} \to \overline{B(0,1)}$ by

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } |f(x)| \leq 1 \\ \frac{f(x)}{|f(x)|} & \text{for } |f(x)| > 1. \end{cases}$$

It is easy to check that the map $\tilde{f}$ is continuous. We will show that $\tilde{f}$ does not have a fixed point thus contradicting the Brouwer fixed point theorem.

Of course in case when $f(x) \in \overline{B(0,1)}$ then also $\tilde{f}(x) = f(x)$ and, because of our assumption $\tilde{f}(x) \neq x$.

On the other hand if $|f(x)| > 1$, then $|\tilde{f}(x)| = 1$. But in this case, $|x| < 1$, because for $|x| = 1$ there is $|f(x) \leq 1$. So also here $\tilde{f}(x) \neq x$.

Thus there exists a continuous map $\tilde{f} : \overline{B(0,1)} \to \overline{B(0,1)}$, which does not have a fixed point, what contradicts the Brouwer fixed point theorem and proves that (1) implies (2).

Problem 12. Assume that $U \subset \mathbb{R}^k$ is an open bounded set, $f \in C(\overline{U})$ and there exists such $y \in U$ that

$$\forall_{x\in\partial U}\ \forall_{\mu>0}\ f(x) - y \neq \mu(x-y)\,.$$

Then there exists the fixed point of f in $\overline{U}$.

Solution: Assume that f does not have a fixed point in the boundary ∂U. Then we can write a homotopy $h : [0,1] \times \overline{U} \to \mathbb{R}^k$ given by

$$h(t,x) = x - y - t\,(f(x) - y)\ .$$

As we can see if $h(t,x) = 0$ for $x \in \partial U$ and $t \in (0,1]$, then

$$0 = f(x) - y - \frac{1}{t}(x-y),$$

what contradicts our assumption. On the other hand for $t = 0$ we have $x = y$, what is not possible since $y \in U$ and $x \in \partial U$.

The homotopy h is well defined, so

$$\deg(h(1,\cdot),U) = \deg(h(0,\cdot),U) = \deg(I - y, U) = 1\,.$$

But $h(1,x) = x - f(x)$ so there must exist such $x \in U$, that $x - f(x) = 0$, i.e. the fixed point of f.