Topological degree and its basic applications.

Finitely dimensional case.

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1 Introduction and basic notation

We are going to present one of the basic tools used by the branch of mathematics that is referred to as *nonlinear analysis*. We can define the branch broadly as the one that deals with solving nonlinear problems – including equations (also differential and integral equations) or optimization problems. We are searching for different methods that may be used for wide classes on nonlinearities so we do concentrate on qualitative results rather then on pointing precisely (or approximately) the solution that we are looking for. The ideas that we are going to present will be tailored to answer the questions like

- does there exists a solution of the equation in the specified set?
- what is the structure of the solution set?
- how does solution set change when we change problem parameters?

We will introduce the notion of the topological degree, known also as the Brouwer degree (in case of finite dimensional spaces) or Leray-Schauder degree (in case of completely continuous vector fields in infinitely dimensional spaces). We are not going to present the precise construction – although it is very beautiful and referring to various mathematical concepts – but we are looking at this tool from the very practical perspective: as a tool that may be useful in numerous applications. That is why we will present the idea by an axiomatic definition. Starting from the set of three axioms we will be deriving numerous theorems and concepts – but actually we must be aware that we will not prove that such animal as the *topological degree* actually exists. The good news is that it really exists – there is nice construction procedure that we mentioned before – but to find out how it works we must direct the Reader to the literature (especially to the classic book of Nirenberg [3], where the construction is presented in even more general setting of maps between two manifolds of the same dimension, but also to [4] and [5]).

We will restrict ourselves to the finitely dimensional case of the k-dimensional Euclidean space $\mathbb{R}^k$ with the inner product

$$\langle x, y\rangle = x_1y_1 + x_2y_2 + ... + x_ky_k,$$

where $x = (x_1, ..., x_k), y = (y_1, ..., y_k) \in \mathbb{R}^k$. The Euclidean norm of the vector x will be denoted by $|x|$.

In what follows $U \subset \mathbb{R}^k$ will always be the open and bounded set. By $\overline{U}$ we will denote the closure and by ∂U the boundary of the set U. The letter I denotes the identity map $I : \overline{U} \to \overline{U}$.

The derivative of the map $f : \overline{U} \to \mathbb{R}^k$ at $x_0 \in U$ (if exists) will be denoted by $Df(x_0)$. The norm of the linear map $Df(x_0)$ will be denoted by $\|Df(x_0)\|$, so

$$\|Df(x_0)\| = \sup_{|h|=1} |Df(x_0)(h)|.$$

2 Axioms

We will consider the space $C(\overline{U})$ of continuous maps $f : \overline{U} \to \mathbb{R}^k$ with the supremum norm

$$\|f\|_\infty = \sup_{x\in\overline{U}} |f(x)|.$$

Since $\overline{U}$ is compact the norm is well defined, and it is well-known that the space $C(\overline{U})$ is the Banach space. Please note that $f(x)$ may be represented by the coordinate functions $f(x) = (f_1(x), ..., f_k(x))$, where $f_i : \overline{U} \to \mathbb{R}$ are continuous functions.

We will be interested in the certain subset of space $C(\overline{U})$, i.e.

$$\mathcal{K}(\overline{U}) = \{f \in C(\overline{U}) : 0 \notin f(\partial U)\}.$$

This means that $f \in \mathcal{K}(\overline{U})$ if and only if $f \in C(\overline{U})$ and $f(x) \neq 0$ for $x \in \partial U$. We will call this family *the set of compact vector fields in* $\overline{U}$.

Let us start with simple but important observations being the consequence of the compactness of the set ∂U and the continuity of the map f.

Proposition 1. *If $f \in \mathcal{K}(\overline{U})$, then there exists such $r > 0$, that $|f(x)| \geq r$ for all $x \in \partial U$.*

This leads to:

Proposition 2. *The set $\mathcal{K}(\overline{U})$ is the open subset of $C(\overline{U})$.*

Proof. Let us take any $f_0 \in \mathcal{K}(\overline{U})$ and let $r > 0$ be such constant that $|f_0(x)| \geq r$ for all $x \in \partial U$. Then for each $\|f - f_0\|_\infty < \frac{r}{2}$ and $x \in \partial U$

$$|f(x)| \geq |f_0(x)| - |f(x) - f_0(x)| \geq r - \frac{r}{2} = \frac{r}{2} > 0,$$

what proves that $f \in \mathcal{K}(\overline{U})$. □

Now we are going to define certain relation in the set $\mathcal{K}(\overline{U})$, that will appear to be one of the most important tools that we will be using.

Definition 1. We call two maps $f, g \in \mathcal{K}(\overline{U})$ homotopic if there exists such continuous map $h : [0,1] \times \overline{U} \to \mathbb{R}^k$, that

- $h(t, \cdot) \in \mathcal{K}(\overline{U})$, for $t \in [0,1]$;
- $h(0, \cdot) = f$;
- $h(1, \cdot) = g$.

We call the map h *homotopy joining maps f and g.*

Let us now look at two simple one-dimensional examples that will show how this concept works.

Example 1. Let $f, g : [-1,1] \to \mathbb{R}$ be given by $f(x) = x^2$ and $g(x) = 2$. As we can see the map $h : [0,1] \times [-1,1] \to \mathbb{R}$ given by

$$h(t,x) = (1-t)f(x) + tg(x)$$

is a valid homotopy joining f and g.

Example 2. Let $f, g : [-1,1] \to \mathbb{R}$ be given by $f(x) = x$ and $g(x) = 2$. The maps are not homotopic. Assume there exists such homotopy $h : [0,1] \times [-1,1] \to \mathbb{R}$ that joins f and g. Hence $h(0,0) = -1$ and $h(1,0) = 2$. The map $h(\cdot, 0) : [0,1] \to \mathbb{R}$ is, of course, a continuous map, that is why for some $t \in [0,1]$, there must be $h(t,0) = 0$, implying that h is not valid homotopy.

Multi-dimensional examples showing the maps that may not be joined by homotopy may seem to be more difficult to prove (we cannot refer to Darboux property as easily), but later we will see that they will emerge somewhat naturally.

Let us now proceed to the axiomatic definition of the topological degree.

Definition 2. By *topological degree* we mean the family of maps $\deg(\cdot, U) : \mathcal{K}(\overline{U}) \to \mathbb{Z}$, defined for open and bounded $U \subset \mathbb{R}^k$ and satisfying the following axioms

(A1) (normalization) If $0 \in U$, then $\deg(I, U) = 1$;

(A2) (additivity) Let $U_1, U_2 \subset U$ be such open sets that $U_1 \cap U_2 = \emptyset$ and $0 \notin f(U \setminus (U_1 \cup U_2))$, then

$$\deg(f, U) = \deg(f|_{\overline{U_1}}, U_1) + \deg(f|_{\overline{U_2}}, U_2);$$

(A3) (homotopy) Let $f, g \in \mathcal{K}(\overline{U})$ be homotopic, then $\deg(f, U) = \deg(g, U)$.

We call the integer value $\deg(f, U)$ the topological degree of the map f relative to U.

The three axioms presented above lead to numerous consequences and very often these consequences are not obvious. As an example we can see that if $\deg(f, U) \neq \deg(g, U)$ then f and g may not be joined by homotopy. We will see below that the identity map on $\overline{B(0,1)}$ may not be joined by homotopy to the constant map – the statement that is far from being trivial in case of maps in dimension $k \geq 2$.

3 Basic properties

We are going to present several simple properties that may be inferred from the set of axioms presented before.

Property 1. *Assume $f, g \in \mathcal{K}(\overline{U})$ are maps satisfying $f(x) = g(x)$ for $x \in \partial U$. Then*

$$\deg(f, U) = \deg(g, U)$$

Proof. Let us define the homotopy $h : [0, 1] \times \overline{U} \to \mathbb{R}^k$ by

$$h(t, x) = (1 - t)f(x) + tg(x).$$

As we can see $h(t, x) = f(x) = g(x)$ for all $t \in [0, 1] \times \partial U$. But as $f \in \mathcal{K}(\overline{U})$, we are sure that $f(x) \neq 0$. This means that maps f and g are homotopic and hence by the homotopy axiom, we can see that

$$\deg(f, U) = \deg(g, U).$$

□

Property 2. $\deg(f, \emptyset) = 0$.

Proof. Let us take $U = U_1 = U_2 = \emptyset$. As we can see, we may apply the additivity axiom and conclude that

$$\deg(f, U) = \deg(f, U_1) + \deg(f, U_2);$$

$$\deg(f, \emptyset) = \deg(f, \emptyset) + \deg(f, \emptyset) = 2\deg(f, \emptyset).$$

Hence $\deg(f, \emptyset) = 0$. □

Property 3 (excision). *Assume* $f \in \mathcal{K}(\overline{U})$ *and* $V \subset U$ *is such open bounded set that* $0 \notin f(\overline{U} \setminus V)$*. Then*

$$\deg(f, U) = \deg(f|_{\overline{V}}, V).$$

Proof. Let us take $U_1 = V$ and $U_2 = \emptyset$. We can see that applying additivity axiom to the sets U, U_1, U_2 we arrive to

$$\deg(f, U) = \deg(f|_{\overline{V}}, V) + \deg(f, \emptyset).$$

Because of the Property 2 we see that $\deg(f, U) = \deg(f|_{\overline{V}}, V)$ what completes the proof. □

Property 4. *Let* $f \in \mathcal{K}(\overline{U})$ *be such that* $0 \notin f(\overline{U})$*. Then* $\deg(f, U) = 0$.

Proof. As we can see we may now make use of the excision property given above for $V = \emptyset$. This shows that

$$\deg(f, U) = \deg(f, \emptyset) = 0,$$

by Property 2. □

Property 5 (existence). *Assume* $\deg(f, U) \neq 0$*. Then there exists such* $x_0 \in U$*, that* $f(x_0) = 0$.

Proof. This is just the logical transposition of the Property 4. □

The last property shows the main power of the topological degree as the tool for solving different problems. By showing that the degree has the nonzero value in the given open set U we may conclude that there must exists zero of the map f somewhere in the open set U. Although we don't know how the value of the degree may be computed yet, we can feel that if this technical issue is overcome, we can have quite nice tool of showing that solution of our problem exists.

We have not stressed the dependence of the degree on the dimension of the Euclidian space which it is defined on – but actually for each $k \in \mathbb{N}$ and space $\mathbb{R}^k$ we have degree defined independently. Now we are going to show how the degrees defined within different spaces may be related. To avoid any misunderstanding we will now stress the dependence on the dimension in our notation, so for sets $U \subset \mathbb{R}^m$ and maps $f : \overline{U} \to \mathbb{R}^m$ we will use $\mathcal{K}_m(\overline{U})$ for a family of vector fields and $\deg_m(f, U)$ for the degree.

In order to show the dependence of the degree $\deg_k$ on $\deg_m$ for $k < m$ we will need certain method of extending continuous functions on open bounded subsets of $\mathbb{R}^m$, to continuous functions on certain open, bounded subsets of $\mathbb{R}^k$. The extension will be quite natural, by means of the the identity map on certain subset of $\mathbb{R}^{k-m}$.

Let $B \subset \mathbb{R}^{m-k}$ denote the open ball $B(0, 1) \subset \mathbb{R}^{m-k}$. Let $U \subset \mathbb{R}^k$ be open and bounded set, and let $f : \overline{U} \to \mathbb{R}^k$ be continuous. Let $\tilde{f} : \overline{U} \times \overline{B} \to \mathbb{R}^k \times \mathbb{R}^{m-k}$ be given by

$$\tilde{f}(x, y) = (f(x), y).$$

Theorem 1. *Let* $k < m$ *be natural numbers and assume there is degree* $\deg_m$ *defined for open bounded subsets* $U \subset \mathbb{R}^m$*. Then the map*

$$\deg_k : \mathcal{K}_k(\overline{U}) \to \mathbb{Z}$$

given by

$$\deg_k(f, U) = \deg_m(\tilde{f}, U \times B),$$

satisfies the axioms (A1)-(A3).

Proof. **(A1).** Let $V \subset \mathbb{R}^k$ be a neighbourhood of 0 and let $f : V \to \mathbb{R}^k$ be the identity map, $f(x) = x$. Then the extension $\tilde{f} : \overline{V} \times \overline{B} \to \mathbb{R}^m$ is also the identity map, meaning

$$\deg_k(f, V) = \deg_m(I, V \times B) = 1.$$

(A2). Let $U \subset \mathbb{R}^k$, $f \in \mathcal{K}_k(\overline{U})$ and $U_1, U_2 \subset U$ by such that

$$0 \notin f(\overline{U \setminus (U_1 \cup U_2)}).$$

Let us look at $\tilde{f} : \overline{U} \times \overline{B} \to \mathbb{R}^m$ and take $x \in \overline{U \setminus (U_1 \cup U_2)}$. As we can see $\tilde{f}(x, y) = (f(x), y) \neq (0, 0)$ for $(x, y) \in \overline{(U \times B) \setminus ((U_1 \times B) \cup (U_2 \times B))}$, so

$$\deg_m(\tilde{f}, U \times B) = \deg_m(\tilde{f}, U_1 \times B) + \deg_m(\tilde{f}, U_2 \times B),$$

implying that

$$\deg_k(f, U) = \deg_k(f, U_1) + \deg_k(f, U_2).$$

(A3). Let us take $f, g \in \mathcal{K}_k(\overline{U})$ and homotopy joining them, i.e. such $h : [0, 1] \times \overline{U} \to \mathbb{R}^k$ that $h(0, x) = f(x)$, $h(1, x) = g(x)$ and $h(t, x) \neq 0$ for $t \in [0, 1]$ and $x \in \partial U$. Let us now define $\tilde{h} : [0, 1] \times \overline{U} \times \overline{B} \to \mathbb{R}^k \times \mathbb{R}^{m-k}$ by

$$\tilde{h}(t, x, y) = (h(t, x), y).$$

As we can see $\tilde{h}(t, x, y) = (0, 0)$ implies $y = 0$ and $h(t, x) = 0$, what means $(x, y) \in U \times B$, so for $(x, y) \in \partial(U \times B)$ we have $\tilde{h}(t, x, y) \neq (0, 0)$ so we may apply the homotopy axiom for $\deg_m$ and observe

$$\deg_k(f, U) = \deg_m(\tilde{f}, U \times B) = \deg_m(\tilde{g}, U \times B) = \deg_k(g, U).$$

□

We will end this section with the practical definition that refers to the case of isolated zeroes of the map f.

Let x_0 be the isolated zero of the map f in U. Let $\mathcal{V}$ be the family of such open sets $V \subset U$ that $V \cap f^{-1}(0) = \{x_0\}$. Obviously when $V_1, V_2 \in \mathcal{V}$, then $V_1 \cup V_2 \in \mathcal{V}$ and

$$\deg(f, V_1) \ = \ \deg(f, V_1 \cup V_2) \ = \ \deg(f, V_2)$$

by the excision property of the degree.

The previous observation allows us to give the following definition

Definition 3. Assume x_0 is the isolated zero of the continuous map f in U. Then the index of the map f relative to the point $x_0 \in U$ is given by

$$i(f, x_0) \ = \ \deg(f, V),$$

where $V \in \mathcal{V}$.

4 Calculations

In this section we are going to show how the degree may be calculated by means of quite natural properties of the map $f \in \mathcal{K}(\overline{U})$. This may also be considered as the comment on the uniqueness of the degree, i.e. that there exists at most one family of maps $\deg(\cdot, U)$ satisfying axioms (A1)-(A3). It is not far from here to the proof of the existence of such family (i.e. that formulas given in this section actually always satisfy the axioms (A1)-(A3)), but as we mentioned before, we are trying to avoid technical difficulties, that do not lead to any potential applications.

Let us first observe that starting from axioms (A1)-(A3) we may calculate the degree of all C^1 maps $f \in \mathcal{K}(\overline{U})$. But we will start with the very simple map $f : [-1,1] \to \mathbb{R}$ given by $f(x) = -x$.

Lemma 1. $\deg(-I, (-1,1)) = -1$.

Proof. First of all, let us note, that by the excision property we have

$$\deg(-I, (-1,1)) = \deg(-I, (-1,3)).$$

Let us also observe that the two maps $I : [-1,3] \to \mathbb{R}$ and $f : [-1,3] \to \mathbb{R}$ given by $f(x) = x - 2$ may be joined by homotopy $h : [-1,1] \times [-1,3] \to \mathbb{R}$ given by

$$h(t,x) = x - 2t.$$

That is why $\deg(f, (-1,3)) = 1$. And, by the excision property, $\deg(f, (1,3)) = 1$.

Let us now define the continuous map $g : [-1,3] \to \mathbb{R}$ given by

$$g(x) = \begin{cases} -x & \text{for } x \in [-1,1] \\ x - 2 & \text{for } x \in [1,3]. \end{cases}$$

As we can see it maybe joined by the homotopy to the constant map (equal 2), so by the Property 4, the equality holds

$$\deg(g, (-1,3)) = 0.$$

But we can identify all zeroes of the function g, which are equal to 0 and 2, so now it is time to refer to the additivity axiom. This gives us

$$0 = \deg(g, (-1,3)) = \deg(-I, (-1,1)) + \deg(f, (1,3)) = \deg(-I, (-1,1)) + 1,$$

what completes the proof. □

Let us now consider the case of $-I : \mathbb{R}^2 \to \mathbb{R}^2$.

Lemma 2. *Let $U \subset \mathbb{R}^2$ be the open and bounded neighbourhood of $0 \in \mathbb{R}^2$. Then*

$$\deg(-I, U) = 1.$$

Proof. Let us first consider $U = B(0,1)$. The case of general U is an easy consequence of the excision property. Let us consider the homotopy $h : [0,1] \times \overline{B(0,1)} \to \mathbb{R}^2$ given by

$$h(t, (x_1, x_2)) = \begin{bmatrix} \cos(\pi t) & \sin(\pi t) \\ -\sin(\pi t) & \cos(\pi t) \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

As we can see the homotopy is well-defined and joins I with $-I$. Hence

$$\deg(-I, B(0,1)) = \deg(I, B(0,1)) = 1.$$

□

Similarly for the identity map in the open and bounded neighbourhood of zero $U \subset \mathbb{R}^k$ we may prove the following lemma.

Lemma 3. *Let $U \subset \mathbb{R}^k$ be the open and bounded neighbourhood of $0 \in \mathbb{R}^2$. Then*

$$\deg(-I, U) = (-1)^k.$$

Proof. Let us first assume that $k = 2n$. By the excision property we may assume that $U = B_2(0,1) \times B_2(0,1) \times ... \times B_2(0,1)$, where $B_2(0,1) \subset \mathbb{R}^2$ is the open ball in $\mathbb{R}^2$. On each two-dimensional component $B_2(0,1)$ the map $-I$ joined by homotopy to identity, what shows that in this case

$$\deg(-I, U) = 1.$$

On the other hand for $k = 2n+1$ let us take $U = B_2(0,1) \times B_2(0,1) \times ... \times B_2(0,1) \times (-1,1)$ and building the homotopy in the same way as above we can show that $-I$ may be joined on $\overline{U}$ by a valid homotopy to the map $\tilde{I}(x_1, x_2, ..., x_{2n}, x_{2n+1}) = (x_1, x_2, ..., x_{2n}, -x_{2n+1})$. Following the ideas of Theorem 1 and Lemma 1 we can see that

$$\deg_{2n+1}(\tilde{I}, U) = \deg_1(-I, (-1,1)) = -1.$$

what completes the proof. □

But what about the general linear map $A : \mathbb{R}^k \to \mathbb{R}^k$? Of course the only nontrivial case which we may consider is the linear isomorphism and calculate the degree relative to the ball containing 0.

Theorem 2. *Let $A : \mathbb{R}^k \to \mathbb{R}^k$ be the linear isomorphism. Then*

$$\deg(A, B(0,r)) = (-1)^\nu,$$

where ν is the sum of the multiplicity of negative eigenvalues of A.

Proof. The known fact from the linear algebra (related to Jordan canonical form of the matrix) says that there exists a decomposition $\mathbb{R}^k = X_1 \oplus X_2$ and $A = A_1 \oplus A_2$, where $A_1 : X_1 \to X_1$, $A_2 : X_2 \to X_2$, and A_2 does not have a negative eigenvalue, while A_1 has only negative eigenvalues, with sum of their multiplicities equal to $\dim X_1$.

Let us write $x = (p,q) \in X_1 \oplus X_2$ and, for the fixed $r > 0$, let us define the homotopy

$$h(t,x) = h(t,(p,q)) = (1-t)(A_1p, A_2q) + t(-p,q)) = (-tp + (1-t)A_1p, tq + (1-t)A_2q).$$

Let us check if it is possible that $h(t,(p,q)) = (0,0)$ for some nonzero vector (p,q). Then

$$\begin{cases} -tp = -(1-t)A_1p \\ tq = -(1-t)A_2q. \end{cases}$$

In case $t = 1$ we must have $p = q = 0$, so let us assume that $t \in [0,1)$. Then

$$\begin{cases} \frac{t}{1-t}p = A_1p \\ \frac{-t}{1-t}q = A_2q. \end{cases}$$

If $p \neq 0$, then $\frac{t}{1-t}$ is the nonnegative eigenvalue of A_1, which is not possible. Hence $p = 0$. On the other hand if $q \neq 0$, then $\frac{-t}{1-t}$ is the eigenvalue of A_2. But A_2 does not have 0 as an eigenvalue (because A is an isomorphism). Moreover, it does not have any negative eigenvalue. Hence also $q = 0$.

As we can see A is homotopic to $(-I|_{X_1}) \oplus I|_{X_2}$. By homotopy property and Theorem 1 we can see that

$$\deg(f, B(0,r)) = \deg(-I|_{X_1}) \oplus I|_{X_2}, B(0,r)) = \deg(-I|_{X_1}, B(0,r) \cap X_1) = (-1)^{\dim x_1} = (-1)^\nu$$

□

Corollary 1. *It is well known that the sign of the determinant of the matrix A equals to $(-1)^\nu$, where ν is the sum of the multiplicity of negative eigenvalues of A. Hence for the linear isomorphism A*

$$\deg(A, B(0,r)) = \operatorname{sign}\det A.$$

Definition 4. Let $f : \overline{U} \to \mathbb{R}^k$ be the C^1 map. We call $y \in \mathbb{R}^k$ a regular value of the map f if for all $x \in f^{-1}(y)$ the derivative $Df(x)$ is a linear isomorphism.

Lemma 4. *If 0 is a regular value of the C^1 map $f \in \mathcal{K}(U)$, then the set $f^{-1}(0) \subset U$ is finite.*

Proof. Let us take such $x_0 \in \overline{U}$, that $f(x_0) = 0$. Because $Df(x_0)$ is the linear isomorphism we know that in some nighbourhood U_{x_0} of x_0 there is no other zero of f. So each zero is isolated in the set $f^{-1}(0)$. Because of the compactness of $f^{-1}(0)$ this implies that the set is finite. □

The additivity axiom, together with the previous lemma, immediately implies what follows:

Lemma 5. *If 0 is a regular value of the C^1 map $f \in \mathcal{K}(U)$, then*

$$\deg(f, U) = \sum_{x_i \in f^{-1}(0)} i(f, x_i),$$

where $i(f, x_i)$ denotes the index od the map f relative to x_i (see Definition 3).

Proof. For each isolated zero x_i of f there exists such ball $B(x_i, r) \subset U$, that $f^{-1}(0) \cap B(x_i, r) = \{x_i\}$. By the additivity axiom we can see that

$$\deg(f, U) = \sum_{x_i \in f^{-1}(0)} \deg(f, B(x_i, r)) = \sum_{x_i \in f^{-1}(0)} i(f, x_i).$$

□

But the local degree (i.e. the index of f relative to the zero x_i) may be expressed by the derivative $Df(x_i)$. This is because in the small neighbourhood of the isolated zero x_i the map f may be joined by homotopy with the linear isomorphism $Df(x_i)$. This leads to the following theorem

Theorem 3. *If 0 is a regular value of the C^1 map $f \in \mathcal{K}(U)$, then*

$$\deg(f, U) = \sum_{x_i \in f^{-1}(0)} \operatorname{sign}\det Df(x_i).$$

Proof. First, we are going to show, that in the ball $\overline{B(x_0, r_0)}$ with radius $r_0 > 0$ small enough the homotopy $h : [0,1] \times \overline{B(x_0, r_0)} \to \mathbb{R}^k$, given by

$$h(t,x) = (1-t)Df(x_0)(x-x_0) + tf(x)$$

is well defined.

We can see that

$$|h(t,x)| \geq \|Df(x_0)\| \cdot |x-x_0| - t|Df(x_0)(x-x_0) - f(x)| =$$

$$\|Df(x_0)\||x-x_0| - t|f(x) - f(x_0) - Df(x_0)(x-x_0)|.$$

Because of the differentiability of f at x_0, there exists such $r_0 > 0$, that

$$|f(x) - f(x_0) - Df(x_0)(x-x_0)| \leq \frac{\|Df(x_0)\|}{2}|x-x_0|,$$

for $|x - x_0| < r_0$.

Then

$$|h(t,x)| \geq \frac{\|Df(x_0)\|}{2}|x-x_0| > 0$$

for $\|x - x_0\| = r_0$, what proves that the homotopy is well defined.

By homotopy axiom and Theorem 2

$$\deg(f, B(x_i, r_0)) = \deg(Df(x_i), B(x_i, r_0)) = \operatorname{sign}\det Df(x_i),$$

what completes the proof. □

But what we can do with the map that does not have 0 as the regular value? It is good to refer to the celebrated Sard's theorem.

Theorem 4 (Sard's theorem)**.** *Let $U \subset \mathbb{R}^k$ be the open and bounded set and $f : U \to \mathbb{R}^k$ the C^1 map. Then the set of critical values of f, i.e.*

$$\{y \in \mathbb{R}^k : \exists_{x\in U} f(x) = y \wedge \det Df(x) = 0\}$$

has Lebesgue measure zero.

Corollary 2. *The set of regular values of f, i.e.*

$$\{y \in \mathbb{R}^k : \forall_{x\in f^{-1}(y)} \det Df(x) \neq 0\},$$

is dense in $\mathbb{R}^k$.

Assume now that 0 is not a regular value of f. In this case from the previous corollary we may conclude that there exists the sequence $\{c_n\} \subset \mathbb{R}^k$ of regular values of f such that $c_n \to 0$. By Proposition 1 there exist such $r > 0$, that $|f(x)| \geq r$ for all $x \in \partial U$. Then we may take the regular value c of the map f satisfying $|c| < r$ and consider the homotopy $h : [0,1] \times \overline{U} \to \mathbb{R}^k$ given by

$$h(t,x) = f(x) - tc.$$

For $x \in \partial U$ we have

$$|h(t,x)| = |f(x) - tc| \geq |f(x)| - t|c| \geq r - |c| > 0,$$

hence the homotopy is well defined. This means that

$$\deg(f, U) = \deg(f_c, U),$$

where $f_c(x) = f(x) - c$ and the value of the degree does not depend on $c \in \mathbb{R}^k$ satisfying $|c| < r$. But we know how the value of $\deg(f_c, U)$ may be calculated:

$$\deg(f_c, U) = \sum_{x \in f_c^{-1}(0)} \operatorname{sign} \det Df(x) = \sum_{x \in f^{-1}(c)} \operatorname{sign} \det Df(x).$$

There is one more step left to the most general case: the continuous map $f \in \mathcal{K}(\overline{U})$. We will refer to the following well-known theorem

Theorem 5. *The set $C^1(\overline{U}) \cap \mathcal{K}(\overline{U})$ is the dense subset of $\mathcal{K}(\overline{U})$.*

Let us now fix $f \in \mathcal{K}(\overline{U})$. By Proposition 1 there exist such $r > 0$ that $|f(x)| \geq r$ for $x \in \partial U$. Hence for any $\tilde{f} \in C^1(\overline{U}) \cap \mathcal{K}(\overline{U})$ satisfying $\|f - \tilde{f}\|_\infty < r$ the homotopy $h : [0,1] \times \overline{U} \to \mathbb{R}^k$ given by

$$h(t, x) = (1 - t)f(x) + t\tilde{f}(x)$$

is well defined. Hence we can see that

$$\deg(f, U) = \deg(\tilde{f}, U),$$

and the value does not depend on the selection of $\tilde{f}$, as long as $\|f - \tilde{f}\|_\infty < r$.

This shows that for each $f \in \mathcal{K}(\overline{U})$ the degree $\deg(f, U)$ may be expressed by the determinant formula

$$\deg(f, U) = \sum_{x \in \tilde{f}^{-1}(0)} \operatorname{sign} \det D\tilde{f}(x),$$

for the properly selected function $\tilde{f}$. This formula allows as to prove various properties

Property 6 (multiplication)**.** *Let $f \in \mathcal{K}_k(\overline{U})$, $g \in \mathcal{K}_m(\overline{V})$. Then*

$$\deg_{k+m}(f \oplus g, U \oplus V) = \deg_k(f, U) \cdot \deg_m(g, V),$$

where $f \oplus g : \overline{U} \oplus \overline{V} \to \mathbb{R}^k \oplus \mathbb{R}^m$ is given by $(f \oplus g)(x, y) = (f(x), g(y))$, and $\deg_k$ and $\deg_m$ are induced by $\deg_{k+m}$ as in Theorem 1.

Proof. The observation is easy for the C^1 maps f, g having 0 as the regular value. We can see that $(f \oplus g)(x, y) = (0, 0)$ iff $f(x) = 0$ and $g(y) = 0$, so

$$\deg_{k+m}(f \oplus g, U \oplus V) = \sum_{x \in f^{-1}(0); y \in g^{-1}(0)} \operatorname{sign} \det D(f \oplus g)(x, y) =$$

$$= \sum_{x \in f^{-1}(0); y \in g^{-1}(0)} \operatorname{sign} \det Df(x) \cdot \operatorname{sign} \det Dg(y) =$$

$$= \deg_k(f, U) \cdot \deg_m(g, V).$$

The general continuous case may be achieved by the approximation of $f \oplus g$ with C^1 maps $\tilde{f} \oplus \tilde{g}$ having $(0, 0)$ as the regular value. ∎

Let us have a look at one more consequence of the determinant formula.

Theorem 6 (change of variables)**.** *Let* $A:\mathbb{R}^k\to\mathbb{R}^k$ *be the linear isomorphism and let* $0\notin f(\partial U)$ *. Then* $\deg(f,U)=\deg(AfA^{-1},A(U))$ *.*

Proof. Let us first assume that $f\in C^1(\overline{U})$ and 0 is the regular value of f. Then

$$\deg\left(AfA^{-1},A(U)\right) = \sum_{x\in(AfA^{-1})^{-1}(0)} \operatorname{sign}\det D(AfA^{-1})(x)$$

Let us observe that $(AfA^{-1})^{-1}(0) = Af^{-1}A^{-1}(0) = Af^{-1}(0)$ and let us substitute $y = A^{-1}(x)$.

$$\begin{aligned}\deg\left(AfA^{-1},A(U)\right) &= \sum_{y\in f^{-1}(0)} \operatorname{sign}\det\left(ADf(y)A^{-1}\right) = \\ &= \sum_{y\in f^{-1}(0)} \operatorname{sign}\det A\cdot\operatorname{sign}\det Df(y)\cdot\operatorname{sign}\det A^{-1} = \\ &= \sum_{y\in f^{-1}(0)} \operatorname{sign}\det Df(y) = \deg(f,U)\end{aligned}$$

Let us now consider $f\in C(\overline{U})$. Then there exists such $g\in C^1(\overline{U})$, with 0 as the regular value, that $\deg(f,U)=\deg(g,U)$ and

$$|f(x)-g(x)|<\varepsilon\,,$$

for all $x\in\overline{U}$ and some $\varepsilon>0$.

Then

$$\begin{aligned}\left|AfA^{-1}(x)-AgA^{-1}(x)\right| &= \left|A(f-g)A^{-1}(x)\right| \le \\ &\le \|A\|\cdot\left|(f-g)A^{-1}(x)\right| < \|A\|\cdot\varepsilon\,.\end{aligned}$$

For ε small enough: $\deg\left(AfA^{-1},A(U),0\right)=\deg\left(AgA^{-1},A(U)\right)$. But, according to what we can see above, $\deg(g,U)=\deg\left(AgA^{-1},A(U)\right)$. Hence we can see that

$$\deg(f,U)=\deg(g,U)=\deg(AgA^{-1},A(U))=\deg(AfA^{-1},A(U)).$$

□

5 Sample applications

Let us look now at a series of theorems that present very general results concerning existence of zeroes (or fixed points) of nonlinear maps. The proofs will mainly refer to the homotopy property, showing that if we are able to define the appropriate deformation of our map, to the map of the known degree (e.g. to the identity map), then we can say something about the existence of the zero of the map.

All balls and other sets used in this section are subsets of $\mathbb{R}^k$.

Theorem 7. *Let* $f:\overline{B(0,r)}\to\mathbb{R}^k$ *be a continuous map such that* $|f(x)|\le r$ *for all* $|x|=r$*. Then there exists a fixed point of* f*.*

Proof. If f has a fixed point in $\partial B(0,r)$, then we are done. So we may assume that $f(x) \neq x$ for $|x| = r$.

Now define the homotopy $h : [0,1] \times \overline{B(0,r)} \to \mathbb{R}^k$ by

$$h(t,x) = x - tf(x).$$

As we can see $h(1,x) \neq x$ by our assumption. And for $t \in [0,1)$ we have

$$t|f(x)| \leq tr < r = |x|,$$

hence $x \neq tf(x)$ and the homotopy is well defined. Thus we may conclude that by axioms (A1) and (A3) we have

$$\deg(I - f, B(0,r)) = \deg(I, B(0,r) = 1$$

what, by existence property 5, implies that there exists zero of the map $I - f$, i.e. the fixed point of f. □

The well-known Brouwer fixed point theorem is the immediate consequence of the Theorem 7 given above.

Theorem 8 (Brouwer fixed point theorem)**.** *Let $f : \overline{B(0,1)} \to \overline{B(0,1)}$ be a continuous map. Then there exists a fixed point of f.*

Similarly we may prove the following:

Theorem 9. *Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be such continuous map that*

$$\lim_{|x| \to +\infty} \frac{|f(x)|}{|x|} = 0,$$

(such maps are called sublinear), then there exists a fixed point of f

Proof. Let us observe that there exists such $R > 0$ that $f(\overline{B(0,R)} \subset B(0,R)$. Otherwise there would exist $|x_n| \to +\infty$ satisfying $|f(x_n)| \geq |x_n|$ what contradicts the sublinearity assumption.

Having such ball $B(0,R)$ we may apply the Brouwer fixed point theorem and conclude that there exists the fixed point of f in $\overline{B(0,R)}$. □

We may also prove the following generalization of the Brouwer fixed point theorem

Theorem 10. *Let $A \subset \mathbb{R}^k$ be the set homeomorphic to $\overline{B(0,1)}$ and $f : A \to A$ any continuous map. Then there exists the fixed point of A.*

The proof is left to the Reader.

Slightly different – more geometrical conditions – may also lead to existence theorems

Theorem 11. *Let U be an open and bounded subset of $\mathbb{R}^k$ and $f : \overline{U} \to \mathbb{R}^k$ be a continuous map. If $\langle f(x), x \rangle > 0$ for all $x \in \partial U$, then there exists zero of the map f.*

Proof. This is, again, an easy application of the homotopy axiom (A3). Let us define the homotopy $h : [0,1] \times \overline{U} \to \mathbb{R}^k$ by

$$h(t,x) = (1-t)f(x) + tx.$$

As we can see, since it is not possible that $f(x) = 0$ or $x = 0$, for $x \in \partial U$

$$\langle h(t,x), h(t,x) \rangle = (1-t)^2 |f(x)|^2 + 2t(1-t)\langle f(x), x \rangle + t^2 |x|^2 > 0,$$

for $x \in \partial U$ and $t \in [0,1]$, what implies that $h(t,x) \neq 0$.

This implies that

$$\deg(f, B(0,r)) = \deg(I, B(0,r)) = 1$$

and there exists 0 of the map f. □

The above theorem leads to an interesting corollary:

Theorem 12. *Let $f : \mathbb{R}^k \to \mathbb{R}^k$ be a continuous map. If*

$$\lim_{|x| \to +\infty} \frac{\langle f(x), x \rangle}{|x|} = +\infty,$$

then f maps $\mathbb{R}^k$ onto $\mathbb{R}^k$ (i.e. $f(\mathbb{R}^k) = \mathbb{R}^k$).

Proof. Let us take any $y \in \mathbb{R}^k$ and define the map $f_y : \mathbb{R}^k \to \mathbb{R}^k$ by $f_y(x) = f(x) - y$. We will show that there exists the zero of the map f_y i.e. such point $x \in \mathbb{R}^k$ that $f(x) = y$.

Let us first observe that for any $x \in \mathbb{R}^k$ we can write

$$\frac{\langle f_y(x), x \rangle}{|x|} = \frac{\langle f(x), x \rangle - \langle y, x \rangle}{|x|} = \frac{\langle f(x), x \rangle}{|x|} - \frac{\langle y, x \rangle}{|x|} \geq \frac{\langle f(x), x \rangle}{|x|} - |y|.$$

Because there exists such $R_y > 0$, that

$$\frac{\langle f(x), x \rangle}{|x|} \geq |y| + 1$$

for $|x| \geq R_y$, we can see that the map f_y satisfies assumptions of the previous Theorem 11 for $U = B(0, R_y)$. So there exists zero of the map f_y, what completes the proof. □

References

[1] W. Krawcewicz, J. Wu, *Theory of Degrees, with Applications to Bifurcations and Differential Equations*, Wiley-Interscience (1997).

[2] Jean Mawhin, Leray-Schauder Degree: a Half Century Of Extensions And Applications, Topological Methods in Nonlinear Analysis, Journal of the Juliusz Schauder Center, Volume 14, 1999, 195Ű228

[3] L. Nirenberg, *Topics in nonlinear functional analysis.*, Courant Institute of Mathematical Sciences New York University, New York (1974).

[4] Donal O'Regan, Yeol Je Cho, Yu Qing Chen, *Topological degree theory and applications*, CRC Press, 2006

[5] M. Schechter, *An Introduction to Nonlinear Analysis*, Cambridge University Press, 2004.