Limit Theorems for Markov operators - the list of exercises

Tomasz Szarek

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Exercise 1. Let $(X_n)_{n\geq 1}$ be a Markov chain. Show that $(f(X_n))_{n\geq 1}$ may not be a Markov chain.

Exercise 2. Suppose that $\{\mathcal{F}_n : n \geq 0\}$ is a filtration over $(\Omega, \mathcal{F}, \mathbf{P})$ such that $\mathcal{F}_0$ is trivial and $\{Z_n : n \geq 1\}$ is a sequence of square integrable martingale differences, i.e. it is $\{\mathcal{F}_n : n \geq 1\}$ adapted, $\mathbf{E}Z_n^2 < \infty$ and $\mathbf{E}[Z_n|\mathcal{F}_{n-1}] = 0$ for all $n \geq 1$. Define the martingale

$$M_n := \sum_{j=1}^{N} Z_j, \quad N \geq 1, \quad M_0 := 0$$

and its quadratic variation $\langle M\rangle_N := \sum_{j=1}^{N} \mathbf{e}[Z_j^2|\mathcal{F}_{j-1}]$ for $N \geq 1$.

Assume that

(M1) for every $\varepsilon > 0$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \mathbf{E}[Z_{j+1}^2, |Z_{j+1}| \geq \varepsilon\sqrt{N}] = 0,$$

(M2) we have

$$\sup_{n\geq 1} \mathbf{E}Z_n^2 < \infty$$

and there exists $\sigma \geq 0$ such that

$$\lim_{K\to\infty} \limsup_{l\to\infty} \frac{1}{l} \sum_{m=1}^{l} \mathbf{E}|K^{-1}\mathbf{E}[\langle M\rangle_{mK} - \langle M\rangle_{(m-1)K}|\mathcal{F}_{(m-1)K}]$$
$$-\sigma^2| = 0.$$

(M3) for every $\varepsilon > 0$

$$\lim_{K\to\infty} \limsup_{l\to\infty} \frac{1}{lK} \sum_{m=1}^{l} \sum_{j=(m-1)K}^{mK-1} \mathbf{E}[1 + Z_{j+1}^2, |M_j - M_{(m-1)K}|$$
$$\geq \varepsilon\sqrt{lK}] = 0.$$

.

Show that we have

$$\lim_{N\to\infty} \frac{\mathbf{E}\langle M\rangle_N}{N} = \sigma^2$$

and

$$\lim_{N\to\infty} \mathbf{E}e^{itM_N/\sqrt{N}} = e^{-\sigma^2 t^2/2} \qquad \text{for all } t \in \mathbb{R}.$$

Exercise 3. Show that the space $\mathcal{M}_{1,1}$ of all probability Borel measures μ on (X, ρ) such that $\int_X \rho(x, x_0)\mu(\mathrm{d}x) < \infty$ for some $x_0 \in X$ with the Wasserstein metric

$$d(\nu_1, \nu_2) = \sup\{|\langle f, \nu_1\rangle - \langle f, \nu_2\rangle| : f : E \to \mathbb{R}, \ \mathrm{Lip}\, f \leq 1\} \quad \text{for } \nu_1, \nu_2 \in \mathcal{M}_{1,1}$$

is a complete metric space.

Exercise 4. Prove that the convergence of $(\mu_n)_{n\geq 1} \subset \mathcal{M}_{1,1}$ to $\mu \in \mathcal{M}_{1,1}$ in the Wasserstein metric is equivalent to the weak convergence.

Exercise 5. Show that any Markov chain $(X_n)_{n\geq 1}$ on (E, ρ) with the transition function π satisfying the following condition: There exists $x \in E$ such that for any $\varepsilon >$ there is a compact set $K \subset E$ such that

$$\pi^n(x, E) \geq 1 - \varepsilon$$

has an invariant measure. (Here π^n is defined by the reccurence:

$$\pi^1 = \pi \qquad \text{and} \qquad \pi^{n+1}(x, A) = \int_E \pi(y, A)\pi^N(x, \mathrm{d}y)$$

for any $x \in E$ and a Borel set $A \subset E$.)

Exercise 6. Let a Markov operaor P be given and let μ_* be an invariant meaure. If $\mu_* \in \mathcal{M}_1$ is invariant we may associate with P the dynamical system $(\Omega, \mathcal{F}, (\Theta_n)_{n\in\mathbb{N}}, \mathbb{P}_{\mu_*})$ given in the following way:

- $\Omega = X^{\mathbb{N}}$,
- $\mathcal{F} = \mathcal{B}(X)^{\mathbb{N}}$,
- $(\Theta_n)_{n\in\mathbb{N}}$ is the semigroup of measurable transformations from Ω to Ω given by the formula:
$$(\Theta_n\omega)(m) = \omega(n+m) \qquad \text{for } n, m \in \mathbb{N},$$
- $\mathbb{P}_{\mu_*}$ is the unique measure, by the Kolmogorov extension theorem, such that
$$\mathbb{P}_{\mu_*}(\{\omega \in \Omega : (\omega(m_1), \dots, \omega(m_n)) \in \Gamma)\}) = \mathbb{P}_{\mu_*}^{\{m_1,\dots,m_n\}}(\Gamma), \text{where}$$
$$P_{\mu_*}^{\{m_1,\dots,m_n\}}(\Gamma) = \int_X \mu(\mathrm{d}x) \int_X \pi^{m_1}(x, \mathrm{d}x_1) \int_X \pi^{m_2-m_1}(x_1, \mathrm{d}x_2) \cdots \int_X \pi^{m_{n-1}-m_{n-2}}(x_{n-2}, \mathrm{d}x_{n-1})$$
$$\times \int_X \pi^{m_n-m_{n-1}}(x_{n-1}, \mathrm{d}x_n)\mathbf{1}_\Gamma(x_1, \dots, x_n) \text{ for } \Gamma \in \mathcal{B}(X)^{\{m_1,\dots,m_n\}} \text{ and } m_1 < m_2 < \dots < m_n.$$

An invariant measure $\mu_* \in \mathcal{M}_1$ is called *ergodic* if the system $(\Omega, \mathcal{F}, (\Theta_n)_{n\in\mathbb{N}}, \mathbb{P}_{\mu_*})$ is ergodic. We say that a measurable set $A \subset X$ is *μ–invariant* if $\pi^n(x, A) = 1$ for μ–almost every $x \in A$ and $n \geq 1$.

Show the following characterization of an ergodic measure is well known: μ is ergodic if and only if every μ–invariant set A is of μ–measure 0 or 1

Exercise 7. Show that if μ_* is a unique invariant measure for some Markov operator P, then μ_* is ergodic.

Exercise 8. Let a Markov operator P be given and let μ_1 and μ_2 be two different invariant ergodic measures. Prove that μ_1 and μ_2 are mutually singular.

Exercise 9. An invariant probability measure for the Markov operator P is ergodic if and only if it is an extremal point of the set of all the invariant probability measures for P.