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Limit Theorems For Markov Operators - a model way of solving exercises

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Exercise 1. Let P be a Markov operaor and P^* its dual. Assume that the following conditions are satisfied:

(H0) the Markov operator satisfies the Feller property, i.e. $P^*(C_b(E)) \subset C_b(E)$.

(H1) for any $\nu \in \mathcal{M}_{1,1}$ we have $P\nu \in \mathcal{M}_{1,1}$.

(H2) there exist $\gamma \in (0, 1)$ and $c > 0$ such that

$$d_W(P^n\nu_1, P^n\nu_2) \le c\gamma^n d(\nu_1, \nu_2)$$

for $n \ge 1, \nu_1, \nu_2 \in \mathcal{M}_{1,1}$.

(H3) there exists $x_0 \in E$ and $\delta > 0$ such that

$$\sup_{n\ge 0} \mathbb{E}_\mu \rho_{x_0}^{2+\delta}(X_n) = \sup_{n\ge 0} \int \rho_{x_0}^{2+\delta} d(P^n\mu) < \infty$$

for some initial distribution $\mu \in \mathcal{M}_{1,1}$ (μ is the distribution of X_0).

Let $f : X \to \mathbb{R}$ be a bounded Lipschitz function such that $\int_X f \mathrm{d}\mu_* = 0$, where μ_* is an invariant measure. Show that

$$\chi(x) = \sum_{i=1}^{\infty} P^{*,i} f(x)$$

is a Lipschitz function.

Solution: From conditions **(H0)**–**(H3)** it follows that P has an invariant measure μ_*. Therefore,

$$\begin{aligned}|\chi(x)| &\le \sum_{i=1}^{\infty} |P^{*,i} f(x)| = \sum_{i=1}^{\infty} |P^{*,i} f(x) - \int_X f \mathrm{d}(P^i\mu_*)| \le \operatorname{Lip} f \sum_{i=1}^{\infty} d_W(P^i\delta_x - P^i\mu_*) \\ &\le c \operatorname{Lip} f \sum_{i=1}^{\infty} q^i < \infty,\end{aligned}$$

by the fact that $\int_X f \mathrm{d}(P^n\mu_*) = \int_X f \mathrm{d}\mu_* = 0$.

Further, we have

$$\begin{aligned}|\chi(x) - \chi(y)| &\le \sum_{i=1}^{\infty} |P^{*,i} f(x) - P^{*,i} f(y)| \le \operatorname{Lip} f \sum_{i=1}^{\infty} d_W(P^i\delta_x - P^i\delta_y) \\ &\le c \operatorname{Lip} f \sum_{i=1}^{\infty} q^i d_W(\delta_x - \delta_y) \le \left(c \operatorname{Lip} f \sum_{i=1}^{\infty} q^i \right) \rho(x, y)\end{aligned}$$

for any $x, y \in X$. $\square$

Exercise 2. Let us introduce random variables

$$M_n := \chi(x_n) - \chi(x_0) + \sum_{i=0}^{n-1} g(x_i) \quad \text{for } n \geq 0$$

and their square integrable differences which are of the form

$$Z_n = \chi(x_n) - \chi(x_{n-1}) + g(x_{n-1}) \quad \text{for } n \geq 1$$

and

$$Z_0 = 0 \quad \text{Prob-a.s.}$$

Show that $(M_n)_{n\geq 0}$ is a martingale on the space $(X^\infty, \otimes_{i=1}^{\infty} B_X, \text{Prob}_{\mu_*})$ with respect to its natural filtration.

Solution: Note that by the Markov property we have

$$E_{\mu_*}\left(g(x_{n+1})|\mathcal{F}_n\right)(\omega) = E_{x_n(\omega)}\left(g\right) = (P^* g)(x_n(\omega))$$

and therefore

$$\begin{aligned}
E_{\mu_*}\left(M_{n+1}|\mathcal{F}_n\right) &= E_{\mu_*}\left(\chi(x_{n+1})|\mathcal{F}_n\right) - \chi(x_0) + \sum_{i=0}^{n} g(x_i) \\
&= \sum_{i=0}^{\infty} P^*(P^{*,i} g)(x_n) - \chi(x_0) + \sum_{i=0}^{n} g(x_i) \\
&= \sum_{i=1}^{\infty} P^{*,i} g(x_n) + g(x_n) - \chi(x_0) + \sum_{i=0}^{n-1} g(x_i) = M_n.
\end{aligned}$$

□

Exercise 3. Let $(P^t)_{t\geq 0}$ be a Markov semigroup. For a given $t > 0$ and $\mu \in \mathcal{M}_1$ define $Q^t\mu := t^{-1}\int_0^t P^s\mu ds$. Let

$$\mathcal{T} := \left\{ x \in X : \text{the family of measures } \left(Q^t(x,\cdot)\right)_{t\geq 0} \text{ is tight } \right\}.$$

Show that if for any Lipschitz function f ($f \in \text{Lip}_b(X)$) and $x \in X$ we have (the e–property)

$$\lim_{y\to x} \sup_{t\geq 0} |P^{*,t} f(y) - P^{*,t} f(x)| = 0,$$

then for any $x \in \mathcal{T}$ the sequence $(Q^t(x,\cdot))_{t\geq 0}$ weakly converges to some invariant measure.

Solution: Fix $x \in \mathcal{T}$ and assume, contrary to our hypothesis, that the sequence $(Q^t(x,\cdot))_{t\geq 0}$ does not converge. Since $(Q^t(x,\cdot))_{t\geq 0}$ is tight, by the Prokhorov theorem we may find at least two different probability measures, say, μ_*, μ^* and two sequences of positive reals $(s_n)_{n\geq 1}$ and $(t_n)_{n\geq 1}$, $\lim_{n\to\infty} s_n = \lim_{n\to\infty} t_n = +\infty$, such that $(Q^{s_n}(x,\cdot))_{n\geq 1}$, $(Q^{t_n}(x,\cdot))_{n\geq 1}$ weakly converge to μ_* and μ^*, respectively.

Choose $f \in \mathcal{L}_b(X)$ such that $\int_X f \mathrm{d}\mu_* \neq \int_X f \mathrm{d}\mu^*$ and let $\varepsilon > 0$ be such that $|\int_X f \mathrm{d}\mu_* - \int_X f \mathrm{d}\mu^*| > \varepsilon$.

Let $K \subset X$ be a compact set such that $Q^t(x,K) > 1 - \varepsilon/(6\|f\|)$ for all $t \geq 0$. Hence both the measure μ_* and μ^* on the set K are grater than $1 - \varepsilon/(6\|f\|)$, by the Alexandrov theorem. Passing to a subsequence if necessary, we may assume that on the set K the sequence $(s_n^{-1}\int_0^{s_n} P_s f \mathrm{d}s)_{n\geq 1}$ converges uniformly to some $\tilde{f}_* \in C(K)$, by the e–property and the Arzela-Ascoli theorem. Let $f_* \in C(X)$ be an extension of $\tilde{f}_*$ such that $\|f_*\| = \|\tilde{f}_*\|_K$. Obviously, $\|f_*\| \leq \|f\|$. From the definition of K we obtain that $|\int_{X\setminus K} f_* \mathrm{d}\mu^*| < \varepsilon/6$ and $|\int_{X\setminus K}(s_n^{-1}\int_0^{s_n} P^{*,s} f \mathrm{d}s)\mathrm{d}\mu^*| < \varepsilon/6$. On the other hand, from the definition of f_* we have $\int_K f_* \mathrm{d}\mu^* = \lim_{n\to\infty}\int_K (s_n^{-1}\int_0^{s_n} P^{*,s} f \mathrm{d}s)\mathrm{d}\mu^*$. Consequently, we obtain

$$\left|\int_X f_* \mathrm{d}\mu^* - \int_X \left(s_n^{-1}\int_0^{s_n} P^{*,s} f \mathrm{d}s\right)\mathrm{d}\mu^*\right| < \varepsilon/3 \tag{1}$$

for n sufficiently large. Since μ^* is invariant, the second integral in the above formula equals $\int_X f \mathrm{d}\mu^*$, which finally gives

$$\left|\int_X f_* \mathrm{d}\mu^* - \int_X f \mathrm{d}\mu^*\right| < \varepsilon/3. \tag{2}$$

Let $N \in \mathbb{N}$ be such that

$$\left|\int_X f_* \mathrm{d}\mu^* - \int_X f_* \mathrm{d}(Q^{t_N}(x,\cdot))\right| < \varepsilon/3. \tag{3}$$

In the same manner as in the proof of formula (1) we may show that

$$\left|\int_X f_* \mathrm{d}(Q^{t_N}(x,\cdot)) - \int_X \left(s_n^{-1}\int_0^{s_n} P^{*,s} f \mathrm{d}s\right)\mathrm{d}(Q^{t_N}(x,\cdot))\right| < \varepsilon/3$$

for n sufficiently large and hence

$$\left|\int_X f_* \mathrm{d}(Q^{t_N}(x,\cdot)) - \int_X f \mathrm{d}(Q^{s_n,t_N}(x,\cdot))\right| < \varepsilon/3$$

for n sufficiently large. We easily check that $\lim_{n\to\infty}\|Q^{s_n,t_N}(x,\cdot) - Q^{s_n}(x,\cdot)\|_{TV} = 0$, and, therefore, we have

$$\left|\int_X f_* \mathrm{d}(Q^{t_N}(x,\cdot)) - \int_X f \mathrm{d}(Q^{s_n}(x,\cdot))\right| < \varepsilon/3$$

for n sufficiently large and consequently

$$\left|\int_X f_* \mathrm{d}(Q^{t_N}(x,\cdot)) - \int_X f \mathrm{d}\mu_*\right| < \varepsilon/3. \tag{4}$$

Combining (2), (3) and (4) we obtain $\left|\int_X f\mathrm{d}\mu^* - \int_X f\mathrm{d}\mu_*\right| < \varepsilon$, contrary to the definition of ε. $\square$

Exercise 4: Recall that $x \in X$ *generates a measure* $\mu \in \mathcal{M}_1$ if $\mu \in \operatorname{cl}\operatorname{conv}\{Q^t\delta_x : t \geq 0\}$, where the closure is taken in the weak topology. Assume that $(P_t)_{t\geq 0}$ has the e–property. Show that if $\mu \in \mathcal{M}_1$ is generated by $x \in \mathcal{T}$, then the sequence $(Q^t\mu)_{t\geq 0}$ has the same limit as $(Q^t(x,\cdot))_{t\geq 0}$.

Solution: Fix $f \in \mathcal{L}_b(X)$ with $||f|| \leq 1$. The e–property gives

$$\lim_{n\to\infty} \sup_{t\geq 0} \left|\int_X f dQ^t\mu_n - \int_X f dQ^t\mu\right| = 0,$$

provided μ_n weakly converges to μ.

Fix $\varepsilon > 0$ and let $\mu \in \mathcal{M}_1$ be generated by x. Then we may find $N \in \mathbb{N}$, $\alpha_1, \ldots, \alpha_N \geq 0$ and $\nu_1, \ldots, \nu_N \in \mathcal{M}_1$ such that $\alpha_1 + \ldots + \alpha_N = 1$ and

$$\sup_{t\geq 0} \left|\int_X f dQ^t(\alpha_1 Q^{t_1}(x,\cdot) + \ldots + \alpha_N Q^{t_N}(x,\cdot)) - \int_X f dQ^t\mu\right| < \varepsilon/2$$

for some $t_1, \ldots, t_N \geq 0$.

Further we have

$$||Q^t Q^{t_i}(x,\cdot) - Q^t(x,\cdot)||_{TV} < \varepsilon/2,$$

for all t large enough, say, $t \geq T$, and $i = 1, \ldots, N$. Therefore,

$$\left|\int_X f\mathrm{d}Q^t\mu - \int_X f\mathrm{d}Q^t(x,\cdot)\right| \leq \left|\int_X f dQ^t\mu - \int_X f dQ^t(\alpha_1 Q^{t_1}(x,\cdot) + \ldots + \alpha_N Q^{t_N}(x,\cdot))\right|$$
$$+ \sum_{i=1}^N \alpha_i ||Q^t Q^{t_i}(x,\cdot) - Q^t(x,\cdot)||_{TV} < \varepsilon$$

for $t \geq T$. Consequently, if one of the sequences $(Q^t\mu)_{t\geq 0}$, $(Q^t(x,\cdot))_{t\geq 0}$ converges, then so does the second and both have the same limit. $\square$

Exercise 5: Show that if $(P_t)_{t\geq 0}$ has the e–property and μ is an ergodic invariant measure, then every $x \in \operatorname{supp}\mu$ generates the measure μ.

Solution: We show this in three steps.

Step I: Fix $x \in \operatorname{supp}\mu$ and let $A \subset X$ be an open neighbourhood of x. Define

$$B = \{y \in X : \pi^t(y, A) = 0 \quad \text{for all } t \geq 0\}.$$

We easily check that

$$B = \bigcap_{q\in\mathbb{Q}_+} \{y \in X : \pi^q(y, A) = 0\}.$$

Now we show that $\mu(B) = 0$. Assume, contrary to our claim, that $\mu(B) > 0$. We shall prove that $P^{*,t}\mathbf{1}_B = \mathbf{1}_B$, hence $\mu(B) = 1$, by the ergodicity of the measure μ. On the other hand, since $B \subset X \setminus A$ and $\mu(A) > 0$, this leads to a contradiction. So, to finish this step we show that $P^{*,t}\mathbf{1_B} \geq \mathbf{1_B}$. Fix $z \in B$. We have $P^{*,t}\mathbf{1}_B(z) = \int_X \mathbf{1}_B(y)\pi^t(z, \mathrm{d}y) = \pi^t(z, B)$. Let $\mathbb{Q}_+ = \{q_1, q_2, \ldots\}$. Set $C_n = \{y : \pi^{q_n}(y, A) > 0\}$. Observe that if $\pi^t(z, X \setminus B) > 0$, then $\pi^t(z, C_m) > 0$ for some $m \in \mathbb{N}$, since $X \setminus B = \bigcup_{i=1}^\infty C_i$. Then $\pi^{t+q_m}(z, A) = \int_X \pi^{q_m}(y, A)\pi^t(z, \mathrm{d}y) \geq \int_{C_m} \pi^{q_m}(y, A)\pi^t(z, \mathrm{d}y) > 0$, contrary to our definition of z. Therefore $\mu(B) = 1$ and we are done.

Step II: Fix an $\varepsilon > 0$. In this step we are going to show that

$$\gamma := \sup \nu(X) = 1,$$

where the supremum is taken over all ν's such that $\mu \geq \nu$ and $\nu = \alpha_1 P^{t_1}\nu_1 + \ldots + \alpha_n P^{t_n}\nu_n$ for some probability measures $\nu_1, \ldots, \nu_n$ supported on $B(x, \varepsilon)$ and $\alpha_1, \ldots, \alpha_n, t_1, \ldots, t_n \geq 0$.

If not, there exist a sequence $(\nu_n)_{n\geq 1}$ such that $1 > \gamma = \lim_{n\to\infty} \nu_n(X)$ and ν_n are as above. Set $\mu_n = \mu - \nu_n$ for $n \geq 1$. Obviously the sequence $(\mu_n)_{n\geq 1}$ is tight, and therefore there exists $\mu_* \neq 0$ such that μ_n's converge weakly to μ_*, passing to a subsequence if necessary. Let $A = B(x, \varepsilon)$. By Step I we may choose $z \in \operatorname{supp} \mu_* \subset \operatorname{supp} \mu$ and $t \geq 0$ such that $\eta = \pi^t(z, A) > 0$. From the Feller property it follows that there exists $\theta > 0$ such that $\pi^t(y, A) \geq \eta/2$ for any $y \in B(z, \theta)$. Denote by $\alpha = \mu_*(B(z, \theta))$. Let $N \in \mathbb{N}$ be such that $\gamma - \nu_N(X) < \eta\alpha/4$ and $\mu_N(B(z, \theta)) > \alpha/2$ by the fact that $(\nu_n)_{n\geq 1}$ converges weakly to μ_* and by the Alexandrov theorem. Then we have

$$P^t\mu_N(A) = \int_X \pi^t(y, A)\mu_N(\mathrm{d}y) \geq \alpha\eta/4$$

and consequently we obtain

$$\mu = P^t\mu \geq P^t\nu_N + (\alpha\eta/4)\tilde{\nu},$$

where $\tilde{\nu}(\cdot) = (P^t\mu - P^t\nu_N)(\cdot \cap A)/(P^t\mu - P^t\nu_N)(A)$. Hence $\gamma \geq \nu_N(X) + \alpha\eta/4$, which contradicts the definition of ν_N.

Step III: From the previous step it easily follows that μ is generated by any $x \in \operatorname{supp} \mu$. Indeed, we see that for an arbitrary $\varepsilon > 0$ we may find $\nu_1, \ldots, \nu_N$ supported on $B(x, \varepsilon)$ and $\alpha_1, \ldots, \alpha_N \geq 0$, $t_1, \ldots, t_N \geq 0$ such that

$$\mu \geq \alpha_1 P^{t_1}\nu_1 + \ldots + \alpha_N P^{t_N}\nu_N$$

and $\alpha_1 + \ldots + \alpha_N > 1 - \varepsilon/2$. Obviously we may choose $T > 0$ such that

$$\|Q^T\nu_i - Q^T(P^{t_i}\nu_i)\|_{TV} < \varepsilon/2 \qquad \text{for all } i \in \{1, \ldots, N\}.$$

Hence we obtain

$$\begin{aligned}
&\|\mu - (\alpha_1 Q^T\nu_1 + \ldots + \alpha_N Q^T\nu_N)\|_{TV} = \|Q^T\mu - (\alpha_1 Q^T\nu_1 + \ldots + \alpha_N Q^T\nu_N)\|_{TV} \\
&\leq \|Q^T\mu - Q^T(\alpha_1 P^{t_1}\nu_1 + \ldots + \alpha_N P^{t_N}\nu_N)\|_{TV} \\
&+ \|(\alpha_1 Q^T P^{t_1}\nu_1 + \ldots + \alpha_N Q^T P^{t_N}\nu_N) - (\alpha_1 Q^T\nu_1 + \ldots \alpha_N Q^T\nu_N)\|_{TV} \\
&\leq \|\mu - (\alpha_1 P^{t_1}\nu_1 + \ldots + \alpha_N P^{t_N}\nu_N)\|_{TV} + \sum_{i=1}^N \alpha_i \|Q^T\nu_i - Q^T P^{t_i}\nu_i\|_{TV} < \varepsilon.
\end{aligned}$$