Limit Theorems For Markov Operators

Tomasz Szarek

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1 The purpose of the lecture

We are aimed at presenting some techniques for proving limit theorems for Markov chains. Limit theorems (the Strong Law of Large Numbers (**SLLN**), the Central Limit Theorem (**CLT**) and the Law of Large Numbers) have been intensively studied for many years. At the beginning these results had been proved for independend random variables. Thereafter, the proofs were extended to martingales which finally were used for proving limit theorems for Markov processes starting with a stationary distribution. Recently it has been shown that instead of considering stationary Markov processes, we may assume that the initial state is arbitrary but the process converges at exponential rate to equilibrium (see [13]) . To exemplify this progress we shall present in these notes the proof of the LIL for Markov chains with spectral gap in the Wasserstein norm.

2 Assumptions and auxiliary results

For every measure $\nu \in \mathcal{M}_1$ the law of the Markov chain $(X_n)_{n\geq 0}$ with transition probability π and initial distribution ν, is the probability measure $\mathbb{P}_\nu$ on $(E^{\mathbb{N}}, \mathcal{B}(E)^{\otimes \mathbb{N}})$ such that:

$$\mathbb{P}_\nu[X_{n+1} \in A | X_n = x] = \pi(x, A) \quad \text{and} \quad \mathbb{P}_\nu[X_0 \in A] = \nu(A),$$

where $x \in E$, $A \in \mathcal{B}(E)$. The expectation with respect to $\mathbb{P}_\nu$ is denoted by $\mathbb{E}_\nu$. For $\nu = \delta_x$, the Dirac measure at $x \in E$, we write just $\mathbb{P}_x$ and $\mathbb{E}_x$.

We will make the following assumption:

(H0) the Markov operator satisfies the Feller property, i.e. $P(C_b(E)) \subset C_b(E)$.

We shall denote by $\mathcal{M}_{1,1}$ the space of all probability measures possessing finite first moment, i.e. $\nu \in \mathcal{M}_{1,1}$ iff $\nu \in \mathcal{M}_1$ and $\int_E \rho(x_0, x)\nu(\mathrm{d}x) < \infty$ for some (thus all) $x_0 \in E$. For abbreviation we shall write $\rho_{x_0}(x) = \rho(x_0, x)$. Observe that every Lipschitz function (even unbounded) is integrable with respect to ν in $\mathcal{M}_{1,1}$. We assume that:

(H1) for any $\nu \in \mathcal{M}_{1,1}$ we have $\nu P \in \mathcal{M}_{1,1}$.

It may be proved that the space $\mathcal{M}_{1,1}$ is a complete metric space when equipped with the Wasserstein metric

$$d(\nu_1, \nu_2) = \sup\{|\langle f, \nu_1\rangle - \langle f, \nu_2\rangle| : f : E \to \mathbb{R},\ \mathrm{Lip}\, f \leq 1\} \quad \text{for } \nu_1, \nu_2 \in \mathcal{M}_{1,1}.$$

Here $\mathrm{Lip}\, f$ denotes the Lipschitz constant of f. The convergence in the Wasserstein metric is equivalent to the weak convergence plus convergence of the first moment, see e.g. [25, Theorem 6.9]. The main assumption made in our note says that the Markov operator P is contractive with respect to the Wasserstein metric, i.e.

(H2) there exist $\gamma \in (0, 1)$ and $c > 0$ such that

$$d(\nu_1 P^n, \nu_2 P^n) \leq c\gamma^n d(\nu_1, \nu_2), \qquad \text{for } n \geq 1,\ \nu_1, \nu_2 \in \mathcal{M}_{1,1}. \tag{2.1}$$

Assumption (H2) is called spectral gap property. Let $\mu \in \mathcal{M}_{1,1}$. From now on we shall assume that the initial distribution of $(X_n)_{n\geq 0}$ is μ. Moreover,

(H3) there exists $x_0 \in E$ and $\delta > 0$ such that

$$\sup_{n\geq 0} \mathbb{E}_\mu \rho_{x_0}^{2+\delta}(X_n) = \sup_{n\geq 0} \int \rho_{x_0}^{2+\delta} d(\mu P^n) < \infty. \tag{2.2}$$

It is easy to prove that under the assumptions (H0)–(H2) there exists a unique invariant (ergodic) measure $\mu_* \in \mathcal{M}_1$. In particular, $\mu_* \in \mathcal{M}_{1,1}$. The proof was given for Markov processes with continuous time in [13] but it still remains valid in the discrete case. In the stationary case (H3) means that $\rho_{x_0}^{2+\delta}$ is in $L_1(\mu_*)$.

Let $n_0 \geq 2$ be such that

$$\gamma_0 = c^2\gamma^{n_0} < 1.$$

We start this part of the paper with a rather technical lemma.

Lemma 2.1. *Let $g_{n,k} : E^{2(k+n)} \to \mathbb{R}$ for arbitrary $k, n \geq 1$, be Lipschitz continuous in each variable with the same Lipschitz constant L. Then there exists a constant $\tilde{L}$ dependent only on L and such that the function*

$$\begin{aligned} H_{n,k}(x) = &\int_E \pi_1(x, \mathrm{d}y_1) \int_E \pi_2(y_1, \mathrm{d}y_2) \cdots \int_E \pi_{2(k+n)-1}(y_{2(k+n)-2}, \mathrm{d}y_{2(k+n)-1}) \\ &\times \int_E \pi_{2(k+n)}(y_{2(k+n)-1}, \mathrm{d}y_{2(k+n)}) g_{n,k}(y_1, \ldots, y_{2(k+n)}), \end{aligned} \tag{2.3}$$

where $\pi_l(y_{l-1}, \mathrm{d}y_l) = \delta_{y_{l-1}} P^{k_l}(\mathrm{d}y_l)$, $k_l \geq 1$ and additionally $k_l \geq n_0 - 1$ for all even l, is Lipschitzean with the Lipschitz constant $\tilde{L}$.

Proof. Define the functions $g_j : E^j \to \mathbb{R}$ by the formula

$$\begin{aligned} g_j(y_0, y_1, \ldots, y_{j-1}) = &\int_E \pi_j(y_{j-1}, \mathrm{d}y_j) \int_E \pi_{j+1}(y_j, \mathrm{d}y_{j+1}) \times \cdots \\ \times &\int_E \pi_{2(k+n)}(y_{2(k+n)-1}, \mathrm{d}y_{2(k+n)}) g_{n,k}(y_1, \ldots, y_{2(k+n)}) \text{ for } j = 1, \ldots, 2(k+n). \end{aligned}$$

Let $\mathcal{L}_{j,l}$ for $j = 1, \ldots, 2(k+n)$ and $l = 0, \ldots, j-1$ denote the Lipschitz constant of g_j with respect to y_l. Then the Lipschitz constant of $H_{n,k}$ is equal to $\mathcal{L}_{1,0}$. It is obvious that $\mathcal{L}_{j,l} \leq L$ for $0 \leq l < j-1$, $j > 1$. To evaluate $\mathcal{L}_{j,j-1}$ fix $y_0, y_1, \ldots, y_{j-2}$ and $\tilde{y}_{j-1}, \hat{y}_{j-1}$.

Then we have

$$\begin{aligned}
& g_j(y_0, y_1, \ldots, y_{j-2}, \hat{y}_{j-1}) - g_j(y_0, y_1, \ldots, y_{j-2}, \tilde{y}_{j-1}) \\
&\qquad = \int_E \pi_j(\hat{y}_{j-1}, \mathrm{d}y_j) g_{j+1}(y_0, y_1, \ldots, \hat{y}_{j-1}, y_j) \\
&\qquad - \int_E \pi_j(\tilde{y}_{j-1}, \mathrm{d}y_j) g_{j+1}(y_0, y_1, \ldots, \tilde{y}_{j-1}, y_j) \\
&\qquad = \int_E \pi_j(\hat{y}_{j-1}, \mathrm{d}y_j)(g_{j+1}(y_0, y_1, \ldots, \hat{y}_{j-1}, y_j) \\
&\qquad - g_{j+1}(y_0, y_1, \ldots, \tilde{y}_{j-1}, y_j)) \\
&\qquad + \int_E \pi_j(\hat{y}_{j-1}, \mathrm{d}y_j) g_{j+1}(y_0, y_1, \ldots, \tilde{y}_{j-1}, y_j) \\
&\qquad - \int_E \pi_j(\tilde{y}_{j-1}, \mathrm{d}y_j) g_{j+1}(y_0, y_1, \ldots, \tilde{y}_{j-1}, y_j)
\end{aligned}$$

and consequently

$$\begin{aligned}
& |g_j(y_0, y_1, \ldots, \hat{y}_{j-1}) - g_j(y_0, y_1, \ldots, \tilde{y}_{j-1})| \\
& \leq \mathcal{L}_{j+1,j-1} \rho(\hat{y}_{j-1}, \tilde{y}_{j-1}) \int_E \pi_j(\hat{y}_{j-1}, \mathrm{d}y_j) \\
& + |\langle P^{k_j} \tilde{g}_{j+1}, \delta_{\hat{y}_j} \rangle - \langle P^{k_j} \tilde{g}_{j+1}, \delta_{\tilde{y}_j} \rangle| \\
& \leq L\rho(\hat{y}_{j-1}, \tilde{y}_{j-1}) + c_j \mathcal{L}_{j+1,j} \rho(\hat{y}_{j-1}, \tilde{y}_{j-1}),
\end{aligned}$$

where $c_j = c\gamma$ if j odd, $c_j = c\gamma^{n_0-1}$ if j even and

$$\tilde{g}_{j+1}(\cdot) = g_{j+1}(y_0, y_1, \ldots, y_{j-2}, \tilde{y}_{j-1}, \cdot).$$

Hence we have

$$\mathcal{L}_{j,j-1} \leq L + c_j \mathcal{L}_{j+1,j} \qquad \text{for } j = 1, \ldots, 2(k+n) - 1.$$

Since $\mathcal{L}_{2(k+n),2(k+n)-1} \leq L$, an easy computation shows that

$$\mathcal{L}_{1,0} \leq \frac{L(c\gamma + 1)}{1 - \gamma_0}.$$

This completes the proof. □

3 The law of the iterated logarithm

3.1 A martingale result

We start with recalling a classical result due to C.C. Heyde and D.J. Scott [10]. Let $\{M_n, \mathcal{F}_n : n \geq 0\}$ be a martingale on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}_0 = \{\Omega, \emptyset\}$ and $\mathcal{F}_n$ is the σ–field generated by $M_1, M_2, \ldots, M_n$ for $n > 0$. Let $Z_0 = M_0 = 0$ $\mathbb{P}$-a.s. and $Z_n = M_n - M_{n-1}$ for $n \geq 1$. Further, let $s_n^2 = \mathbb{E}M_n^2 < \infty$.

We consider the metric space $(C, \tilde{\rho})$ of all real-valued continuous functions on $[0, 1]$ with

$$\tilde{\rho}(x, y) = \sup_{0 \le t \le 1} |x(t) - y(t)| \qquad \text{for } x, y \in C.$$

Let K be the set of absolutely continuous functions $x \in C$ such that $x(0) = 0$ and $\int_0^1 (x'(t))^2 \mathrm{dt} \le 1$.

Define the real function g on $[0, \infty)$ by $g(s) = \sup\{n : s_n^2 \le s\}$. We define a sequence of real random functions η_n on $[0, 1]$, for $n > g(e)$, by

$$\eta_n(t) = \frac{M_k + (s_n^2 t - s_k^2)(s_{k+1}^2 - s_k^2)^{-1} Z_{k+1}}{\sqrt{2 s_n^2 \log\log s_n^2}}$$

if $s_k^2 \le s_n^2 t \le s_{k+1}^2$, $k = 1, \dots, n-1$ and

$$\eta_n(t) = 0 \qquad \text{for } n \le g(e).$$

Proposition 3.1. (Theorem 1 in [10]) *Under the above notations for the square integrable martingale $\{M_n\}$, if $s_n^2 \to \infty$ and*

$$\sum_{n=1}^{\infty} s_n^{-4} \mathbb{E}[Z_n^4 \mathbf{1}_{\{|Z_n| < \gamma s_n\}}] < \infty \qquad \textit{for some } \gamma > 0, \tag{3.1}$$

$$\sum_{n=1}^{\infty} s_n^{-1} \mathbb{E}[|Z_n| \mathbf{1}_{\{|Z_n| \ge \epsilon s_n\}}] < \infty \qquad \textit{for all } \epsilon > 0, \tag{3.2}$$

$$s_n^{-2} \sum_{k=1}^{n} Z_k^2 \to 1 \quad \mathbb{P}\textit{-a.s. as } n \to \infty \tag{3.3}$$

hold, then $\{\eta_n\}_{n \ge 1}$ is relatively compact in C and the set of its limit points coincides with K.

3.2 Application to Markov chains

Let (E, ρ) be a Polish space, $\{X_n\}$ a Markov chain with state space E, transition operator P satisfying conditions (H0)–(H2), and initial probability μ satisfying (H3). Let $\psi : E \to \mathbb{R}$ be a Lipschitz function with Lipschitz constant $L > 0$, such that $\langle \psi, \mu_* \rangle = 0$ (otherwise we could consider $\tilde{\psi} = \psi - \langle \psi, \mu_* \rangle$).

For every n and any Lipschitz function ψ, Minkowski's inequality in $L_{2+\delta}(\mu P^n)$ and (H3) yield

$$\begin{aligned}(\mathbb{E}_\mu[|\psi(X_n)|^{2+\delta}])^{1/(2+\delta)} &= \left(\int_E |\psi(x)|^{2+\delta} d(\mu P^n)\right)^{1/(2+\delta)} \\ &\le |\psi(x_0)| + L \left(\int_E \rho_{x_0}^{2+\delta} d(\mu P^n)\right)^{1/(2+\delta)} < \infty\end{aligned} \tag{3.4}$$

and consequently $\sup_{n\geq 0} \mathbb{E}_\mu[|\psi(X_n)|^{2+\delta}] < \infty$.

We have

$$\sum_{i=0}^{\infty} |P^i\psi(x)| = \sum_{i=0}^{\infty} |\langle \psi, \delta_x P^i\rangle - \langle \psi, \mu_* P^i\rangle| \leq cd(\delta_x, \mu_*) \sum_{i=0}^{\infty} \gamma^i < \infty,$$

by (H2). Thus we may define the function

$$\chi(x) := \sum_{i=0}^{\infty} P^i\psi(x) \qquad \text{for } x \in E.$$

We easily check that χ is a Lipschitz function (see Lemma 4.5 of [13]) and satisfies the Poisson equation $\chi - P\chi = \psi$.

It is well known, e.g. [9], that

$$M_n = \chi(X_n) - \chi(X_0) + \sum_{i=0}^{n-1} \psi(X_i) \qquad \text{for } n \geq 0$$

is a martingale on the space $(E^{\mathbb{N}}, \mathcal{B}(E)^{\otimes\mathbb{N}}, \mathbb{P}_\mu)$ with respect to the natural filtration; its square integrable martingale differences are of the form

$$Z_n = \chi(X_n) - \chi(X_{n-1}) + \psi(X_{n-1}) \quad \text{for } n \geq 1.$$

Observe that $\mathbb{E}_{\mu_*} Z_1^2 < \infty$. Indeed, we easily check that $x \to \mathbb{E}_x(Z_1^2 \wedge k)$ for any $k \geq 1$ is a bounded continuous function. Further, since $\mathbb{E}_{\mu P^n}(Z_1^2 \wedge k) = \int_E \mathbb{E}_x(Z_1^2 \wedge k)\mu P^n(\mathrm{d}x) \to \mathbb{E}_{\mu_*}(Z_1^2 \wedge k)$ for any $k \geq 1$ as $n \to \infty$ and $\sup_{n\geq 0} \mathbb{E}_{\mu P^n}(Z_1^2) < \infty$, we obtain that $\mathbb{E}_{\mu_*}(Z_1^2) < \infty$.

Set

$$\sigma^2 := \mathbb{E}_{\mu_*} Z_1^2.$$

We have

$$\lim_{n\to\infty} \mathbb{E}_\mu Z_n^2 = \lim_{n\to\infty} \mathbb{E}_{\mu P^n} Z_1^2 = \sigma^2. \tag{3.5}$$

Remark. The variance σ^2 defined above and the variance which appears as the variance of the limiting normal distribution in the CLT (see eg. [6] and [9]) are the same. One can prove that the function χ, which solves the Poisson equation for ψ, is in $L_2(\mu_*)$ in a similar way we proved $E_{\mu_*}(Z_1^2) < \infty$. By [9] the martingale differences are in fact $Z_n = \chi(X_n) - P\chi(X_{n-1})$, and it can be easily computed that

$$\sigma^2 = E_{\mu_*}(Z_1^2) = E_{\mu_*}[\chi(X_1) - P\chi(X_0)]^2 = \int_E \chi^2 d\mu_* - \int_E (P\chi)^2 d\mu_* \tag{3.6}$$

and the last term is precisely the variance of the limiting normal distribution given in [9]. The fact that χ is in $L_2(\mu_*)$ also yields for μ_*-almost every x the **CLT** in $(\Omega, \mathcal{F}, \mathbb{P}_x)$, by [2, Theorem IV.8.1].

In fact, since χ and ψ are Lipschitzean, we have $\sup_{n\geq 1}\mathbb{E}_\mu|Z_n|^{2+\delta}<\infty$, by Minkowski's inequality and (3.4). Further, observe that

$$\sup_{n\geq 1}\mathbb{E}_\mu(Z_n^2\mathbf{1}_{\{|Z_n|^2\geq k\}})\leq k^{-\delta/2}\sup_{n\geq 1}\mathbb{E}_\mu|Z_n|^{2+\delta}\to 0$$

as $k\to\infty$. Therefore, condition (3.5) follows from the fact that $\mathbb{E}_{\mu P^n}(Z_1^2\wedge k)\to\mathbb{E}_{\mu_*}(Z_1^2\wedge k)$ as $n\to\infty$ for any $k\geq 1$. Finally, we obtain by orthogonality of the martingale differences

$$\lim_{n\to\infty}\frac{s_n^2}{n}=\lim_{n\to\infty}\frac{\mathbb{E}_\mu M_n^2}{n}=\lim_{n\to\infty}\frac{\sum_{i=1}^n\mathbb{E}_\mu Z_i^2}{n}=\sigma^2.$$

Lemma 3.2. *The square integrable martingale differences $(Z_n)_{n\geq 1}$ satisfy the following condition:*

$$\frac{1}{n}\sum_{l=1}^n Z_l^2\to\sigma^2\quad\mathbb{P}_\mu\text{-a.s. as } n\to\infty\tag{3.7}$$

and consequently if $\sigma^2>0$ condition (3.3) holds.

Proof. First observe that to finish the proof it is enough to show that for any $i\in\{1,\dots,n_0\}$ we have

$$\frac{1}{n}\sum_{l=1}^n Z_{i+ln_0}^2\to\sigma^2\quad\mathbb{P}_\mu\text{-a.s. as } n\to\infty.$$

If we show that both the functions

$$x\to\mathbb{E}_x(|\liminf_{n\to\infty}(1/n\sum_{l=1}^n Z_{i+ln_0}^2)-\sigma^2|\wedge 1)$$

and

$$x\to\mathbb{E}_x(|\limsup_{n\to\infty}(1/n\sum_{l=1}^n Z_{i+ln_0}^2)-\sigma^2|\wedge 1)$$

are continuous, we shall be done. Indeed, then we have

$$\begin{aligned}
&\mathbb{E}_\mu(|\liminf_{n\to\infty}(1/n\sum_{l=1}^n Z_{i+ln_0}^2)-\sigma^2|\wedge 1)\\
&\qquad=\int_E\mathbb{E}_x(|\liminf_{n\to\infty}(1/n\sum_{l=1}^n Z_{i+ln_0}^2)-\sigma^2|\wedge 1)\mu(\mathrm{d}x)\\
&\qquad=\int_E\mathbb{E}_x(|\liminf_{n\to\infty}(1/n\sum_{l=1}^n Z_{i+ln_0}^2)-\sigma^2|\wedge 1)\mu P^{i+mn_0}(\mathrm{d}x)\\
&\qquad\to\mathbb{E}_{\mu_*}(|\liminf_{n\to\infty}(1/n\sum_{l=1}^n Z_{i+ln_0}^2)-\sigma^2|\wedge 1),
\end{aligned}$$

as $m \to +\infty$, by the fact that μP^{i+mn_0} converges weakly to μ_* as $m \to +\infty$. On the other hand, from the Birkhoff individual ergodic theorem we have

$$\mathbb{E}_{\mu_*}(|\liminf_{n\to\infty}(1/n\sum_{l=1}^{n} Z^2_{i+ln_0}) - \sigma^2| \wedge 1) = 0$$

and consequently

$$\mathbb{E}_{\mu}(|\liminf_{n\to\infty}(1/n\sum_{l=1}^{n} Z^2_{i+ln_0}) - \sigma^2| \wedge 1) = 0,$$

which, in turn, gives

$$\liminf_{n\to\infty}(1/n\sum_{l=1}^{n} Z^2_{i+ln_0}) = \sigma^2 \quad \mathbb{P}_{\mu}\text{-a.s.}$$

Analogously we may show that

$$\limsup_{n\to\infty}(1/n\sum_{l=1}^{n} Z^2_{i+ln_0}) = \sigma^2 \quad \mathbb{P}_{\mu}\text{-a.s.}$$

The remainder of the proof is devoted to showing the continuity of the relevant functions. Again, we restrict to the first function, since the proof for the second one goes in almost the same manner.

Observe that

$$\begin{aligned}
&\mathbb{E}_x(|\liminf_{n\to\infty}(1/n\sum_{l=1}^{n} Z^2_{i+ln_0}) - \sigma^2| \wedge 1) \\
&= \lim_{n\to\infty}\lim_{k\to\infty}\mathbb{E}_x\left(\left|\min\left\{\frac{1}{n}\sum_{l=1}^{n} Z^2_{i+ln_0} - \sigma^2, \dots, \frac{1}{n+k}\sum_{l=1}^{n+k} Z^2_{i+ln_0} - \sigma^2\right\}\right| \wedge 1\right) \\
&= \lim_{n\to\infty}\lim_{k\to\infty} H_{n,k}(x),
\end{aligned}$$

where

$$\begin{aligned}
H_{n,k}(x) &= \mathbb{E}_x(|\min\{1/n(\sum_{l=1}^{n} Z^2_{i+ln_0} \wedge n(1+\sigma^2)) - \sigma^2, \dots, \\
&\qquad 1/(n+k)(\sum_{l=1}^{n+k} Z^2_{i+ln_0} \wedge (n+k)(1+\sigma^2) - \sigma^2)\}| \wedge 1) \\
&= \mathbb{E}_x(|\min\{1/n(\sum_{l=1}^{n}(\chi(X_{i+ln_0}) - \chi(X_{i-1+ln_0}) + \psi(X_{i-1+ln_0}))^2 \wedge n(1+\sigma^2)) - \sigma^2, \\
&\dots, 1/(n+k)(\sum_{l=1}^{n+k}(\chi(X_{i+ln_0}) - \chi(X_{i-1+ln_0}) + \psi(X_{i-1+ln_0}))^2 \wedge (n+k)(1+\sigma^2) \\
&\qquad - \sigma^2)\}| \wedge 1).
\end{aligned}$$

Set

$$
\begin{aligned}
&g_{n,k}(y_1,\dots,y_{2(n+k)})\\
&= |\min\{1/n(\sum_{l=1}^{n}(\chi(y_{2l})-\chi(y_{2l-1})+\psi(y_{2l-1}))^2 \wedge n(1+\sigma^2)) - \sigma^2, \dots,\\
&1/(n+k)(\sum_{l=1}^{n+k}(\chi(y_{2l})-\chi(y_{2l-1})+\psi(y_{2l-1}))^2 \wedge (n+k)(1+\sigma^2) - \sigma^2)\}| \wedge 1
\end{aligned}
$$

so that

$$
H_{n,k}(x) = \mathbb{E}_x(g_{n,k}(X_{i+n_0-1}, X_{i+n_0}, X_{i+2n_0-1}, X_{i+2n_0}, \dots, X_{i+2(n+k)n_0})).
$$

Observe that $H_{n,k}$ is given by formula (2.3). If we show that there exists L such that $g_{n,k}$ is Lipschitz continuous in each variable with the Lipschitz constant L (independent of n,k), then all $H_{n,k}$ are Lipschitzean with the same Lipschitz constant $\tilde{L}$, by Lemma 1. Consequently $\lim_{n\to\infty}\lim_{k\to\infty} H_{n,k}$ is Lipschitzean and in particular continuous. Since minimum of any finite family of functions which are Lipschitz continuous in each variable with the Lipschitz constant L is Lipschitz continuous in each variable with the same Lipschitz constant L, to finish the proof it is enough to observe that the function

$$
(y_1,\dots,y_{2p}) \to 1/p(\sum_{l=1}^{p}(\chi(y_{2l})-\chi(y_{2l-1})+\psi(y_{2l-1}))^2 \wedge p(1+\sigma^2)) - \sigma^2
$$

is Lipschitz continuous in each variable with the Lipschitz constant L for fixed $L > 0$. On the other hand, each term in the above sum is Lipschitz continuous in each variable with the Lipschitz constant

$$
(1/p)(\operatorname{Lip}\chi + \operatorname{Lip}\psi)2p(1+\sigma^2) = 2(\operatorname{Lip}\chi + \operatorname{Lip}\psi)(1+\sigma^2).
$$

Observe that each variable appears in one term in the above sum. Hence $L \le 2(\operatorname{Lip}\chi + \operatorname{Lip}\psi)(1+\sigma^2)$, which finishes the proof. □

Note that following the proof of our previous lemma we are able to show that the considered Markov chain satisfies the strong law of large numbers **(SLLN)**. This result however directly follows from Theorem 2.1 in [20].

Lemma 3.3. *Let $\sigma^2 > 0$. Under the assumptions (H0)–(H3) the square integrable martingale differences $(Z_n)_{n\ge1}$ satisfy conditions (3.1), (3.2).*

Proof. Since $\sup_{n\ge1}\mathbb{E}_\mu|Z_n|^{2+\delta} < \infty$, δ is the constant given in (H3), we have

$$
\begin{aligned}
\sum_{n=1}^{\infty} s_n^{-4}\mathbb{E}_\mu[Z_n^4\mathbf{1}_{\{|Z_n|<\gamma s_n\}}] &\le \sum_{n=1}^{\infty} s_n^{-4}\gamma^{2-\delta}s_n^{2-\delta}\mathbb{E}_\mu|Z_n|^{2+\delta}\\
&\le \gamma^{2-\delta}\sup_{n\ge1}\mathbb{E}_\mu|Z_n|^{2+\delta}\sum_{n=1}^{\infty}s_n^{-2-\delta}.
\end{aligned}
$$

On the other hand, the condition $s_n^2/n \to \sigma^2$ as $n \to \infty$ gives $\sum_{n=1}^{\infty} s_n^{-2-\delta} < \infty$, which completes the proof of condition (3.1).

To show condition (3.2) observe that

$$\sum_{n=1}^{\infty} s_n^{-1} \mathbb{E}_\mu[|Z_n| \mathbf{1}_{\{|Z_n| \geq \epsilon s_n\}}] \leq \sum_{n=1}^{\infty} s_n^{-1} \mathbb{E}_\mu[|Z_n|^{2+\delta}/(\epsilon s_n)^{1+\delta}]$$

$$\leq \epsilon^{-1-\delta} \sup_{n \geq 1} \mathbb{E}_\mu |Z_n|^{2+\delta} \sum_{n=1}^{\infty} s_n^{-2-\delta} < \infty.$$

The proof is complete. □

3.3 The Law of Iterated Logarithm for Markov chains

Theorem 3.4. *Let (E, ρ) be a Polish space, $\{X_n\}$ a Markov chain with state space E, transition operator P satisfying conditions (H0)–(H2), and initial probability μ satisfying (H3). If ψ is a Lipschitz function with $\langle \psi, \mu_* \rangle = 0$ and $\sigma^2 > 0$, then $\mathbb{P}_\mu$-a.s. the sequence*

$$\theta_n(t) = \frac{\sum_{i=1}^{k} \psi(X_i) + (nt - k)\psi(X_{k+1})}{\sigma \sqrt{2n \log \log n}}$$

if $k \leq nt \leq k+1$, $k = 1, \ldots, n-1$ for $t > 0$, $n > e$ and $\theta_n(t) = 0$ otherwise is relatively compact in C and the set of its limit points coincides with K.

Proof. First observe that since $s_n^2/n \to \sigma^2 > 0$ as $n \to \infty$ we have

$$\frac{\sqrt{2 s_n^2 \log \log s_n^2}}{\sigma \sqrt{2n \log \log n}} \to 1 \quad \text{as } n \to \infty.$$

Consequently, from Lemmas 2 and 3 it follows that the sequence

$$\eta_n(t) = \frac{M_k + (s_n^2 t - s_k^2)(s_{k+1}^2 - s_k^2)^{-1} Z_{k+1}}{\sigma \sqrt{2n \log \log n}}$$

if $s_k^2 \leq s_n^2 t \leq s_{k+1}^2$, $k = 1, \ldots, n-1$ for $t > 0, n > e$ and $\eta_n(t) = 0$ otherwise is relatively compact in C and the set of its limit points coincides with K, due to Heyde and Scott [10]. Let $t \in (0, 1]$ and $n \geq 1$. Observe that if $k \leq nt \leq k+1$, then

$$\frac{k\sigma^2}{s_k^2} s_k^2 \leq \frac{n\sigma^2}{s_n^2} t s_n^2 \leq \frac{(k+1)\sigma^2}{s_{k+1}^2} s_{k+1}^2.$$

Set

$$\hat{\eta}_n(t) = \frac{M_k + (nt - k) Z_{k+1}}{\sigma \sqrt{2n \log \log n}},$$

where $k \geq 1$ such that $k \leq nt \leq k+1$. Since $(n\sigma^2)/s_n^2 \to 1$ as $n \to \infty$ for any $\varepsilon > 0$ holds

$$(1 - \varepsilon) s_k^2 \leq (1 + \varepsilon) s_n^2 t \leq (1 + \varepsilon)^2 (1 - \varepsilon)^{-1} s_{k+1}^2$$

for all n large enough. Hence there is $t_* \in [t(1-\varepsilon)(1+\varepsilon)^{-1}, t(1+\varepsilon)(1-\varepsilon)^{-1}]$ such that $s_k^2 \le s_n^2 t_* \le s_{k+1}^2$. On the other hand, the diameter of the interval $[s_k^2/s_n^2, s_{k+1}^2/s_n^2]$ for a fixed $k = 1, \dots, n-1$ converges to 0 as $n \to \infty$. Consequently, for any $t > 0$ and $n > e$ there exists $t_n > 0$ such that $\hat{\eta}_n(t) = \eta_n(t_n)$ and $t_n \to t$ as $n \to \infty$. Since the sequence $(\eta_n(t))_{n>e}$ is relatively compact in C and the set of its limit points coincides with K, the sequence $(\hat{\eta}_n(t))_{n>e}$ is also relatively compact and has the same set of limits points.

Fix $\varepsilon > 0$. Define the sets

$$A_n = \left\{ \omega \in \Omega : \frac{|M_n - \sum_{i=1}^{n-1} \psi(X_i)|}{\sqrt{n}} \ge \varepsilon/2 \right\} \cup \left\{ \omega \in \Omega : \frac{|Z_{n+1} - \psi(X_n)|}{\sqrt{n}} \ge \varepsilon/2 \right\} \quad \text{for } n \ge 1.$$

Now we are going to show that $\sum_{n=1}^{\infty} \mathbb{P}_\mu(A_n) < \infty$. Indeed, keeping in mind that χ is Lipschitzean, by the Chebyshev inequality we obtain

$$\begin{aligned} &\mathbb{P}_\mu \left(\left\{ \omega \in \Omega : \frac{|M_n - \sum_{i=1}^{n-1} \psi(X_i)|}{\sqrt{n}} \ge \varepsilon/2 \right\} \right) \\ &= \mathbb{P}_\mu \left(\left\{ \omega \in \Omega : \frac{|\chi(X_n) - \chi(X_0)|}{\sqrt{n}} \ge \varepsilon/2 \right\} \right) \\ &\le c_0 \frac{\mathbb{E}(\rho_{x_0}(X_n))^{2+\delta} + \mathbb{E}(\rho_{x_0}(X_0))^{2+\delta}}{n^{1+\delta/2}} \le \frac{\tilde{c}}{n^{1+\delta/2}}, \end{aligned}$$

by (H3) for some constant $\tilde{c} > 0$ independent of n.

Analogously, we may check that there exists a positive constant $\tilde{C}$ (independent of n) such that

$$\begin{aligned} &\mathbb{P}_\mu \left(\left\{ \omega \in \Omega : \frac{|Z_{n+1} - \psi(X_n)|}{\sqrt{n}} \ge \varepsilon/2 \right\} \right) \\ &= \mathbb{P}_\mu \left(\left\{ \omega \in \Omega : \frac{|\chi(X_{n+1}) - \chi(X_n)|}{\sqrt{n}} \ge \varepsilon/2 \right\} \right) \\ &\le \frac{\tilde{C}}{n^{1+\delta/2}}, \end{aligned}$$

by (H3) and the Lipschitz property of the function χ. Thus the series $\sum_{n=1}^{\infty} \mathbb{P}_\mu(A_n)$ is convergent.

Finally, from the Borel–Cantelli lemma it follows that $\mathbb{P}_\mu$-a.s.

$$\limsup_{n\to\infty} \sup_{0\le t\le 1} \left| \frac{M_k + (nt-k)Z_{k+1}}{\sigma\sqrt{2n\log\log n}} - \frac{\sum_{i=1}^{k} \psi(X_i) + (nt-k)\psi(X_{k+1})}{\sigma\sqrt{2n\log\log n}} \right| < \varepsilon,$$

where $k \le nt \le k+1$. Since $\varepsilon > 0$ was arbitrary, the proof is complete. □

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