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Limit Theorems For Markov Operators

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Introduction

For every measure $\nu \in \mathcal{M}_1$ the law of the Markov chain $(X_n)_{n\geq 0}$ on some Polish space (E,ρ) with transition probability π and initial distribution ν, is the probability measure $\mathbb{P}_\nu$ on $(E^{\mathbb{N}}, \mathcal{B}(E)^{\otimes \mathbb{N}})$ such that:

$$\mathbb{P}_\nu[X_{n+1} \in A | X_n = x] = \pi(x, A) \quad \text{and} \quad \mathbb{P}_\nu[X_0 \in A] = \nu(A),$$

where $x \in E$, $A \in \mathcal{B}(E)$. (Here $\mathcal{B}(E)$ denotes the family of all Borel sets.)
The expectation with respect to $\mathbb{P}_\nu$ is denoted by $\mathbb{E}_\nu$. For $\nu = \delta_x$, the Dirac measure at $x \in E$, we write just $\mathbb{P}_x$ and $\mathbb{E}_x$.

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The Markov operator P^* is defined by the formula

$$P^* f(x) = \int_E f(y)\pi(x, \mathrm{d}y)$$

for every bounded Borel measurable function f on E.
We shall denote by P, the associated transfer operator describing the evolution of the law of X_n. To be precise, $P\mu$ is defined by the formula $\int_E f(x)P\mu(\mathrm{d}x) = \int_E P^* f(x)\mu(\mathrm{d}x)$ for any $f \in B_b(E)$ and $\mu \in \mathcal{M}_1$.

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We shall denote by $\mathcal{M}_{1,1}$ the space of all probability measures possessing finite first moment, i.e. $\nu \in \mathcal{M}_{1,1}$ iff $\nu \in \mathcal{M}_1$ and $\int_E \rho(x_0, x)\nu(\mathrm{d}x) < \infty$ for some (thus all) $x_0 \in E$. For abbreviation we shall write $\rho_{x_0}(x) = \rho(x_0, x)$. Observe that every Lipschitz function (even unbounded) is integrable with respect to ν in $\mathcal{M}_{1,1}$.

It may be proved that the space $\mathcal{M}_{1,1}$ is a complete metric space when equipped with the Wasserstein metric

$$d(\nu_1, \nu_2) = \sup\left\{\left|\int_E f\mathrm{d}\nu_1 - \int_E f\mathrm{d}\nu_2\right| : f : E \to \mathbb{R},\ \operatorname{Lip} f \le 1\right\}$$

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Assumptions

We will make the following assumptions:

(H0) the Markov operator satisfies the Feller property, i.e. $P^*(C_b(E)) \subset C_b(E)$.

(H1) for any $\nu \in \mathcal{M}_{1,1}$ we have $P\nu \in \mathcal{M}_{1,1}$.

(H2) there exist $\gamma \in (0,1)$ and $c > 0$ such that

$$d(P^n\nu_1, P^n\nu_2) \le c\gamma^n d(\nu_1, \nu_2) \qquad (1)$$

for $n \ge 1, \nu_1, \nu_2 \in \mathcal{M}_{1,1}$.

(H3) there exists $x_0 \in E$ and $\delta > 0$ such that

$$\sup_{n\ge 0} \mathbb{E}_\mu \rho_{x_0}^{2+\delta}(X_n) = \sup_{n\ge 0} \int \rho_{x_0}^{2+\delta} d(P^n\mu) < \infty \qquad (2)$$

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The Central Limit Theorem

Let μ_* be a unique invariant measure for a given Markov operator P and let $g : E \to \mathbb{R}$ be a Lipschitz function such that $\int_X g\, d\mu_* = 0$. Let $(X_n)_{n\geq 1}$ be a Markov chain with some initial distribution μ. Define

$$S_n = S_n^{\mu}(g) := g(X_1) + g(X_2) + \cdots + g(X_n), \qquad \hat{S}_n := \frac{1}{\sqrt{n}} S_n.$$

We say that the Markov chain $(g(X_n))_{n\geq 0}$ satisfies the **Central Limit Theorem** if

$$\frac{\hat{S}_n^{\mu}}{\sqrt{n}} \Longrightarrow Z \quad \text{as } n \to \infty,$$

where Z is a random vector with normal distribution $\mathcal{N}(0, \sigma^2)$ and the convergence is understood in law.

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$$\limsup_{n\to\infty} \frac{g(X_n)}{\sigma\sqrt{2n\log\log n}} = 1$$

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The CLT - stationary case; Maxwell–Woodroofe Theorem

Theorem 1. Maxwell-Woodroofe, Ann. Prob. 28 (2), 2000

Let $(X_n)_{n\geq 0}$ be a Markov chain with the stationary initial distribution μ_*. If $g \in L^2(X, \mu_*)$ is a Lipschitz function such that $\int_X g\, d\mu_* = 0$ and

$$\sum_{n=1}^{\infty} n^{-3/2} \| \sum_{k=0}^{n-1} P^{*k} g \|_{L^2(X,\mu_*)} < \infty, \tag{3}$$

then $\lim_{n\to\infty} \mathbb{E}_{\mu^*} \left[\hat{S}_n^2 \right] := \sigma^2$ exists and and $(g(X_n))_{n\geq 1})$ satisfies the CLT.

The CLT - stationary case

Theorem 2.

Assume that the operator P describing the evolution of the law of $(X_n)_{n\geq 0}$ with the initial stationary distribution μ_* satisfies conditions **(H0)–(H2)**. If $g \in L^2(X, \mu_*)$ is a Lipschitz function such that $\int_X g \, d\mu_* = 0$, then $(g(X_n)_{n\geq 1})$ satisfies the CLT.

Proof: It is easy to show that condition (**??**) holds. □

The CLT - general case

Theorem 3.

Assume that the operator P describing the evolution of the law of $(X_n)_{n\geq 0}$ with the initial distribution $\mu \in \mathcal{M}_{1,1}$ satisfies conditions **(H0)–(H3)**. If $g \in L^2(X, \mu_*)$ is a Lipschitz function such that $\int_X g\, d\mu_* = 0$, then $(g(X_n))_{n\geq 0}$ satisfies the CLT.

Proof: From Theorem 2 we know that the CLT is satisfied when $\mu = \mu_*$. In particular,

$$\mathbb{E}_{\mu_*} \exp[it\hat{S}_n(g)] \to e^{-\sigma^2 t^2/2}.$$

With our assumption we may check that

$$|\mathbb{E}_{\mu_*} \exp[it\hat{S}_n(g)] - \mathbb{E}_{\mu} \exp[it\hat{S}_n(g)]| \to 0.$$

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