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Markov operators and their stability

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Definitions

Let (X, ρ) be a Polish space. Let $\mathcal{B}(X)$ be the space of all Borel subsets of X and let $B(X)$ (resp. $C(X)$) be the Banach space of all bounded, measurable (resp. continuous) functions on X equipped with the supremum norm $\|\cdot\|_\infty$. We denote by $\mathrm{Lip}_b(X)$ the space of all bounded Lipschitz continuous functions on X. By $\mathcal{M}$ and $\mathcal{M}_1$ we denote the family of Borel measures such that $\mu(X) < \infty$ for $\mu \in \mathcal{M}$ and $\mu(X) = 1$ for $\mu \in \mathcal{M}_1$.

An operator $P : \mathcal{M} \to \mathcal{M}$ will be called a *Markov operator* if it satisfies the following two conditions

- positive linearity: $P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2$ for $\lambda_1, \lambda_2 \geq 0$; $\mu_1, \mu_2 \in \mathcal{M}$;
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We start with two examples of Markov operators:

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Let $S : X \to X$ be a Borel measurable transformation. Then the operator $P : \mathcal{M} \to \mathcal{M}$ defined by

$$P\mu(A) = \mu(S^{-1}(A)) \qquad \text{for } A \in \mathcal{B}(X)$$

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We consider the system

$$x_{n+1} = T(x_n, \xi_n) \qquad \text{for } n = 0, 1, \dots,$$

where T is a given transformation and the ξ_n are independent random vectors. We make the following assumptions:

- T is defined on the space $X \times W$ with values in X, where W is a closed subset of $\mathbb{R}$. For every fixed $y \in W$ the function $T(x, y)$ is continuous in x and for every fixed $x \in X$ it is measurable in y.
- The random variables $\xi_0, \xi_1, \dots,$ have values in W and have te same distribution, that is, the measure

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Indeed, $\mu_{n+1} = P\mu_n$, where

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for any $f \in C(X)$ and $\mu \in \mathcal{M}$.

We say that a sequence of measures $(\mu_n)_{n\geq 1}$ *converges weakly* to some measure μ_* if

$$\lim_{n\to\infty} \int_X f(x)\mu_n(\mathrm{d}x) = \int_X f(x)\mu(\mathrm{d}x) \quad \text{for any } f \in C(X).$$

Let P be a Markov operator; a measure $\mu \in \mathcal{M}$ is called *invariant* if $P\mu = \mu$. A Markov operator P is called *asymptotically stable* if there exists a stationary measure $\mu_* \in \mathcal{M}_1$ such that the sequence $(P^n\mu)_{n\geq 1}$ for any $\mu \in \mathcal{M}_1$ converges weakly to μ_*.

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Nonnegative functionals

By $C^*(X)$ and $C^*_+(X)$ we denote the space of all bounded linear functionals and the subspace of all nonnegative functionals, respectively.

We say that $\varphi \in C^*_+(X)$ is a *Riesz functional* if there exists a Borel measure μ such that

$$\varphi(f) = \int_X f(x)\mu(dx) \qquad \text{for } f \in C(X).$$

We say that $\varphi_* \in C^*_+(X)$ is a *Banach functional* if for every Borel measure μ the condition

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Nonnegative functionals

We may formulate the following decomposition theorem for $\varphi \in C_+^*(X)$.

Theorem 1.

For every functional $\varphi \in C_+^*(X)$ there exists a Riesz functional φ_0 and a Banach functional φ_* such that

$$\varphi = \varphi_0 + \varphi_*.$$

This decomposition is unique, namely the measure μ corresponding to φ_0 is given by the formula

$$\mu(K) = \inf\{\varphi(f) : f \in C(X),\ f \geq \mathbf{1}_K\},$$

where $K \subset X$ is an arbitrary compact set.

Banach limits

Let l^∞ denote the space of all bounded sequences. A functional $\mathbb{L} : l^\infty \to \mathbb{R}$ is called a *Banach limit* if it is nonnegative, bounded with the norm 1 and such that

$$\mathbb{L}((a_1, a_2, \ldots)) = \mathbb{L}((a_2, a_3, \ldots)) \quad \text{for all } (a_1, a_2, \ldots) \in l^\infty.$$

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Invariant measures for Markov operators

Theorem 2.

Let $P : \mathcal{M} \to \mathcal{M}$ be a Feller operator. Assume that there is a compact set $Y \subset X$ and a measure $\mu_0 \in \mathcal{M}_1$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} P^k \mu_0(Y) > 0. \quad (1)$$

Then there exists an invariant measure $\mu_* \in \mathcal{M}_1$.

Corollary 1

Any Markov operator defined on a compact space possesses an invariant measure.

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Sketch of the Proof of Theorem 2: Let $\mathbb{L} : l^\infty \to \mathbb{R}$ be a Banach limit.

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From (1) and Theorem 1 it follows that the Riesz part φ_0 is nontrivial. On the other hand, from the properties of Banach limits we obtain that $\varphi(f) = \varphi(Uf)$, where U is dual to P. Hence the measure μ_* corresponding to φ_0 is invariant. □

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Wasserstein norm

In the space $\mathcal{M}_{sig} := \mathcal{M} - \mathcal{M}$ we introduce the **Wasserstein norm**

$$\|\mu - \nu\| = \sup\left\{\left|\int_X f \mathrm{d}(\mu - \nu)\right| : \|f\|_\infty \leq 1, \operatorname{Lip} f \leq 1\right\}$$

for $\mu, \nu \in \mathcal{M}$.

- Convergence in the Wasserstein norm is equivalent to the weak convergence, i.e.

$$\lim_{n\to\infty} \|\mu_n - \mu\| = 0 \text{ iff } \lim_{n\to\infty} \int_X f(x)\mu_n(\mathrm{d}x) = \int_X f(x)\mu(\mathrm{d}x)$$

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Nonexpansive Markov operators

A Markov operator P will be called *nonexpansive* if

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Theorem 3.

Any nonexpansive Markov operator is a Feller operator.

Sketch of the Proof of Theorem 3: Define the operator U by the formula:

$$Uf(x) = \int_X f(y) P\delta_x(\mathrm{d}y),$$

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Stability of Markov operators

Theorem 3.

Let $P : \mathcal{M} \to \mathcal{M}$ be a nonexpansive Markov operator. Assume that for every $\varepsilon > 0$ there is a Borel set A with $\operatorname{diam} A \le \varepsilon$ and a number $\alpha > 0$ such that

$$\liminf_{n\to\infty} P^n\mu(A) \ge \alpha \qquad \text{for } \mu \in \mathcal{M}_1.$$

Then P is asymptotically stable.

E–property

We say that a Markov operator P satisfies the *e–property* at $z \in X$ if it is a Feller operator with dual U and for any bounded Lipschitz function f we have

$$\lim_{\rho(x,z)\to 0} \sup\{|U^n f(x) - U^n f(z)| : n \geq 1\} = 0.$$

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Invariant measures for P with the e–property.

In applications the phase space (X, ρ) is usally some space of functions. Then it is very hard to verify condition (1). For Markov operators with the e–property we may weaken this condition.

The following theorem holds:

Theorem 4.

Let $P : \mathcal{M} \to \mathcal{M}$ be a Markov operator. Assume that there exists $z \in X$ such that for every $\varepsilon > 0$ we have

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} P^i \mu(B(z, \varepsilon)) > 0$$

for some measure $\mu \in \mathcal{M}$. If P satisfies the e–property at z, then P admits an invariant measure.

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