# Markov operators and their stability - the list of exercises

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**Exercise 1.** Let $(X, \rho)$ be a Polish space and let $\mu \in \mathcal{M}$. Show that for any $\varepsilon > 0$ there is a compact set $K \subset X$ such that

$$\mu(X \setminus K) \leq \varepsilon.$$

**Exercise 2.** Let $(X, \rho)$ be a Polish space and let $P$ be a Markov operator. Assume that $P$ is asymptotically stable. Show that for any measure $\mu \in \mathcal{M}$ there exists a compact set $K \subset X$ such that

$$\max_{n \geq 1} P^n \mu(X \setminus K) \leq \varepsilon.$$

**Exercise 3.** Let $\mathcal{M}_{sig} = \mathcal{M} - \mathcal{M}$. Prove that

$$\|\nu\| := \sup \left\{ \int_X f(x)\nu(\mathrm{d}x) : \|f\|_\infty \leq 1 \text{ and } \operatorname{Lip} f \leq 1 \right\}$$

is a norm in $\mathcal{M}_{sig}$.

**Exercise 4.** Let $l^\infty$ denote the space of all bounded sequences. A functional $\mathbb{L} : l^\infty \to \mathbb{R}$ is called a *Banach limit* if it is nonnegative, bounded with the norm 1 and such that

$$\mathbb{L}((a_1, a_2, \ldots)) = \mathbb{L}((a_2, a_3, \ldots)) \quad \text{for all } (a_1, a_2, \ldots) \in l^\infty.$$

- Using the Hahn–Banach theorem prove that a Banach limit exists.
- Show that if $\mathbb{L}$ is a Banach limit, then

  $$\liminf_{n \to \infty} a_n \leq \mathbb{L}((a_1, a_2, \ldots)) \leq \limsup_{n \to \infty} a_n$$

  for $(a_1, a_2, \ldots) \in l^\infty$.

**Exercise 5.** Prove that the space $(\mathcal{M}_1, \| \cdot \|)$ is complete and separable.

**Exercise 6.** Prove that any nonexpansive Markov operator is a Feller operator.

**Exercise 7.** Consider $N$ continuous transformations

$$S_i : X \to X \qquad \text{for } i = 1, \ldots, N$$

and let

$$(p_1, p_2, \ldots, p_N), \quad p_i \geq 0 \quad \sum_{i=1}^{N} p_i = 1,$$

be a probabilistic vestor.

Define the Markov chain $(X_n)_{n\geq 1}$ in the following way: if an initial point $x_0 \in X$ is chosen we toss an $N$–sided die and if the number $i_0$ is drawn we define $x_1 = S_{i_0}(x_0)$. Then we toss up the die again and if the number $i_1$ is drawn we define $x_2 = S_{i_1}(x_1)$, and so on.

This procedure can be easily formalized: consider the sequence of independent random variables $\xi_0, \xi_1, \dots$ such that

$$\text{prob}(\xi_n = i) = p_i \qquad \text{for } i = 1, \dots, N.$$

The dynamical system is defined by the formula

$$x_{n+1} = S_{\xi_n}(x_n) \qquad \text{for } n = 1, 2, \dots$$

is called an iterated function system.

Let $\mu_n(A) = \text{prob}(x_n^{-1} \in A$ for an arbitrary Borel set $A$.

- Prove that

  $$\mu_{n+1} = P\mu_n$$

  where $P$ is of the form

  $$P\mu(A) = \sum_{i=1}^{N} p_i \mu(S_i^{-1}(A)). \tag{1}$$

- Show that $P$ is a Markov operator.
- Show that $P$ is a Feller operator and its dual is of the form

  $$Uf(x) = \sum_{i=1}^{N} p_i f(S_i(x)) \quad \text{for } f \in C(X).$$

**Exercise 8.** Consider an iterated function system $(S_1, \dots, S_N; p_1, \dots, p_N)$. Assume that $S_i$ is a Lipschitz function with the Lipschitz constant $L_i$ for $i = 1, \dots, N$. Show that if

$$\sum_{i=1}^{N} p_i L_i < 1,$$

then the operator $P$ given by (1) is asymptotically stable.